Dynamic Asset Pricing Theory
(Provisional Manuscript)

Darrell Duffie
Graduate School of Business
Stanford University

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Contents

Preface

THIS BOOK IS an introduction to the theory of portfolio choice and asset pricing in multiperiod settings under uncertainty. An alternate title might be "Arbitrage, Optimality, and Equilibrium," because the book is built around the three basic constraints on asset prices: absence of arbitrage, single-agent optimality, and market equilibrium. The most important unifying principle is that any of these three conditions implies that there are "state prices," meaning positive discount factors, one for each state and date, such that the price of any security is merely the state-price weighted sum of its future payoffs. This idea can be traced to Kenneth Arrow's (1953) invention of the general equilibrium model of security markets. Identifying the state prices is the major task at hand. Technicalities are given relatively little emphasis so as to simplify these concepts and to make plain the similarities between discrete and continuous-time models. All continuous-time models are based on Brownian motion, despite the fact that most of the results extend easily to the case of a general abstract information filtration.

To someone who came out of graduate school in the mid-eighties, the decade spanning roughly 1969-79 seems like a golden age of dynamic asset pricing theory. Robert Merton started continuous-time financial modeling with his explicit dynamic programming solution for optimal portfolio and consumption policies. This set the stage for his 1973 general equilibrium model of security prices, another milestone. His next major contribution was his arbitrage-based proof of the option pricing formula introduced by Fisher Black and Myron Scholes in 1973, and his continual development of that approach to derivative pricing. The Black-Scholes model now seems to be, by far, the most important single breakthrough of this "golden decade," and ranks alone with the Modigliani-Miller (1958) Theorem and the Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965) in its overall importance for financial theory and practice. A tremendously influential simplification of the Black-Scholes model appeared in the "binomial" option pricing model of Cox, Ross, and Rubinstein (1979), who drew on an insight of Bill Sharpe.

Working with discrete-time models, LeRoy (1973), Rubinstein (1976), and Lucas (1978) developed multiperiod extensions of the CAPM. To this day, the "Lucas model" is the "vanilla flavor" of equilibrium asset pricing models. The simplest multiperiod representation of the CAPM finally appeared in Doug Breeden's continuous-time consumption based CAPM, published in 1979. Although not published until 1985, the Cox-Ingersoll-Ross model of the term structure of interest rates appeared in the mid-seventies and is still the premier textbook example of a continuous-time general equilibrium asset pricing model with practical applications. It also ranks as one of the key breakthroughs of that decade. Finally, extending the ideas of Cox and Ross (1976) and Ross (1978), Harrison and Kreps (1979) gave an almost definitive conceptual structure to the whole theory of dynamic security prices.

The period since 1979, with relatively few exceptions, has been a mopping-up operation. On the theoretical side, assumptions have been weakened, there have been noteworthy extensions and illutrative models, and the various problems have become much more unified under the umbrella of the Harrison-Kreps model. On the applied side, markets have experienced an explosion of new valuation techniques, hedging applications, and security innovation, much of this based on the Black-Scholes and related arbitrage models. No major investment bank, for example, lacks the experts or computer technology required to implement advanced mathematical models of the term structure. Because of the wealth of new applications, there has been a significant development of special models to treat stochastic volatility, jump behavior including default, and the term structure of interest rates, along with many econometric advances designed to take advantage of the resulting improvements in richness and tractability.

Although it is difficult to predict where the theory will go next, in order to promote faster progress by people coming into the field it seems wise to have some of the basics condensed into a textbook. This book is designed to be a streamlined course text, not a research monograph. Much generality is sacrificed for expositional reasons, and there is relatively little emphasis on mathematical rigor or on the existence of general equilibrium. As its title indicates, I am treating only the theoretical side of the story. Although it might be useful to tie the theory to the empirical side of asset pricing, we have excellent treatments of the econometric modeling of financial data by

Campbell, Lo, and MacKinlay [1994] as well as Gourieroux, Scaillet, and Szafarz (1997). The story told by this book also leaves out some important aspects of functioning security markets such as asymmetric information, and transactions costs. I have chosen to develop only some of the essential ideas of dynamic asset pricing, and even these are more than enough to put into one book or into a one-semester course.

Other books whose treatments overlap with some of the topics treated here include Avelleneda and Laurence (1999), Bjork (1998), Dana and Jeanblanc (1998), Demange and Rochet (1992), Dewynne and Wilmott (1994), Dixit and Pindyck (1993), Dothan (1990), Duffie (1988), Harris (1987), Huang (1994), Huang and Litzenberger (1988), Ingersoll (1987), Jarrow (1988), Karatzas (1997), Karatzas and Shreve (1998), Lamberton and Lapeyre (1997), Magill and Quinzii (1994), Malliaris (1982), Merton (1990b), Musiela and Rutkowski (1998), Stokey and Lucas (1989), and Willmott, Howison, and Dewynne (1995). Each has its own aims and themes. There are also survey articles by Constantinides (1989), Cox and Huang (1988), Huang (1987b), Karatzas (1987), Marimon (1987), Merton (1990a, 1993, 1994a,b), and Rothschild (1986). I hope that readers will find some advantage in having yet another perspective.

A reasonable way to teach a shorter course on continuous-time asset pricing out of this book is to begin with Chapter 1 as an introduction to the basic notion of state prices and then to go directly to Chapters 5 through 10. Chapter 11, on numerical methods, could be skipped at some cost in the student's ability to implement the results. There is no direct dependence of any results in Chapters 5 through 11 on the first four chapters.

For mathematical preparation, little beyond undergraduate analysis, as in Bartle (1976), and linear algebra, as in O'Nan (1976), is assumed. Further depth, for example, by study of Rudin (1973) or a similar text on functional analysis and measure theory, would be useful. Some background in microeconomics would be useful, at the level of Kreps (1990), Luenberger (1995), or Varian (1984). Familiarity with probability theory at a level approaching Billingsley (1986), for example, would also speed things along, although measure theory is not used heavily. In any case, a series of appendices supplies all of the required concepts and definitions from probability theory and stochastic calculus. Additional useful references in this regard are Arnold (1974), Brémaud (1981), Chung and Williams (1990), Elliott (1982), Karatzas and Shreve (1988), Karr (1991), Kopp (1984), Oksendal (1985), Rogers and Williams (1987), and Stroock and Varadhan (1979).

Students seem to learn best by doing problem exercises. Each chapter has exercises and notes to the literature. I have tried to be thorough in giving sources for results whenever possible and plead that any cases in which I have mistaken or missed sources be brought to my attention for correction. The notation and terminology throughout is fairly standard. I use $\mathbb{R}$ to denote the real line and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ for the extended real line. For any set Z and positive integer n, I use Z^n for the set of n-tuples of the form $(z_1, \ldots, z_n)$ with z_i in Z for all i. For example, think of $\mathbb{R}^n$. The conventions used for inequalities in any context are

- $x \geq 0$ means that x is nonnegative. For x in $\mathbb{R}^n$, this is equivalent to $x \in \mathbb{R}^n_+$;
- $x > 0$ means that x is nonnegative and not zero, but not necessarily strictly positive in all coordinates;
- $x \gg 0$ means x is strictly positive in every possible sense. The phrase "x is strictly positive" means the same thing. For x in $\mathbb{R}^n$, this is equivalent to $x \in \mathbb{R}^n_{++} \equiv \text{ int } (\mathbb{R}^n_+)$.

Although warnings will be given at appropriate times, it should be kept in mind that $X = Y$ will be used to mean equality almost everywhere or almost surely, as the case may be. The same caveat applies to each of the above inequalities. A function F on an ordered set (such as $\mathbb{R}^n$) is *increasing* if $F(x) \geq F(y)$ whenever $x \geq y$ and *strictly increasing* if $F(x) > F(y)$ whenever $x > y$. When the domain and range of a function are implicitly obvious, the notation "$x \mapsto F(x)$" means the function that maps x to $F(x)$; for example, $x \mapsto x^2$ means the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x^2$. Also, while warnings appear at appropriate places, it is worth pointing out again here that for ease of exposition, a continuous-time "process" will be defined throughout as a jointly measurable function on $\Omega \times [0, T]$, where $[0, T]$ is the given time interval and $(\Omega, \mathcal{F}, P)$ is the given underlying probability space.

This first four chapters take place in a discrete-time setting with a discrete set of states. This should ease the development of intuition for the models to be found in chapters 5 through 11. The three pillars of the theory, *arbitrage*, *optimality*, and *equilibrium*, are developed repeatedly in different settings. Chapter 1 is the basic single-period model. Chapter 2 extends the results of Chapter 1 to many periods. Chapter 3 specializes Chapter 2 to a Markov setting and illustrates dynamic programming as an alternate solution technique. The Ho-and-Lee and Black-Derman-Toy term-structure models are included as exercises. Chapter 4 is an infinite-horizon counterpart to Chapter 3 that has become known as the *Lucas model*.

The focus of the theory is the notion of state prices, which specify the price of any security as the state-price weighted sum or expectation of the security's state-contingent dividends. In a finite-dimensional setting, there exist state prices if and only if there is no arbitrage. The same fact is true in infinite-dimensional settings under mild technical regularity conditions. Given an agent's optimal portfolio choice, a state-price vector is given by that agent's utility gradient. In an equilibrium with Pareto optimality, a state-price vector is likewise given by a representative agent's utility gradient at the economy's aggregate consumption process.

Chapters 5 through 11 develop a continuous-time version of the theory in which uncertainty is generated by Brownian motion. The results are somewhat richer and more delicate than those in Chapters 1 through 4, with a greater dependence on mathematical technicalities. It is wiser to focus on the parallels than on these technicalities. Once again, the three basic forces behind the theory are arbitrage, optimality, and equilibrium.

Chapter 5 introduces the continuous-trading model and develops the Black-Scholes partial differential equation (PDE) for arbitrage-free prices of derivative securities. The Harrison-Kreps model of equivalent martingale measures is presented in Chapter 6 in parallel with the theory of state prices in continuous time. Chapter 7 presents models of the term structure of interest rates, including the Black-Derman-Toy, Vasicek, Cox-Ingersoll-Ross, and Heath-Jarrow-Morton models, as well as extensions. Chapter 8 presents specific classes of derivative securities, such as futures, forwards, American options, and lookback options. Chapter 8 also introduces models of option pricing with stochastic volatility. Chapter 9 is a summary of optimal continuous-time portfolio choice, using both dynamic programming and an approach involving equivalent martingale measures or state prices. Chapter 10 is a summary of security pricing in an equilibrium setting. Included are such well-known models as Breeden's consumption-based capital asset pricing model and the general equilibrium version of the Cox-Ingersoll-Ross model of the term structure of interest rates. Chapter 11 outlines three numerical methods for calculating derivative security prices in a continuous-time setting: binomial approximation, Monte Carlo simulation of a discrete-time approximation of security prices, and finite-difference solution of the associated PDE for the asset price or the fundamental solution.

In my preparation of the first edition, I relied on help from many people, in addition to those mentioned above who developed this theory. In 1982, Michael Harrison gave a class at Stanford that had a major effect on my understanding and research goals. Beside me in that class was Chi-fu Huang; we learned much of this material together, becoming close friends and collaborators. I owe him a lot. I am grateful to Niko and Vana Skiadas, who treated me with overwhelming warmth and hospitality at their home on Skiathos, where parts of the first draft were written. Useful comments on subsequent drafts have been provided by Howie Corb, Rui Kan, John Overdeck, Christina Shannon, Philippe Henrotte, Chris Avery, Pinghua Young, Don Iglehart, Rohit Rahi, Shinsuke Kambe, Marco Scarsini, Kerry Back, Heracles Polemarchakis, John Campbell, Ravi Myneni, Michael Intriligator, Robert Ashcroft, and Ayman Hindy. I thank Kingston Duffie, Ravi Myneni, Paul Bernstein, and Michael Boulware for coding and running some numerical examples. In writing the book, I have benefited from research collaboration over the years with George Constantinides, Larry Epstein, Mark Garman, John Geanakoplos, Chi-fu Huang, Matt Jackson, Pierre-Louis Lions, Andreu Mas-Colell, Andy McLennan, Philip Protter, Tony Richardson, Wayne Shafer, Ken Singleton, Costis Skiadas, Richard Stanton, and Bill Zame. At Princeton University Press, Jack Repcheck was a friendly, helpful, and supportive editor. I owe a special debt to Costis Skiadas, whose generous supply of good ideas has had a big influence on the result.

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Chapter 1

Introduction to State Pricing

This chapter introduces the basic ideas of the course in a finite-state one-period setting. In many basic senses, each subsequent chapter merely repeats this one from a new perspective. The objective is a characterization of security prices in terms of "state prices," one for each state of the world. The price of a given security is simply the state-price weighted sum of its payoffs in the different states. One can treat a state price as the "shadow price," or Lagrange multiplier, for wealth contingent on a given state of the world. We obtain a characterization of state prices, first based on the absence of arbitrage, then based on the first-order conditions for optimal portfolio choice of a given agent, and finally from the first-order conditions for Pareto optimality in an equilibrium with complete markets. State prices are connected with the "beta" model for excess expected returns, a special case of which is the Capital Asset Pricing Model (CAPM). Many readers will find this chapter to be a review of standard results. In most cases, here and throughout, technical conditions are imposed that give up much generality so as to simplify the exposition.

1A Arbitrage and State Prices

Uncertainty is represented here by a finite set $\{1, \ldots, S\}$ of states, one of which will be revealed as true. The N securities are given by an $N \times S$ matrix D, with D_{ij} denoting the number of units of account paid by security i in state j. The security prices are given by some q in $\mathbb{R}^N$. A *portfolio* $\theta \in \mathbb{R}^N$ has *market value* $q \cdot \theta$ and *payoff* $D^\top \theta$ in $\mathbb{R}^S$. An *arbitrage* is a portfolio θ in $\mathbb{R}^N$ with $q \cdot \theta \leq 0$ and $D^\top \theta > 0$, or $q \cdot \theta < 0$ and $D^\top \theta \geq 0$. An arbitrage is therefore, in effect, a portfolio offering "something for nothing." Not surprisingly, it will later be shown that an arbitrage is naturally ruled out, and this gives a characterization of security prices as follows. A *state-price* vector is a vector ψ in $\mathbb{R}^S_{++}$ with $q = D\psi$. We can think of ψ_j as the marginal cost of obtaining an additional unit of account in state j.

Theorem. *There is no arbitrage if and only if there is a state-price vector.*

Proof: The proof is an application of the Separating Hyperplane Theorem. Let $L = \mathbb{R} \times \mathbb{R}^S$ and $M = \{(-q \cdot \theta, D^\top \theta) : \theta \in \mathbb{R}^N\}$, a linear subspace of L. Let $K = \mathbb{R}_+ \times \mathbb{R}^S_+$, which is a cone (meaning that if x is in K, then λx is in K for each strictly positive scalar λ.) Both K and M are closed and convex subsets of L. There is no arbitrage if and only if K and M intersect precisely at 0, as pictured in Figure 1.1.

Suppose $K \cap M = \{0\}$. The Separating Hyperplane Theorem (in a version for closed cones that is found in Appendix B) implies the existence of a linear functional $F : L \to \mathbb{R}$ such that $F(z) < F(x)$ for all z in M and nonzero x in K. Since M is a linear space, this implies that $F(z) = 0$ for all z in M and that $F(x) > 0$ for all nonzero x in K. The latter fact implies that there is some $\alpha > 0$ in $\mathbb{R}$ and $\psi \gg 0$ in $\mathbb{R}^S$ such that $F(v, c) = \alpha v + \psi \cdot c$, for any $(v, c) \in L$. This in turn implies that $-\alpha q \cdot \theta + \psi \cdot (D^\top \theta) = 0$ for all θ in $\mathbb{R}^N$. The vector ψ / α is therefore a state-price vector.

Conversely, if a state-price vector ψ exists, then for any θ, we have $q \cdot \theta = \psi^\top D^\top \theta$. Thus, when $D^\top \theta \geq 0$ we have $q \cdot \theta \geq 0$, and when $D^\top \theta > 0$ we have $q \cdot \theta > 0$, so there is no arbitrage. ∎

1B Risk-Neutral Probabilities

We can view any p in $\mathbb{R}^S_+$ with $p_1 + \cdots + p_S = 1$ as a vector of probabilities of the corresponding states. Given a state-price vector ψ for the dividend-price pair (D, q), let $\psi_0 = \psi_1 + \cdots + \psi_S$ and, for any state j, let $\hat{\psi}_j = \psi_j / \psi_0$. We now have a vector $(\hat{\psi}_1, \ldots, \hat{\psi}_S)$ of probabilities and can write, for an arbitrary security i,

$$\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{S} \hat{\psi}_j D_{ij},$$

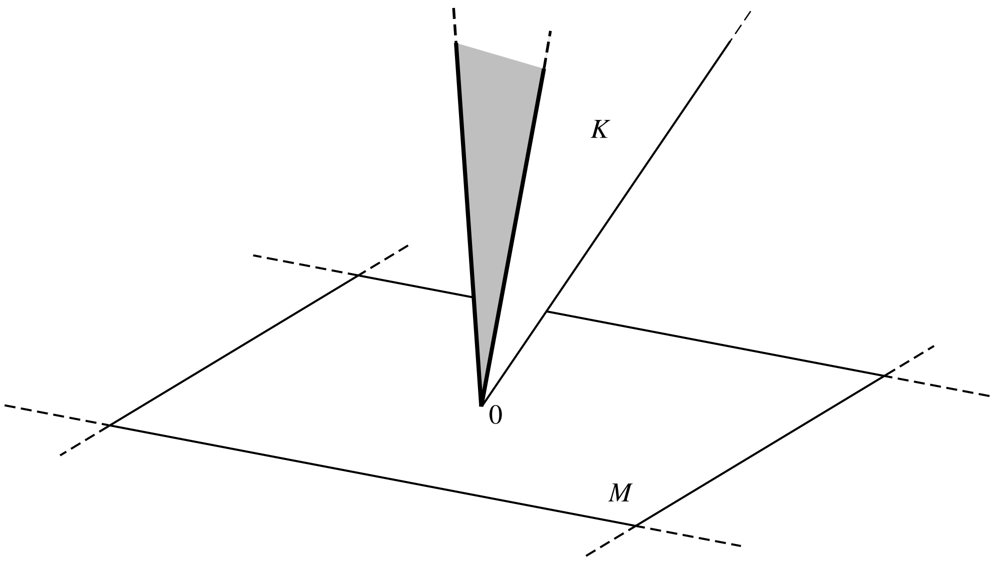

Figure1.1: Separating a Cone from a Linear Subspace

viewing the normalized price of the security as its expected payoff under specially chosen "risk-neutral" probabilities. If there exists a portfolio $\overline{\theta}$ with $D^\top\overline{\theta} = (1, 1, \ldots, 1)$, then $\psi_0 = \overline{\theta} \cdot q$ is the discount on riskless borrowing and, for any security i, $q_i = \psi_0 \hat{E}(D_i)$, showing any security's price to be its discounted expected payoff in this sense of artificially constructed probabilities.

1C Optimality and Asset Pricing

Suppose the dividend-price pair (D, q) is given. An *agent* is defined by a strictly increasing *utility function* $U : \mathbb{R}^S_+ \to \mathbb{R}$ and an *endowment* e in $\mathbb{R}^S_+$. This leaves the *budget-feasible set*

$$X(q, e) = \left\{ e + D^\top\theta \in \mathbb{R}^S_+ : \theta \in \mathbb{R}^N,\, q \cdot \theta \le 0 \right\},$$

and the problem

$$\sup_{c \in X(q,e)} U(c). \tag{1}$$

We will suppose for this section that there is some some portfolio θ^0 with payoff $D^\top\theta^0 > 0$. Because U is strictly increasing, the wealth constraint

$q \cdot \theta \leq 0$ is then binding at an optimum. That is, if $c^* = e + D^\top \theta^*$ solves (1), then $q \cdot \theta^* = 0$.

Proposition. *If there is a solution to (1), then there is no arbitrage. If U is continuous and there is no arbitrage, then there is a solution to (1).*

Proof is left as an exercise.

Theorem. *Suppose that c^* is a strictly positive solution to (1), that U is continuously differentiable at c^*, and that the vector $\partial U(c^*)$ of partial derivatives of U at c^* is strictly positive. Then there is some scalar $\lambda > 0$ such that $\lambda \partial U(c^*)$ is a state-price vector.*

Proof: The first-order condition for optimality is that for any θ with $q \cdot \theta = 0$, the marginal utility for buying the portfolio θ is zero. This is expressed more precisely in the following way: The strict positivity of c^* implies that for any portfolio θ, there is some scalar $k > 0$ such that $c^* + \alpha D^\top \theta \geq 0$ for all α in $[-k, k]$. Let $g_\theta : [-k, k] \to \mathbb{R}$ be defined by

$$g_\theta(\alpha) = U(c^* + \alpha D^\top \theta).$$

Suppose $q \cdot \theta = 0$. The optimality of c^* implies that g_θ is maximized at $\alpha = 0$. The first-order condition for this is that $g'_\theta(0) = \partial U(c^*)^\top D^\top \theta = 0$. We can conclude that, for any θ in $\mathbb{R}^N$, if $q \cdot \theta = 0$, then $\partial U(c^*)^\top D^\top \theta = 0$. From this, there is some scalar μ such that $\partial U(c^*)^\top D^\top = \mu q$.

By assumption, there is some portfolio θ^0 with $D^\top \theta^0 > 0$. From the existence of a solution to (1), there is no arbitrage, implying that $q \cdot \theta^0 > 0$. We have

$$\mu q \cdot \theta^0 = \partial U(c^*)^\top D^\top \theta^0 > 0.$$

Thus $\mu > 0$. We let $\lambda = 1/\mu$, obtaining

$$q = \lambda\, D \partial U(c^*), \tag{2}$$

implying that $\lambda \partial U(c^*)$ is a state-price vector. ∎

Although we have assumed that U is strictly increasing, this does not necessarily mean that $\partial U(c^*) \gg 0$. If U is concave and strictly increasing, however, it is always true that $\partial U(c^*) \gg 0$.

Corollary. *Suppose U is concave and differentiable at some $c^* = e + D^\top \theta^* \gg 0$, with $q \cdot \theta^* = 0$. Then c^* is optimal if and only if $\lambda \partial U(c^*)$ is a state-price vector for some scalar $\lambda > 0$.*

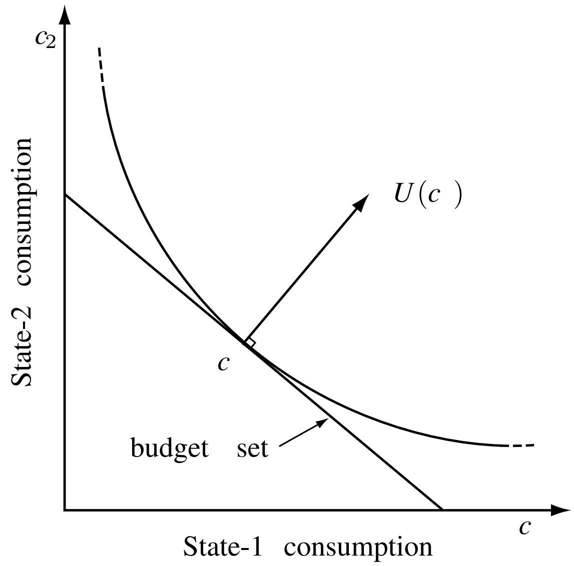

Figure1.2: First-Order Conditions for Optimal Consumption Choice

This follows from the sufficiency of the first-order conditions for concave objective functions. The idea is illustrated in Figure 1.2. In that figure, there are only two states, and a state-price vector is a suitably normalized nonzero positive vector orthogonal to the set $B = \{D^\top\theta : \; q\cdot\theta = 0\}$ of budget-neutral consumption adjustments. The first-order condition for optimality of c^* is that movement in any feasible direction away from c^* has negative or zero marginal utility, which is equivalent to the statement that the budget-neutral set is tangent at c^* to the preferred set $\{c : U(c) \geq U(c^*)\}$, as shown in the figure. This is equivalent to the statement that $\partial U(c^*)$ is orthogonal to B, consistent with the last corollary. Figure 1.3 illustrates a strictly suboptimal consumption choice c, at which the derivative vector $\partial U(c)$ is not co-linear with the state-price vector ψ.

We consider the special case of an *expected utility* function U, defined by a given vector p of probabilities and by some $u : \mathbb{R}_+ \to \mathbb{R}$ according to

$$U(c) = E\left[u(c)\right] \equiv \sum_{j=1}^{S} p_j u(c_j). \tag{3}$$

One can check that for $c \gg 0$, if u is differentiable, then $\partial U(c)_j = p_j u'(c_j)$.

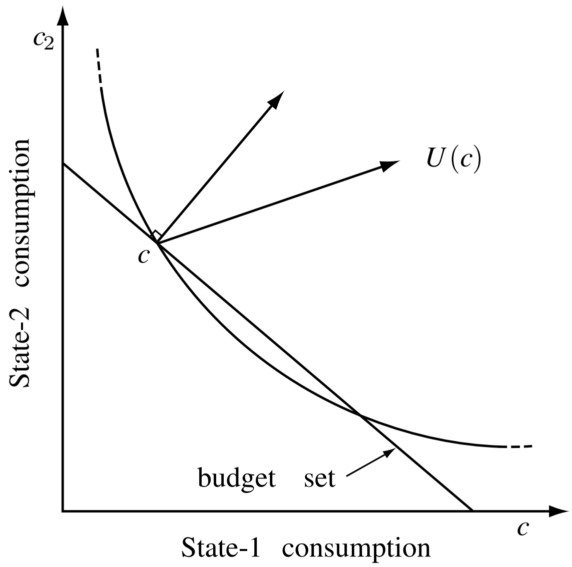

Figure1.3: A Strictly Suboptimal Consumption Choice

For this expected utility function, (2) therefore applies if and only if

$$q = \lambda\, E\left[Du'(c^*)\right], \tag{4}$$

with the obvious notational convention. As we saw in Section 1B, one can also write (2) or (4), with the "risk-neutral" probability $\hat{\psi}_j = u'(c_j^*)p_j/E[u'(c^*)]$, in the form

$$\frac{q_i}{\psi_0} = \hat{E}(D_i) \equiv \sum_{j=1}^{S} D_{ij}\hat{\psi}_j, \qquad 1 \leq i \leq N. \tag{5}$$

1D Efficiency and Complete Markets

Suppose there are m agents, defined as in Section 1C by strictly increasing utility functions $U_1, \ldots, U_m$ and by endowments $e^1, \ldots, e^m$. An *equilibrium* for the *economy* $[(U_i, e^i), D]$ is a collection $(\theta^1, \ldots, \theta^m, q)$ such that, given the security-price vector q, for each agent i, θ^i solves $\sup_\theta U_i(e^i + D^\top \theta)$ subject to $q \cdot \theta \leq 0$, and such that $\sum_{i=1}^m \theta^i = 0$. The existence of equilibrium is treated in the exercises and in sources cited in the notes.

With $\text{span}(D) \equiv \{D^\top \theta : \theta \in \mathbb{R}^N\}$ denoting the set of possible portfolio payoffs, markets are *complete* if $\text{span}(D) = \mathbb{R}^S$, and are otherwise *incomplete*.

Let $e = e^1 + \cdots + e^m$ denote the aggregate endowment. A consumption *allocation* $(c^1, \ldots, c^m)$ in $\left(\mathbb{R}^S_+\right)^m$ is *feasible* if $c^1 + \cdots + c^m \leq e$. A feasible allocation $(c^1, \ldots, c^m)$ is *Pareto optimal* if there is no feasible allocation $(\hat{c}^1, \ldots, \hat{c}^m)$ with $U_i(\hat{c}^i) \geq U_i(c^i)$ for all i and with $U_i(\hat{c}^i) > U_i(c^i)$ for some i. Complete markets and the Pareto optimality of equilibrium allocations are almost equivalent properties of any economy.

Proposition. *Suppose markets are complete and $(\theta^1, \ldots, \theta^m, q)$ is an equilibrium. Then the associated equilibrium allocation is Pareto optimal.*

This is sometimes known as *The First Welfare Theorem.* The proof, requiring only the strict monotonicity of utilities, is left as an exercise. We have established the sufficiency of complete markets for Pareto optimality. The necessity of complete markets for the Pareto optimality of equilibrium allocations does not always follow. For example, if the initial endowment allocation $(e^1, \ldots, e^m)$ happens by chance to be Pareto optimal, then any equilibrium allocation is also Pareto optimal, regardless of the span of securities. It would be unusual, however, for the initial endowment to be Pareto optimal. Although beyond the scope of this book, it can be shown that with incomplete markets and under natural assumptions on utility, for almost every endowment, the equilibrium allocation is not Pareto optimal.

1E Optimality and Representative Agents

Aside from its allocational implications, Pareto optimality is also a convenient property for the purpose of security pricing. In order to see this, consider, for each vector $\lambda \in \mathbb{R}^m_+$ of "agent weights," the utility function $U_\lambda : \mathbb{R}^S_+ \to \mathbb{R}$ defined by

$$U_\lambda(x) = \sup_{(c^1, \ldots, c^m)} \quad \sum_{i=1}^m \lambda_i \, U_i(c^i) \qquad \text{subject to } c^1 + \cdots + c^m \leq x. \tag{6}$$

Lemma. *Suppose that, for all i, U_i is concave. An allocation $(c^1, \ldots, c^m)$ that is feasible is Pareto optimal if and only if there is some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(c^1, \ldots, c^m)$ solves (6) at $x = e = c^1 + \cdots + c^m$.*

Proof: Suppose that $(c^1, \ldots, c^m)$ is Pareto optimal. Let $U(x) = (U_1(x^1), \ldots, U_m(x^m))$, for any allocation x, and let

$$\mathcal{U} = \{U(x) - U(c) - z \,:\, x \in \mathcal{A},\, z \in \mathbb{R}^m_+\} \subset \mathbb{R}^m,$$

where $\mathcal{A}$ is the set of feasible allocations. Let $J = \{y \in \mathbb{R}^m_+ : y \neq 0\}$. Since $\mathcal{U}$ is convex (by the concavity of utility functions) and $J \cap \mathcal{U}$ is empty (by Pareto optimality), the Separating Hyperplane Theorem (Appendix B) implies that there is a nonzero vector λ in $\mathbb{R}^m$ such that $\lambda \cdot y \leq \lambda \cdot z$ for each y in $\mathcal{U}$ and each z in J. Since $0 \in \mathcal{U}$, we know that $\lambda \geq 0$, proving the first part of the result. The second part is easy to show as an exercise. ▮

Proposition. *Suppose that for all i, U_i is concave. Suppose that markets are complete and that $(\theta^1, \ldots, \theta^m, q)$ is an equilibrium. Then there exists some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(0, q)$ is a (no-trade) equilibrium for the single-agent economy $[(U_\lambda, e), D]$ defined by (6). Moreover, the equilibrium consumption allocation $(c^1, \ldots, c^m)$ solves the allocation problem (6) at the aggregate endowment. That is, $U_\lambda(e) = \sum_i \lambda_i U_i(c^i)$.*

Proof: Since there is an equilibrium, there is no arbitrage, and therefore there is a state-price vector ψ. Since markets are complete, this implies that the problem of any agent i can be reduced to

$$\sup_{c \in \mathbb{R}^S_+} U_i(c) \qquad \text{subject to} \;\; \psi \cdot c \leq \psi \cdot e^i.$$

We can assume that e^i is not zero, for otherwise $c^i = 0$ and agent i can be eliminated from the problem without loss of generality. By the Saddle Point Theorem of Appendix B, there is a Lagrange multiplier $\alpha_i \geq 0$ such that c^i solves the problem

$$\sup_{c \in \mathbb{R}^S_+} U_i(c) - \alpha_i \left(\psi \cdot c - \psi \cdot e^i\right).$$

(The Slater condition is satisfied since e^i is not zero and $\psi \gg 0$.) Since U_i is strictly increasing, $\alpha_i > 0$. Let $\lambda_i = 1/\alpha_i$. For any feasible allocation $(x^1, \ldots, x^m)$, we have

$$\begin{aligned}
\sum_{i=1}^m \lambda_i U_i(c^i) &= \sum_{i=1}^m \left[\lambda_i U_i(c^i) - \lambda_i \alpha_i \left(\psi \cdot c^i - \psi \cdot e^i\right)\right] \\
&\geq \sum_{i=1}^m \lambda_i \left[U_i(x^i) - \alpha_i \left(\psi \cdot x^i - \psi \cdot e^i\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \lambda_i U_i(x^i) - \psi \cdot \sum_{i=1}^{m} (x^i - e^i) \\
&\geq \sum_{i=1}^{m} \lambda_i U_i(x^i).
\end{aligned}
$$

This shows that $(c^1, \ldots, c^m)$ solves the allocation problem (6). We must also show that no trade is optimal for the single agent with utility function U_λ and endowment e. If not, there is some x in $\mathbb{R}_+^S$ such that $U_\lambda(x) > U_\lambda(e)$ and $\psi \cdot x \leq \psi \cdot e$. By the definition of U_λ, this would imply the existence of an allocation $(x^1, \ldots, x^m)$, not necessarily feasible, such that $\sum_i \lambda_i U_i(x^i) > \sum_i \lambda_i U_i(c^i)$ and

$$
\sum_i \lambda_i \alpha_i \psi \cdot x^i = \psi \cdot x \leq \psi \cdot e = \sum_i \lambda_i \alpha_i \psi \cdot c^i.
$$

Putting these two inequalities together, we have

$$
\sum_{i=1}^{m} \lambda_i \left[U_i(x^i) - \alpha_i \psi \cdot (x^i - e^i) \right] > \sum_{i=1}^{m} \lambda_i \left[U_i(c^i) - \alpha_i \psi \cdot (c^i - e^i) \right],
$$

which contradicts the fact that, for each agent i, (c^i, α_i) is a saddle point for that agent's problem. ∎

Corollary 1. *If, moreover, $e \gg 0$ and U_λ is continuously differentiable at e, then λ can be chosen so that $\partial U_\lambda(e)$ is a state-price vector, meaning*

$$
q = D \partial U_\lambda(e). \tag{7}
$$

The differentiability of U_λ at e is implied by the differentiability, for some agent i, of U_i at c^i. (See Exercise 10 (C).)

Corollary 2. *Suppose there is a fixed vector p of state probabilities such that for all i, $U_i(c) = E[u_i(c)] \equiv \sum_{j=1}^{S} p_j u_i(c_j)$, for some $u_i(\,\cdot\,)$. Then $U_\lambda(c) = E[u_\lambda(c)]$, where, for each y in $\mathbb{R}_+$,*

$$
u_\lambda(y) = \max_{x \in \mathbb{R}_+^m} \sum_{i=1}^{m} \lambda_i u_i(x_i) \qquad \text{subject to } x_1 + \cdots + x_m \leq y.
$$

In this case, (7) is equivalent to $q = E[D u_\lambda'(e)]$.

Extensions of this *representative-agent* asset pricing formula will crop up frequently in later chapters.

1F State-Price Beta Models

We fix a vector $p \gg 0$ in $\mathbb{R}^S$ of probabilities for this section, and for any x in $\mathbb{R}^S$ we write $E(x) = p_1 x_1 + \cdots + p_S x_S$. For any x and π in $\mathbb{R}^S$, we take $x\pi$ to be the vector $(x_1\pi_1, \ldots, x_S\pi_S)$. The following version of the *Riesz Representation Theorem* can be shown as an exercise.

Lemma. *Suppose $F : \mathbb{R}^S \to \mathbb{R}$ is linear. Then there is a unique π in $\mathbb{R}^S$ such that, for all x in $\mathbb{R}^S$, we have $F(x) = E(\pi x)$. Moreover, F is strictly increasing if and only if $\pi \gg 0$.*

Corollary. *A dividend-price pair (D, q) admits no arbitrage if and only if there is some $\pi \gg 0$ in $\mathbb{R}^S$ such that $q = E(D\pi)$.*

Proof: Given a state-price vector ψ, let $\pi_s = \psi_s / p_s$. Conversely, if π has the assumed property, then $\psi_s = p_s \pi_s$ defines a state-price vector ψ. ∎

Given (D, q), we refer to any vector π given by this result as a *state-price deflator*. (The terms *state-price density* and *state-price kernel* are often used synonymously with state-price deflator.) For example, the representative-agent pricing model of Corollary 2 of Section 1E shows that we can take $\pi_s = u'_\lambda(e_s)$.

For any x and y in $\mathbb{R}^S$, the *covariance* $\text{cov}(x, y) \equiv E(xy) - E(x)E(y)$ is a measure of covariation between x and y that is useful in asset pricing applications. For any such x and y with $\text{var}(y) \equiv \text{cov}(y, y) \neq 0$, we can always represent x in the form $x = \alpha + \beta y + \epsilon$, where $\beta = \text{cov}(y, x)/\text{var}(y)$, where $\text{cov}(y, \epsilon) = E(\epsilon) = 0$, and where α is a scalar. This *linear regression* of x on y is uniquely defined. The coefficient β is called the associated *regression coefficient*.

Suppose (D, q) admits no arbitrage. For any portfolio θ with $q \cdot \theta \neq 0$, the *return* on θ is the vector R^θ in $\mathbb{R}^S$ defined by $R^\theta_s = (D^\top \theta)_s / q \cdot \theta$. Fixing a state-price deflator π, for any such portfolio θ, we have $E(\pi R^\theta) = 1$. Suppose there is a *riskless portfolio*, meaning some portfolio θ with constant return R^0. We then call R^0 the *riskless return*. A bit of algebra shows that for any portfolio θ with a return, we have

$$E(R^\theta) - R^0 = -\frac{\text{cov}(R^\theta, \pi)}{E(\pi)}.$$

Thus, covariation with π has a negative effect on expected return, as one might expect from the interpretation of state prices as shadow prices for wealth.

The *correlation* between any x and y in $\mathbb{R}^S$ is zero if either has zero variance, and is otherwise defined by

$$\operatorname{corr}(x, y) = \frac{\operatorname{cov}(x, y)}{\sqrt{\operatorname{var}(x)\operatorname{var}(y)}}.$$

There is always a portfolio θ^* solving the problem

$$\sup_{\theta} \operatorname{corr}(D^\top \theta, \pi). \tag{8}$$

If there is such a portfolio θ^* with a return R^* having nonzero variance, then it can be shown as an exercise that for any return R^θ,

$$E(R^\theta) - R^0 = \beta_\theta \left[E(R^*) - R^0 \right], \tag{9}$$

where

$$\beta_\theta = \frac{\operatorname{cov}(R^*, R^\theta)}{\operatorname{var}(R^*)}.$$

If markets are complete, then R^* is of course perfectly correlated with the state-price deflator.

Formula (9) is a *state-price beta model*, showing excess expected returns on portfolios to be proportional to the excess return on a portfolio having maximal correlation with a state-price deflator, where the constant of proportionality is the associated regression coefficient. The formula can be extended to the case in which there is no riskless return. Another exercise carries this idea, under additional assumptions, to the *Capital Asset Pricing Model*, or *CAPM*.

Exercises

Exercise 1.1 The dividend-price pair (D, q) of Section 1A is defined to be *weakly arbitrage-free* if $q \cdot \theta \geq 0$ whenever $D^\top \theta \geq 0$. Show that (D, q) is weakly arbitrage-free if and only if there exists ("weak" state prices) $\psi \in \mathbb{R}^S_+$ such that $q = D\psi$. This fact is known as *Farkas's Lemma*.

Exercise 1.2 Prove the assertion in Section 1A that (D, q) is arbitrage-free if and only if there exists some $\psi \in \mathbb{R}^S_{++}$ such that $q = D\psi$. Instead of following the proof given in Section 1A, use the following result, sometimes known as the *Theorem of the Alternative*.

Stiemke's Lemma. *Suppose A is an $m \times n$ matrix. Then one and only one of the following is true:*

(a) There exists x in $\mathbb{R}^n_{++}$ with $Ax = 0$.

(b) There exists y in $\mathbb{R}^m$ with $y^\top A > 0$.

Exercise 1.3 Show for $U(c) \equiv E[u(c)]$ as defined by (3) that (2) is equivalent to (4).

Exercise 1.4 Prove the existence of an equilibrium as defined in Section 1D under these assumptions: There exists some portfolio θ with payoff $D^\top \theta > 0$ and, for all i, $e^i \gg 0$ and U_i is continuous, strictly concave, and strictly increasing. This is a demanding exercise, and calls for the following general result.

Kakutani's Fixed Point Theorem. *Suppose Z is a nonempty convex compact subset of $\mathbb{R}^n$, and for each x in Z, $\varphi(x)$ is a nonempty convex compact subset of Z. Suppose also that $\{(x, y) \in Z \times Z : x \in \varphi(y)\}$ is closed. Then there exists x^* in Z such that $x^* \in \varphi(x^*)$.*

Exercise 1.5 Prove Proposition 1D. Hint: The maintained assumption of strict monotonicity of $U_i(\,\cdot\,)$ should be used.

Exercise 1.6 Suppose that the endowment allocation $(e^1, \ldots, e^m)$ is Pareto optimal.

(A) Show, as claimed in Section 1D, that any equilibrium allocation is Pareto optimal.

(B) Suppose that there is some portfolio θ with $D^\top \theta > 0$ and, for all i, that U_i is concave and $e^i \gg 0$. Show that $(e^1, \ldots, e^m)$ is itself an equilibrium allocation.

Exercise 1.7 Prove Proposition 1C. Hint: A continuous real-valued function on a compact set has a maximum.

Exercise 1.8 Prove Corollary 1 of Proposition 1E.

Exercise 1.9 Prove Corollary 2 of Proposition 1E.

Exercise 1.10 Suppose, in addition to the assumptions of Proposition 1E, that

(a) $e = e^1 + \cdots + e^m$ is in $\mathbb{R}^S_{++}$;

(b) for all i, U_i is concave and twice continuously differentiable in $\mathbb{R}^S_{++}$;

(c) for all i, c^i is in $\mathbb{R}^S_{++}$ and the Hessian matrix $\partial^2 U(c^i)$, which is negative semi-definite by concavity, is in fact negative definite.

Property (c) can be replaced with the assumption of *regular preferences*, as defined in a source cited below in the Notes.

(A) Show that the assumption that U_λ is continuously differentiable at e is justified and, moreover, that for each i there is a scalar $\gamma_i > 0$ such that $\partial U_\lambda(e) = \gamma_i \partial U_i(c^i)$. (This co-linearity is known as "equal marginal rates of substitution," a property of any Pareto optimal allocation.) Hint: Use the following:

Implicit Function Theorem. *Suppose for given m and n that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is C^k (k times continuously differentiable) for some $k \geq 1$. Suppose also that the $n \times n$ matrix $\partial_y f(\overline{x}, \overline{y})$ of partial derivatives of f with respect to its second argument is non-singular at some $(\overline{x}, \overline{y})$. If $f(\overline{x}, \overline{y}) = 0$, then there exist scalars $\epsilon > 0$ and $\delta > 0$ and a C^k function $Z : \mathbb{R}^m \to \mathbb{R}^n$ such that if $\|x - a\| < \epsilon$, then $f[x, Z(x)] = 0$ and $\|Z(x) - b\| < \delta$.*

(B) Show that the negative definite part of condition (c) is satisfied if $e \gg 0$ and, for all i, U_i is an expected utility function of the form $U_i(c) = E[u_i(c)]$, where u_i is strictly concave with an unbounded derivative on $(0, \infty)$.

(C) Obtain the result of Part (A) without assuming the existence of second derivatives of the utilities. (You would therefore not exploit the Hessian matrix or Implicit Function Theorem.) As the first (and main) step, show the following. Given a concave function $f : \mathbb{R}^S_+ \to \mathbb{R}$, the *superdifferential* of f at some x in $\mathbb{R}^S_+$ is

$$\partial f(x) = \left\{ z \in \mathbb{R}^S : f(y) \leq f(x) + z \cdot (y - x), \quad y \in \mathbb{R}^S_+ \right\}.$$

For any feasible allocation $(c^1, \ldots, c^m)$ and $\lambda \in \mathbb{R}^m_+$ satisfying $U_\lambda(e) = \sum_i \lambda_i U_i(c^i)$,

$$\partial U_\lambda(e) = \cap_{i=1}^m \lambda_i \partial U_i(c_i).$$

Exercise 1.11 (Binomial Option Pricing) As an application of the results in Section 1A, consider the following two-state ($S = 2$) option-pricing problem. There are $N = 3$ securities:

(a) a stock, with initial price $q_1 > 0$ and dividend $D_{11} = Gq_1$ in state 1 and dividend $D_{12} = Bq_1$ in state 2, where $G > B > 0$ are the "good" and "bad" gross returns, respectively;

(b) a riskless bond, with initial price $q_2 > 0$ and dividend $D_{21} = D_{22} = Rq_2$ in both states (that is, R is the riskless return and R^{-1} is the discount);

(c) a *call option* on the stock, with initial price $q_3 = C$ and dividend $D_{3j} = (D_{1j} - K)^+ \equiv \max(D_{1j} - K, 0)$ for both states $j = 1$ and $j = 2$, where $K \geq 0$ is the *exercise price* of the option. (The call option gives its holder the right, but not the obligation, to pay K for the stock, with dividend, after the state is revealed.)

(A) Show necessary and sufficient conditions on G, B, and R for the absence of arbitrage involving only the stock and bond.

(B) Assuming no arbitrage for the three securities, calculate the call-option price C explicitly in terms of q_1, G, R, B, and K. Find the state-price probabilities $\hat{\psi}_1$ and $\hat{\psi}_2$ referred to in Section 1B in terms of G, B, and R, and show that $C = R^{-1}\hat{E}(D_3)$, where $\hat{E}$ denotes expectation with respect to $(\hat{\psi}_1, \hat{\psi}_2)$.

Exercise 1.12 (CAPM) In the setting of Section 1D, suppose $(c^1, \ldots, c^m)$ is a strictly positive equilibrium consumption allocation. For any agent i, suppose utility is of the expected-utility form $U_i(c) = E[u_i(c)]$. For any agent i, suppose there are fixed positive constants $\overline{c}$ and b_i such that for any state j, we have $c^i_j < \overline{c}$ and $u_i(x) = x - b_i x^2$ for all $x \leq \overline{c}$.

(A) In the context of Corollary 2 of Section 1E, show that $u'_\lambda(e) = k - Ke$ for some positive constants k and K. From this, derive the CAPM

$$q = AE(D) - B\text{cov}(D, e), \tag{10}$$

for positive constants A and B, where $\text{cov}(D, e) \in \mathbb{R}^N$ is the vector of covariances between the security dividends and the aggregate endowment.

Suppose for a given portfolio θ that each of the following is well defined:

- the return $R^\theta \equiv D^\top \theta / q \cdot \theta$;
- the return R^M on a portfolio M with payoff $D^\top M = e$;
- the return R^0 on a portfolio θ^0 with $\text{cov}(D^\top \theta^0, e) = 0$;
- $\beta_\theta = \text{cov}(R^\theta, R^M)/\text{var}(R^M)$.

The return R^M is sometimes called the *market return*. The return R^0 is called the *zero-beta return* and is the return on a riskless bond if one exists. Prove the "beta" form of the CAPM

$$E(R^\theta - R^0) = \beta_\theta E(R^M - R^0). \tag{11}$$

(B) Part (A) relies on the completeness of markets. Without any such assumption, but assuming that the equilibrium allocation $(c^1, \ldots, c^m)$ is strictly positive, show that the same beta form (11) applies, provided we extend the definition of the market return R^M to be the return on any portfolio solving

$$\sup_{\theta \in \mathbb{R}^N} \text{corr}\,(R^\theta, e). \tag{12}$$

For complete markets, $\text{corr}\,(R^M, e) = 1$, so the result of part (A) is a special case.

(C) The CAPM applies essentially as stated without the quadratic expected-utility assumption provided that each agent i is *strictly variance-averse*, in that $U_i(x) > U_i(y)$ whenever $E(x) = E(y)$ and $\text{var}(x) < \text{var}(y)$. Formalize this statement by providing a reasonable set of supporting technical conditions. We remark that a common alternative formulation of the CAPM allows security portfolios in initial endowments $\hat\theta^1, \ldots, \hat\theta^m$ with $\sum_{i=1}^m \hat\theta^i_j = 1$ for all j. In this case, with the total endowment e redefined by $e = \sum_{i=1}^m \left(e^i + D^\top \hat\theta^i\right)$, the same CAPM (11) applies. If $e^i = 0$ for all i, then even in incomplete markets, $\text{corr}\,(R^M, e) = 1$, since (12) is solved by $\theta = (1, 1, \ldots, 1)$. The Notes below provide references.

Exercise 1.13 An *Arrow-Debreu equilibrium* for $[(U_i, e^i), D]$ is a nonzero vector ψ in $\mathbb{R}^S_+$ and a feasible consumption allocation $(c^1, \ldots, c^m)$ such that for each i, c^i solves $\sup_c U_i(c)$ subject to $\psi \cdot c^i \le \psi \cdot e^i$. Suppose that markets are complete, in that $\text{span}(D) = \mathbb{R}^S$. Show that $(c^1, \ldots, c^m)$ is an Arrow-Debreu consumption allocation if and only if it is an equilibrium consumption allocation in the sense of Section 1D.

Exercise 1.14 Suppose (D, q) admits no arbitrage. Show that there is a unique state-price vector if and only if markets are complete.

Exercise 1.15 (Aggregation) For the "representative-agent" problem (6), suppose for all i that $U_i(c) = E[u(c)]$, where $u(c) = c^\gamma/\gamma$ for some nonzero scalar $\gamma < 1$.

(A) Show for any nonzero agent weight vector $\lambda \in \mathbb{R}^m_+$ that $U_\lambda(c) = E[kc^\gamma/\gamma]$ for some scalar $k > 0$ and that (6) is solved by $c^i = k_i x$ for some scalar $k_i \geq 0$ that is nonzero if and only if λ_i is nonzero.

(B) With this special utility assumption, show that there exists an equilibrium with a Pareto efficient allocation, without the assumption that markets are complete, but with the assumption that $e^i \in \text{span}(D)$ for all i. Calculate the associated equilibrium allocation.

Exercise 1.16 (State-Price Beta Model) This exercise is to prove and extend the state-price beta model (9) of Section 1F.

(A) Show problem (8) is solved by any portfolio θ such that $\pi = D^\top\theta + \epsilon$, where $\text{cov}(\epsilon, D^j) = 0$ for any security j, where $D^j \in \mathbb{R}^S$ is the payoff of security j.

(B) Given a solution θ to (8) such that R^θ is well defined with nonzero variance, prove (9).

(C) Reformulate (9) for the case in which there is no riskless return by redefining R^0 to be the expected return on any portfolio θ such that R^θ is well defined and $\text{cov}(R^\theta, \pi) = 0$, assuming such a portfolio exists.

Exercise 1.17 Prove the Riesz representation lemma of Section 1F. The following hint is perhaps unnecessary in this simple setting but allows the result to be extended to a broad variety of spaces called *Hilbert spaces*. Given a vector space L, a function $(\cdot \mid \cdot) : L \times L \to \mathbb{R}$ is called an *inner product* for L if for any x, y, and z in L and any scalar α we have the five properties:

(a) $(x \mid y) = (y \mid x)$

(b) $(x + y \mid z) = (x \mid z) + (y \mid z)$

(c) $(\alpha x \mid y) = \alpha(x \mid y)$

(d) $(x \mid x) \geq 0$

(e) $(x \mid x) = 0$ if and only if $x = 0$.

Suppose a finite-dimensional vector space L has an inner product $(\cdot \mid \cdot)$. (This defines a special case of a Hilbert space.) Two vectors x and y are defined to be *orthogonal* if $(x \mid y) = 0$. For any linear subspace H of L and any x in L, it can be shown that there is a unique y in H such that $(x - y \mid z) = 0$ for all z in H. This vector y is the orthogonal projection in L of x onto H, and solves the problem $\min_{h \in H} \|x - h\|$. Let $L = \mathbb{R}^S$. For any x and y in L, let $(x \mid y) = E(xy)$. We must show that given a linear functional F, there is a unique π with $F(x) = (\pi \mid x)$ for all x. Let $J = \{x : F(x) = 0\}$. If $J = L$, then F is the zero functional, and the unique representation is $\pi = 0$. If not, there is some z such that $F(z) = 1$ and $(z \mid x) = 0$ for all x in J. Show this using the idea of orthogonal projection. Then show that $\pi = z/(z \mid z)$ represents F, using the fact that for any x, we have $x - F(x)z \in J$.

Exercise 1.18 Suppose there are $m = 2$ consumers, A and B, with identical utilities for consumption c_1 and c_2 in states 1 and 2 given by $U(c_1, c_2) = 0.2\sqrt{c_1} + 0.5 \log c_2$. There is a total endowment of $e_1 = 25$ units of consumption in state 1.

(A) Suppose that markets are complete and that, in a given equilibrium, consumer A's consumption is 9 units in state 1 and 10 units in state 2. What is the total endowment e_2 in state 2?

(B) Continuing under the assumptions of part (A), suppose there are two securities. The first is a riskless bond paying 10 units of consumption in each state. The second is a risky asset paying 5 units of consumption in state 1 and 10 units in state 2. In equilibrium, what is the ratio of the price of the bond to that of the risky asset?

Exercise 1.19 There are two states of the world, labeled 1 and 2, two agents, and two securities, both paying units of the consumption numeraire good. The risky security pays a total of 1 unit in state 1 and pays 3 units in state 2. The riskless security pays 1 unit in each state. Each agent is initially endowed with half of the total supply of the risky security. There are no other endowments. (The riskless security is in zero net supply.) The two agents assign equal probabilities to the two states. One of the agents is risk-neutral, with utility function $E(c)$ for state-contingent consumption c, and can consume negatively or positively in both states. The other, risk-averse, agent has utility $E(\sqrt{c})$ for non-negative state-contingent consumption. Solve for the equilibrium allocation of the two securities in a competitive equilibrium.

Exercise 1.20 Consider a setting with two assets A and B, only, both paying off the same random variable X, whose value is non-negative in every state and non-zero with strictly positive probability. Asset A has price p, while asset B has price q. An arbitrage is then a portfolio $(\alpha, \beta) \in \mathbb{R}^2$ of the two assets whose total payoff $\alpha X + \beta X$ is non-negative and whose initial price $\alpha p + \beta q$ is strictly negative, or whose total payoff is non-zero with strictly positive probability and always non-negative, and whose initial price is negative or zero.

(A) Assuming no restrictions on portfolios, and no transactions costs or frictions, state the set of arbitrage-free prices (p, q). (State precisely the appropriate subset of $\mathbb{R}^2$.)

(B) Assuming no short sales ($\alpha \geq 0$ and $\beta \geq 0$), state the set of arbitrage-free prices (p, q).

(C) Now suppose that A and B can be short sold, but that asset A can only be short sold only by paying an extra fee of $\phi > 0$ per unit sold short. There are no other fees of any kind. Provide the obvious new definition of "no arbitrage" in precise mathematical terms, and state the set of arbitrage-free prices.

Notes

The basic approach of this chapter follows Arrow [1953], taking a general-equilibrium perspective originating with Walras [1877]. Black [1995] offers a persepctive on the general equilibrium approach and a critique of other approaches. The state-pricing implications of no arbitrage found in Section 1A originate with Ross [1978]. The idea of "risk-neutral probabilities" apparently originates with Arrow [1970], a revision of Arrow [1953], and appears as well in Drèze [0 71]. Proposition 1D is the First Welfare Theorem of Arrow [1951] and Debreu [1954]. The generic inoptimality of incomplete markets equilibrium allocations can be gleaned from sources cited by Geanakoplos [1990]. Indeed, Geanakoplos and Polemarchakis [1986] show that even a reasonable notion of constrained optimality generically fails in certain incomplete markets settings. See, however, Kajii [1994] and references cited in the Notes of Chapter 2 for mitigating results. Mas-Colell [1987] and Werner [1991] also treat constrained optimality.

The "representative-agent" approach goes back, at least, to Negishi [1960]. The existence of a representative agent is no more than an illustrative simplification in this setting, and should not be confused with the more demanding notion of aggregation of Gorman [1953] found in Exercise 1.15. In Chapter 9, the existence of a representative agent with smooth utility, based on Exercise 1.11, is important for technical reasons.

Debreu [1972] provides preference assumptions that substitute for the existence of a negative-definite Hessian matrix of each agent's utility function at the equilibrium allocation. For more on regular preferences and the differential approach to general equilibrium, see Mas-Colell [1985] and Balasko [1989]. Kreps [1988] reviews the theory of choice and utility representation of preferences. For Farkas's and Stiemke's Lemmas, and other forms of the Theorem of the Alternative, see Gale [1960].

Arrow and Debreu [1954] and, in a slightly different model, McKenzie [1954] are responsible for a proof of the existence of complete-markets equilibria. Debreu [1982] surveys the existence problem. Standard introductory treatments of general equilibrium theory are given by Debreu [1959] and Hildenbrand and Kirman [1989]. In this setting, with incomplete markets, Polemarchakis and Siconolfi [1991] address the failure of existence unless one has a portfolio θ with payoff $D^{\top}\theta > 0$. Geanakoplos [1990] surveys other literature on the existence of equilibria in incomplete markets, some of which takes the alternative of defining security payoffs in nominal units of account, while allowing consumption of multiple commodities. Most of the literature allows for an initial period of consumption before the realization of the uncertain state. For a survey, see Magill and Shafer [1991]. Additional results on incomplete markets equilibrium include those ofBoyle and Wang [1999], Weil [1992], Araujo and Monteiro [1989], Berk [1994], For related results in multiperiod settings, references are cited in the Notes of Chapter 2.

The superdifferentiability result of Exercise 10(C) is due to Skiadas [1995].

Hellwig [1996], Mas-Colell and Monteiro [1996], and Monteiro [1996] have recently shown existence with a continuum of states. Geanakoplos and Polemarchakis [1986] and Chae [1988] show existence in a model closely related to that studied in this chapter. Grodal and Vind [1988] and Yamazaki [1991] show existence with alternative formulations. With multiple commodities or multiple periods, existence is not guaranteed under any natural conditions, as shown by Hart [1975], who gives a counterexample. For these more delicate cases, the literature on generic existence is cited in the Notes of Chapter 2.

The CAPM is due to Sharpe [1964] and Lintner [1965]. The version without a riskless asset is due to Black [1972]. Allingham [1991], Berk [1992], and Nielsen [990a], and Nielsen [990b] address the existence of equilibrium in the CAPM. Characterization of the mean-variance model and two-fund separation is provided by Bottazzi, Hens, and Löffler [1994] , Nielsen [993a] and Nielsen [993b]. Löffler [1994] provides sufficient conditions for variance aversion in terms of mean-variance preferences. Ross [1976] introduced the *arbitrage pricing theory*, a multifactor model of asset returns that in terms of expected returns can be thought of as an extension of the CAPM. In this regard, see also Bray [994a], Bray [994b] and Gilles and LeRoy [1991]. Balasko and Cass [1986] and Balasko, Cass, and Siconolfi [1990] treat equilibrium with constrained participation in security trading. See, also, Hara [1994].

The binomial option-pricing formula of Exercise 1.11 is from an early edition of Sharpe [1985], and is extended in Chapter 2 to a multiperiod setting. The hint given for the demonstration of the Riesz representation exercise is condensed from the proof given by Luenberger [1969] of the *Riesz-Frechet Theorem*: For any Hilbert space H with inner product $(\cdot \mid \cdot)$, any continuous linear functional $F : H \to \mathbb{R}$ has a unique π in H such that $F(x) = (\pi \mid x)$, $x \in H$. The Fixed Point Theorem of Exercise 1.4 is from Kakutani [1941].

On the role of default and collateralization, see andGeanakoplosZ:99. Gottardi and Kajii [1995] study the role and existence of sunspot equilibria. Pietra [1992] treats indeterminacy.

Chapter 2

The Basic Multiperiod Model

THIS CHAPTER EXTENDS the results of Chapter 1 on arbitrage, optimality, and equilibrium to a multiperiod setting. A connection is drawn between state prices and martingales for the purpose of representing security prices. The exercises include the consumption-based capital asset pricing model and the multiperiod "binomial" option pricing model.

2A Uncertainty

As in Chapter 1, there is some finite set, say Ω, of states. In order to handle multiperiod issues, however, we will treat uncertainty a bit more formally as a *probability space* $(\Omega, \mathcal{F}, P)$, with $\mathcal{F}$ denoting the *tribe* of subsets of Ω that are *events* (and can therefore be assigned a probability), and with P a *probability measure* assigning to any event B in $\mathcal{F}$ its probability $P(B)$. Those not familiar with the definition of a probability space can consult Appendix A.

There are $T+1$ dates: $0, 1, \ldots, T$. At each of these, a tribe $\mathcal{F}_t \subset \mathcal{F}$ denotes the set of events corresponding to the information available at time t. That is, if an event B is in $\mathcal{F}_t$, then at time t this event is known to be true or false. (A definition of tribes in terms of "partitions" of Ω is given in Exercise 2.11.) We adopt the usual convention that $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $t \leq s$, meaning that events are never "forgotten." For simplicity, we also take it that every event in $\mathcal{F}_0$ has probability 0 or 1, meaning roughly that there is no information at time $t = 0$. Taken altogether, the *filtration* $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$ represents how information is revealed through time. For any random variable Y, we let $E_t(Y) = E(Y \mid \mathcal{F}_t)$ denote the conditional expectation of Y given $\mathcal{F}_t$. (Appendix A provides definitions of random variables and of conditional expectation.) An *adapted process* is a sequence $X = \{X_0, \ldots, X_T\}$ such that for each t, X_t is a random variable with respect to $(\Omega, \mathcal{F}_t)$. Informally, this means that at time t, the outcome of X_t is known. An adapted process X is a *martingale* if, for any times t and $s > t$, we have $E_t(X_s) = X_t$. As we shall see, martingales are useful in the characterization of security prices. In order to simplify things, for any two random variables Y and Z, we always write $Y = Z$ if the probability that $Y \neq Z$ is zero.

2B Security Markets

A security is a claim to an adapted *dividend process*, say δ, with δ_t denoting the dividend paid by the security at time t. Each security has an adapted *security-price process* S, so that S_t is the price of the security, *ex dividend*, at time t. That is, at each time t, the security pays its dividend δ_t and is then available for trade at the price S_t. This convention implies that δ_0 plays no role in determining ex-dividend prices. The *cum-dividend* security price at time t is $S_t + \delta_t$.

Suppose there are N securities defined by the $\mathbb{R}^N$-valued adapted dividend process $\delta = (\delta^{(1)}, \ldots, \delta^{(N)})$. These securities have some adapted price process $S = (S^{(1)}, \ldots, S^{(N)})$. A *trading strategy* is an adapted process θ in $\mathbb{R}^N$. Here, $\theta_t = (\theta_t^{(1)}, \ldots, \theta_t^{(N)})$ represents the portfolio held after trading at time t. The dividend process δ^θ *generated* by a trading strategy θ is defined by

$$\delta_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t) - \theta_t \cdot S_t, \tag{1}$$

with "θ_{-1}" taken to be zero by convention.

2C Arbitrage, State Prices, and Martingales

Given a dividend-price pair (δ, S) for N securities, a trading strategy θ is an *arbitrage* if $\delta^\theta > 0$. (The reader should become convinced that this is the same notion of arbitrage defined in Chapter 1.) Let Θ denote the space of trading strategies. For any θ and φ in Θ and scalars a and b, we have $a\delta^\theta + b\delta^\varphi = \delta^{a\theta + b\varphi}$. Thus the *marketed subspace* $M = \{\delta^\theta : \theta \in \Theta\}$ of dividend processes generated by trading strategies is a linear subspace of the space L of adapted processes.

Proposition. *There is no arbitrage if and only if there is a strictly increasing linear function $F : L \to \mathbb{R}$ such that $F(\delta^\theta) = 0$ for any trading strategy θ.*

Proof: The proof is almost identical to the first part of the proof of Theorem 1A. Let $L_+ = \{c \in L : c \geq 0\}$. There is no arbitrage if and only if the cone L_+ and the marketed subspace M intersect precisely at zero. Suppose there is no arbitrage. The Separating Hyperplane Theorem, in a form given in Appendix B for cones, implies the existence of a nonzero linear functional F such that $F(x) < F(y)$ for each x in M and each nonzero y in L_+. Since M is a linear subspace, this implies that $F(x) = 0$ for each x in M, and thus that $F(y) > 0$ for each nonzero y in L_+. This implies that F is strictly increasing. The converse is immediate. ∎

The following result gives a convenient *Riesz representation* of a linear function on the space of adapted processes. Proof is left as an exercise, extending the single-period Riesz representation lemma of Section 1F.

Lemma. *For each linear function $F : L \to \mathbb{R}$ there is a unique π in L, called the Riesz representation of F, such that*

$$F(x) = E\left(\sum_{t=0}^{T} \pi_t x_t\right), \qquad x \in L.$$

If F is strictly increasing, then π is strictly positive.

For convenience, we call any strictly positive adapted process a *deflator.* A deflator π is a *state-price deflator* if, for all t,

$$S_t = \frac{1}{\pi_t} E_t\left(\sum_{j=t+1}^{T} \pi_j \delta_j\right). \tag{2}$$

For $t = T$, the right-hand side of (2) is zero, so $S_T = 0$ whenever there is a state-price deflator. The notion here of a state-price deflator is a natural extension of that of Chapter 1. It can be shown as an exercise that a deflator π is a state-price deflator if and only if, for any trading strategy θ,

$$\theta_t \cdot S_t = \frac{1}{\pi_t} E_t\left(\sum_{j=t+1}^{T} \pi_j \delta_j^\theta\right), \qquad t < T, \tag{3}$$

meaning roughly that the market value of a trading strategy is, at any time, the state-price discounted expected future dividends generated by the strategy. The *cum-dividend* value process V^θ of a trading strategy θ is defined by $V_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t)$. If π is a state-price deflator, we have

$$V_t^\theta = \frac{1}{\pi_t} E_t \left(\sum_{j=t}^{T} \pi_j \delta_j^\theta \right).$$

The *gain process* G for (δ, S) is defined by $G_t = S_t + \sum_{j=1}^{t} \delta_j$. Given a deflator γ, the *deflated gain process* G^γ is defined by $G_t^\gamma = \gamma_t S_t + \sum_{j=1}^{t} \gamma_j \delta_j$. We can think of deflation as a change of numeraire.

Theorem. *The dividend-price pair (δ, S) admits no arbitrage if and only if there is a state-price deflator. A deflator π is a state-price deflator if and only if $S_T = 0$ and the state-price-deflated gain process G^π is a martingale.*

Proof: It can be shown as an easy exercise that that a deflator π is a state-price deflator if and only if $S_T = 0$ and the state-price-deflated gain process G^π is a martingale.

Suppose there is no arbitrage. Then $S_T = 0$, for otherwise the strategy θ is an arbitrage when defined by $\theta_t = 0$, $t < T$, $\theta_T = -S_T$. The previous proposition implies that there is some strictly increasing linear function $F : L \to \mathbb{R}$ such that $F(\delta^\theta) = 0$ for any strategy θ. By the previous lemma, there is some deflator π such that $F(x) = E(\sum_{t=0}^{T} x_t \pi_t)$ for all x in L. This implies that $E(\sum_{t=0}^{T} \delta_t^\theta \pi_t) = 0$ for any strategy θ.

We must prove (2), or equivalently, that G^π is a martingale. From Appendix A, an adapted process X is a martingale if and only if $E(X_\tau) = X_0$ for any finite-valued stopping time $\tau \le T$. Consider, for an arbitrary security n and an arbitrary finite-valued stopping time $\tau \le T$, the trading strategy θ defined by $\theta^{(k)} = 0$ for $k \ne n$ and $\theta_t^{(n)} = 1$, $t < \tau$, with $\theta_t^{(n)} = 0$, $t \ge \tau$. Since $E(\sum_{t=0}^{T} \pi_t \delta_t^\theta) = 0$, we have

$$E\left(-S_0^{(n)} \pi_0 + \sum_{t=1}^{\tau} \pi_t \delta_t^{(n)} + \pi_\tau S_\tau^{(n)} \right) = 0,$$

implying that the deflated gain process $G^{n\pi}$ of security n satisfies $G_0^{n,\pi} = E\left(G_\tau^{n,\pi}\right)$. Since τ is arbitrary, $G^{n,\pi}$ is a martingale, and since n is arbitrary, G^π is a martingale.

This shows that absence of arbitrage implies the existence of a state-price deflator. The converse is easy. ▮

2D Individual Agent Optimality

We introduce an agent, defined by a strictly increasing utility function U on the set L_+ of nonnegative adapted "consumption" processes, and by an *endowment process* e in L_+. Given a dividend-price process (δ, S), a trading strategy θ leaves the agent with the total consumption process $e + \delta^\theta$. Thus the agent has the *budget-feasible consumption set*

$$X = \{e + \delta^\theta \in L_+ : \theta \in \Theta\},$$

and the problem

$$\sup_{c \in X} U(c). \tag{4}$$

The existence of a solution to (4) implies the absence of arbitrage. Conversely, it can be shown as an exercise that if U is continuous, then the absence of arbitrage implies that there exists a solution to (4). For purposes of checking continuity or the closedness of sets in L, we will say that c_n converges to c if $E[\sum_{t=0}^{T} |c_n(t) - c(t)|] \to 0$. Then U is continuous if $U(c_n) \to U(c)$ whenever $c_n \to c$.

Suppose that (4) has a strictly positive solution c^* and that U is continuously differentiable at c^*. We can use the first-order conditions for optimality (which can be reviewed in Appendix B) to characterize security prices in terms of the derivatives of the utility function U at c^*. Specifically, for any c in L, the derivative of U at c^* in the direction c is the derivative $g'(0)$, where $g(\alpha) = U(c^* + \alpha c)$ for any scalar α sufficiently small in absolute value. That is, $g'(0)$ is the marginal rate of improvement of utility as one moves in the direction c away from c^*. This derivative is denoted $\nabla U(c^*; c)$. Because U is continuously differentiable at c^*, $c \mapsto \nabla U(c^*, c)$ is linear function on L into $\mathbb{R}$. Since δ^θ is a budget-feasible direction of change for any trading strategy θ, the first-order conditions for optimality of c^* imply that

$$\nabla U(c^*; \delta^\theta) = 0, \qquad \theta \in \Theta.$$

We now have a characterization of a state-price deflator.

Proposition. *Suppose that (4) has a strictly positive solution c^* and that U has a strictly positive continuous derivative at c^*. Then there is no arbitrage and a state-price deflator is given by the Riesz representation π of $\nabla U(c^*)$:*

$$\nabla U(c^*; x) = E\left(\sum_{t=0}^{T} \pi_t x_t\right), \qquad x \in L.$$

Despite our standing assumption that U is strictly increasing, $\nabla U(c^*;\,\cdot\,)$ need not in general be strictly increasing, but is so if U is concave.

As an example, suppose U has the *additive* form

$$U(c) = E\left[\sum_{t=0}^{T} u_t(c_t)\right], \qquad c \in L_+, \tag{5}$$

for some $u_t : \mathbb{R}_+ \to \mathbb{R}$, $t \geq 0$. It is an exercise to show that if $\nabla U(c)$ exists, then

$$\nabla U(c; x) = E\left[\sum_{t=0}^{T} u_t'(c_t)x_t\right]. \tag{6}$$

If, for all t, u_t is concave with an unbounded derivative and e is strictly positive, then any solution c^* to (4) is strictly positive.

Corollary. *Suppose U is defined by (5). Under the conditions of the Proposition, for any times t and $\tau \geq t$,*

$$S_t = \frac{1}{u_t'(c_t^*)} E_t\left[S_\tau u_\tau'(c_\tau^*) + \sum_{j=t+1}^{\tau} \delta_j u_j'(c_j^*)\right].$$

For the case $\tau = t+1$, this result is often called the *stochastic Euler equation.* Extending this classical result for additive utility, the exercises include other utility examples such as *habit-formation* utility and *recursive* utility. As in Chapter 1, we now turn to the multi-agent case.

2E Equilibrium and Pareto Optimality

Suppose there are m agents; agent i is defined as above by a strictly increasing utility function $U_i : L_+ \to \mathbb{R}$ and an endowment process $e^i \in L_+$. Given a dividend process δ for N securities, an *equilibrium* is a collection $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$, where S is a security-price process and, for each i, θ^i is a trading strategy solving

$$\sup_{\theta \in \Theta} U_i(c) \qquad \text{subject to } c = e^i + \delta^\theta \in L_+, \tag{7}$$

with $\sum_{i=1}^m \theta^i = 0$.

We define markets to be *complete* if, for each process x in L, there is some trading strategy θ with $\delta_t^\theta = x_t$, $t \geq 1$. Complete markets thus means that any consumption process x can be obtained by investing some amount at time 0 in a trading strategy that generates the dividend x_t in each period t after 0. With the same definition of Pareto optimality, Proposition 1D carries over to this multiperiod setting. Any equilibrium $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ has an associated feasible consumption allocation $(c^{(1)}, \ldots, c^{(m)})$ defined by letting $c^i - e^i$ be the dividend process generated by θ^i.

Proposition. *Suppose* $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ *is an equilibrium and markets are complete. Then the associated consumption allocation is Pareto optimal.*

The completeness of markets depends on the security-price process S itself. Indeed, the dependence of the marketed subspace on S makes the existence of an equilibrium a nontrivial issue. We ignore existence here and refer to the Notes for some relevant sources.

2F Equilibrium Asset Pricing

Again following the ideas in Chapter 1, we define for each λ in $\mathbb{R}^m_+$ the utility function $U_\lambda : L_+ \to \mathbb{R}$ by

$$U_\lambda(x) = \sup_{(c^{(1)}, \ldots, c^{(m)})} \sum_{i=1}^{m} \lambda_i \, U_i(c^i) \qquad \text{subject to } c^{(1)} + \cdots + c^{(m)} \leq x. \tag{8}$$

Proposition. *Suppose for all* i *that* U_i *is concave and strictly increasing. Suppose that* $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ *is an equilibrium and that markets are complete. Then there exists some nonzero* $\lambda \in \mathbb{R}^m_+$ *such that* $(0, S)$ *is a (no-trade) equilibrium for the one-agent economy* $[(U_\lambda, e), \delta]$*, where* $e = e^{(1)} + \cdots + e^{(m)}$*. With this* λ *and with* $x = e = e^{(1)} + \cdots + e^{(m)}$*, problem (8) is solved by the equilibrium consumption allocation.*

Proof is assigned as an exercise. The result is essentially the same as Proposition 1E. A method of proof, as well as the intuition for this proposition, is that with complete markets, a state-price deflator π represents Lagrange multipliers for consumption in the various periods and states for all of the agents simultaneously, as well as for the *representative agent* (U_λ, e).

Corollary 1. *If, moreover,* U_λ *is differentiable at* e*, then* λ *can be chosen so that for any times* t *and* $\tau \geq t$*, there is a state-price deflator* π *equal to the Riesz representation of* $\nabla U_\lambda(e)$*.*

Differentiability of U_λ at e can be shown by exactly the arguments used in Exercise 1.10.

Corollary 2. *Suppose for each i that U_i is of the additive form*

$$U_i(c) = E\left[\sum_{t=0}^{T} u_{it}(c_t)\right].$$

Then U_λ is also additive, with

$$U_\lambda(c) = E\left[\sum_{t=0}^{T} u_{\lambda t}(c_t)\right],$$

where

$$u_{\lambda t}(y) = \sup_{x\in\mathbb{R}^m_+} \sum_{i=1}^{m} \lambda_i\, u_{it}(x_i) \qquad \text{subject to } x_1 + \cdots + x_m \leq y.$$

In this case, the differentiability of U_λ at e implies that for any times t and $\tau \geq t$,

$$S_t = \frac{1}{u'_{\lambda t}(e_t)}\, E_t\left[u'_{\lambda\tau}(e_\tau)S_\tau + \sum_{j=t+1}^{\tau} u'_{\lambda j}(e_j)\delta_j\right]. \tag{9}$$

2G Arbitrage and Martingale Measures

This section shows the equivalence between the absence of arbitrage and the existence of a probability measure Q with the property, roughly speaking, that the price of a security is the sum of Q-expected discounted dividends.

There is *short-term riskless borrowing* if, for each given time $t < T$ if there is a security trading strategy θ with $\delta^\theta_{t+1} = 1$ and with $\delta^\theta_s = 0$ for $s < t$ and $s > t+1$. The associated *discount* is $d_t = \theta_t \cdot S_t$. If there is no arbitrage, the discount d_t is uniquely defined and strictly positive, and we may define the associated *short-rate* r_t by $1 + r_t = 1/d_t$. This means that at any time $t < T$, one may invest one unit of account in order to receive $1 + r_t$ units of account at time $t+1$. We refer to $\{r_0, r_1, \ldots, r_{T-1}\}$ as the associated "short-rate process," even though r_T is not defined.

Let us suppose throughout this section that there is short-term riskless borrowing at some uniquely defined short-rate process r. We can define, for any times t and $\tau < T$,

$$R_{t,\tau} = (1 + r_t)(1 + r_{t+1}) \cdots (1 + r_{\tau-1}),$$

the payback at time τ of one unit of account borrowed risklessly at time t and rolled over in short-term borrowing repeatedly until date τ.

It would be a simple situation, both computationally and conceptually, if any security's price were merely the expected discounted dividends of the security. Of course, this is unlikely to be the case in a market with risk-averse investors. We can nevertheless come close to this sort of characterization of security prices by adjusting the original probability measure P. For this, we define a new probability measure Q to be *equivalent* to P if Q and P assign zero probabilities to the same events. An equivalent probability measure Q is an *equivalent martingale measure* if

$$S_t = E_t^Q \left(\sum_{j=t+1}^{T} \frac{\delta_j}{R_{t,j}} \right), \qquad t < T,$$

where E^Q denotes expectation under Q, and likewise $E_t^Q(x) = E^Q(x \mid \mathcal{F}_t)$ for any random variable x.

It is easy to show that Q is an equivalent martingale measure if and only if, for any trading strategy θ,

$$\theta_t \cdot S_t = E_t^Q \left(\sum_{j=t+1}^{T} \frac{\delta_j^\theta}{R_{t,j}} \right), \qquad t < T. \tag{10}$$

If interest rates are deterministic, (10) is merely the total discounted expected dividends, after substituting Q for the original measure P. We will show that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure.

The deflator γ defined by $\gamma_t = R_{0,t}^{-1}$ defines the *discounted gain process* G^γ. The word "martingale" in the term "equivalent martingale measure" comes from the following equivalence.

Lemma. *A probability measure Q equivalent to P is an equivalent martingale measure for (δ, S) if and only if $S_T = 0$ and the discounted gain process G^γ is a martingale with respect to Q.*

We already know from Theorem 2C that the absence of arbitrage is equivalent to the existence of a state-price deflator π. Fixing π, let Q be the probability measure defined, as explained in Appendix A, by the Radon-Nikodym derivative given by

$$\xi_T = \frac{\pi_T R_{0,T}}{\pi_0}.$$

That is, Q is defined by letting $E^Q(Z) = E(\xi_T Z)$ for any random variable Z. Because ξ_T is strictly positive, Q and P are equivalent probability measures. The *density process* ξ for Q is defined by $\xi_t = E_t(\xi_T)$. Relation (A.2) of Appendix A implies that for any times t and $j > t$ and any $\mathcal{F}_j$-measurable random variable Z_j,

$$E_t^Q(Z_j) = \frac{1}{\xi_t} E_t(\xi_j Z_j). \tag{11}$$

Fixing some time $t < T$, consider a trading strategy θ that invests one unit of account at time t and repeatedly rolls the value over in short-term riskless borrowing until time T, with final value $R_{t,T}$. That is, $\theta_t \cdot S_t = 1$ and $\delta_T^\theta = R_{t,T}$. Relation (3) then implies that

$$\pi_t = E_t(\pi_T R_{t,T}) = \frac{E_t(\pi_T R_{0,T})}{R_{0,t}} = \frac{E_t(\xi_T \pi_0)}{R_{0,t}} = \frac{\xi_t \pi_0}{R_{0,t}}. \tag{12}$$

From (11), (12), and the definition of a state-price deflator, (10) is satisfied, so Q is indeed an equivalent martingale measure. We have shown the following result.

Theorem. *There is no arbitrage if and only if there exists an equivalent martingale measure. Moreover, π is a state-price deflator if and only if an equivalent martingale measure Q has the density process ξ defined by $\xi_t = R_{0,t}\pi_t/\pi_0$.*

Proposition. *Suppose that $\mathcal{F}_T = \mathcal{F}$ and there is no arbitrage. Then markets are complete if and only if there is a unique equivalent martingale measure.*

Proof: Suppose that markets are complete and let Q_1 and Q_2 be two equivalent martingale measures. We must show that $Q_1 = Q_2$. Let A be any event. Since markets are complete, there is a trading strategy θ with dividend process δ^θ such that $\delta_T^\theta = R_{0,T} 1_A$ and $\delta_t^\theta = 0$, $0 < t < T$. By (10), we have $\theta_0 \cdot S_0 = Q_1(A) = Q_2(A)$. Since A is arbitrary, $Q_1 = Q_2$.

Exercise 2.18 outlines a proof of the converse part of the result. ∎

This martingale approach simplifies many asset-pricing problems that might otherwise appear to be quite complex, such as the American option-pricing problem to follow in Section 2I. This martingale approach also applies much more generally than indicated here. For example, the assumption of short-term borrowing is merely a convenience. More generally, one can typically obtain an equivalent martingale measure after normalizing prices and dividends by the price of some particular security (or trading strategy).

2H Valuation of Redundant Securities

Suppose that the given dividend-price pair (δ, S) is arbitrage-free, with an associated state-price deflator π. Now consider the introduction of a new security with dividend process $\hat{\delta}$ and price process $\hat{S}$. We say that $\hat{\delta}$ is *redundant* given (δ, S) if there exists a trading strategy θ, with respect to only the original security dividend-price process (δ, S), that *replicates* $\hat{\delta}$, in the sense that $\delta_t^\theta = \hat{\delta}_t$, $t \geq 1$. In this case, the absence of arbitrage for the "larger" dividend-proce process $[(\delta, \hat{\delta}), (S, \hat{S})]$ implies that

$$\hat{S}_t = Y_t \equiv \frac{1}{\pi_t} E_t \left(\sum_{j=t+1}^{T} \pi_j \hat{\delta}_j \right), \qquad t < T.$$

If this were not the case, there would be an arbitrage, as follows. For example, suppose that for some stopping time τ, we have $\hat{S}_\tau > Y_\tau$, and that $\tau \leq T$ with strictly positive probability. We can then define the strategy:

(a) Sell the redundant security $\hat{\delta}$ at time τ for $\hat{S}_\tau$, and hold this position until T.

(b) Invest $\theta_\tau \cdot S_\tau$ at time τ in the replicating strategy θ, and follow this strategy until T.

Since the dividends generated by this combined strategy (a)–(b) after τ are zero, the only dividend is at τ for the amount $\hat{S}_\tau - Y_\tau > 0$, which means that this is an arbitrage. Likewise, if $\hat{S}_\tau < Y_\tau$ for some finite-valued stopping time τ, the opposite strategy is an arbitrage. We have shown the following.

Proposition. *Suppose (δ, S) is arbitrage-free with state-price deflator π. Let $\hat{\delta}$ be a redundant dividend process with price process $\hat{S}$. Then the combined dividend-price pair $[(\delta, \hat{\delta}), (S, \hat{S})]$ is arbitrage-free if and only if it has π as a state-price deflator.*

In applications, it is often assumed that (δ, S) generates complete markets, in which case any additional security is redundant. Exercise 2.1 gives a classical example in which the redundant security is an option on one of the original securities.

2I American Exercise Policies and Valuation

We will now extend our pricing framework to include a family of securities, called "American," for which there is discretion regarding the timing of cash flows.

Given an adapted process X, each finite-valued stopping time τ generates a dividend process $\delta^{X,\tau}$ defined by $\delta_t^{X,\tau} = 0$, $t \neq \tau$, and $\delta_\tau^{X,\tau} = X_\tau$. In this context, a finite-valued stopping time is an *exercise policy*, determining the time at which to accept payment. Any exercise policy τ is constrained by $\tau \leq \overline{\tau}$, for some *expiration time* $\overline{\tau} \leq T$. (We may take $\overline{\tau}$ to be a stopping time in the following, which is usfeul for the case of certain *knockout options*, as shown for example in Exercise 2.1.) We say that $(X, \overline{\tau})$ defines an *American security*. The exercise policy is selected by the holder of the security. Once exercised, the security has no remaining cash flows. A standard example is an American put option on a security with price process p. The American put gives the holder of the option the right, but not the obligation, to sell the underlying security for a fixed exercise price at any time before a given expiration time $\overline{\tau}$. If the option has an exercise price K and expiration time $\overline{\tau} < T$, then $X_t = (K - p_t)^+$, $t \leq \overline{\tau}$, and $X_t = 0$, $t > \overline{\tau}$.

We will suppose that in addition to an American security $(X, \overline{\tau})$, there are securities with an arbitrage-free dividend-price process (δ, S) that generates complete markets. The assumption of complete markets will dramatically simplify our analysis since it implies, for any exercise policy τ, that the dividend process $\delta^{X,\tau}$ is redundant given (δ, S). For notational convenience, we assume that $0 < \overline{\tau} < T$.

Let π be a state-price deflator associated with (δ, S). From Proposition 2H, given any exercise policy τ, the American security's dividend process $\delta^{X,\tau}$ has an associated cum-dividend price process, say V^τ, which, in the absence of arbitrage, satisfies

$$V_t^\tau = \frac{1}{\pi_t} E_t\left(\pi_\tau X_\tau\right), \qquad t \leq \tau.$$

This value does not depend on which state-price deflator is chosen because, with complete markets, state-price deflators are all equal up to a positive rescaling, as one can see from the theorem and proposition of Section 2G.

We consider the optimal stopping problem

$$V_0^* \equiv \max_{\tau \in \mathcal{T}(0)} \qquad V_0^\tau, \tag{13}$$

where, for any time $t \leq \overline{\tau}$, we let $\mathcal{T}(t)$ denote the set of stopping times bounded below by t and above by $\overline{\tau}$. A solution to (13) is called a *rational exercise policy* for the American security X, in the sense that it maximizes the initial arbitrage-free value of the security.

We claim that in the absence of arbitrage, the actual initial price V_0 for the American security must be V_0^*. In order to see this, suppose first that $V_0^* > V_0$. Then one could buy the American security, adopt for it a rational exercise policy τ, and also undertake a trading strategy replicating $-\delta^{X,\tau}$. Since $V_0^* = E(\pi_\tau X_\tau)/\pi_0$, this replication involves an initial payoff of $-\delta_0^{X,\tau} = V_0^*$, and the net effect is a total initial dividend of $V_0^* - V_0 > 0$ and zero dividends after time 0, which defines an arbitrage. Thus the absence of arbitrage easily leads to the conclusion that $V_0 \geq V_0^*$. It remains to show that the absence of arbitrage also implies the opposite inequality $V_0 \leq V_0^*$.

Suppose that $V_0 > V_0^*$. One could sell the American security at time 0 for V_0. We will show that for an initial investment of V_0^*, one can "super-replicate" the payoff at exercise demanded by the holder of the American security, *regardless of the exercise policy used.* Specifically, a *super-replicating trading strategy* for $(X, \overline{\tau}, \delta, S)$ is a trading strategy θ involving only the securities with dividend-price process (δ, S) that has the properties:

(a) $\delta_t^\theta = 0$ for $0 < t < \overline{\tau}$, and

(b) $V_t^\theta \geq X_t$ for all $t \leq \overline{\tau}$,

where, we recall, V_t^θ is the cum-dividend value of θ at time t. Regardless of the exercise policy τ used by the holder of the security, the payment of X_τ demanded at time τ is covered by the market value V_t^θ of a super-replicating strategy θ. (In effect, one modifies θ by liquidating the portfolio θ_τ at time τ, so that the actual trading strategy φ associated with the arbitrage is defined by $\varphi_t = \theta_t$ for $t < \tau$ and $\varphi_t = 0$ for $t \geq \tau$.) By these properties (a)–(b), if $V_0 > V_0^*$ then the strategy of selling the American security and adopting a super-replicating strategy is an arbitrage provided $V_0^\theta = V_0^*$.

This notion of arbitrage for American securities, an extension of the notion of arbitrage used earlier in the chapter, is reasonable because a super-replicating strategy does not depend on the exercise policy adopted by the holder (or sequence of holders over time) of the American security. It would be unreasonable to call a strategy involving a short position in the American security an "arbitrage" if, in carrying it out, one requires knowledge of the

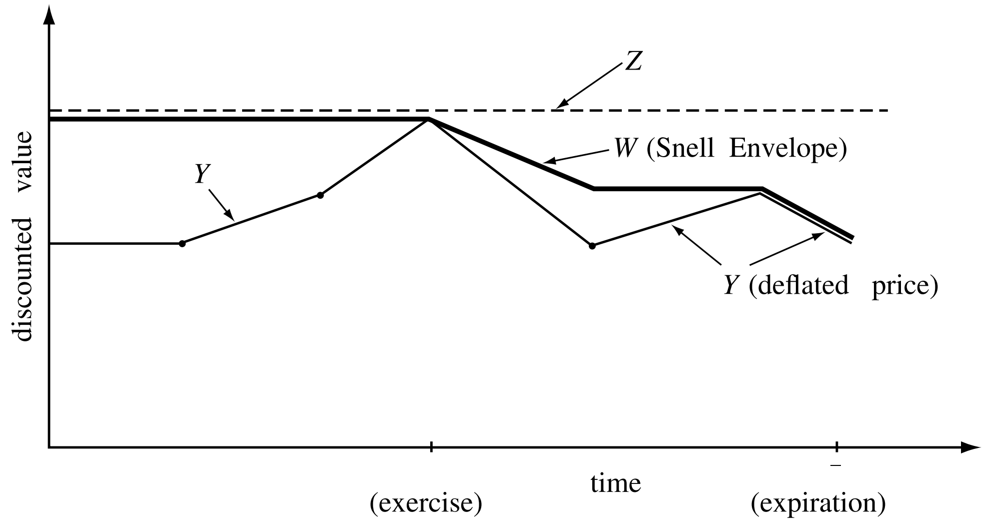

Figure2.1: Snell Envelope and Optimal Stopping Rule: Deterministic Case

exercise policy for the American security that will be adopted by other agents that hold the security over time, who may after all act "irrationally."

Proposition. *There is a super-replicating trading strategy θ for $(X, \overline{\tau}, \delta, S)$ with the initial value $V_0^\theta = V_0^*$.*

In order to construct a super-replicating strategy, we will make a short excursion into the theory of optimal stopping. For any process Y in L, *Snell envelope* W of Y is defined by

$$W_t = \max_{\tau \in \mathcal{T}(t)} E_t(Y_\tau), \qquad 0 \leq t \leq \overline{\tau}.$$

It can be shown as an exercise that for any $t < \overline{\tau}$, $W_t = \max[Y_t, E_t(W_{t+1})]$. Thus $W_t \geq E_t(W_{t+1})$, implying that W is a supermartingale, meaning that we can decompose W in the form $W = Z - A$ for some martingale Z and some increasing adapted process A with $A_0 = 0$. This decomposition is illustrated in Figure 2.1 for the case in which Y is a deterministic process, which implies that W, Z, and A are also deterministic.

We take Y to be defined by $Y_t = X_t \pi_t$, and let W, Z, and A be defined as above. By the definition of complete markets, there is a trading strategy θ with the property that

- $\delta_t^\theta = 0$ for $0 < t < \overline{\tau}$;
- $\delta_{\overline{\tau}}^\theta = Z_{\overline{\tau}} / \pi_{\overline{\tau}}$;
- $\delta_t^\theta = 0$ for $t > \overline{\tau}$.

Property (a) of a super-replicating strategy is satsifed by this strategy θ. From the fact that Z is a martingale and the definition of a state-price deflator, the cum-dividend value V^θ of the trading strategy θ satisfies

$$\pi_t V_t^\theta = E_t(\pi_{\overline{\tau}} \delta_{\overline{\tau}}^\theta) = E_t(Z_{\overline{\tau}}) = Z_t, \qquad t \leq \overline{\tau}. \tag{14}$$

From (14) and the fact that $A_0 = 0$, we know that $V_0^\theta = V_0^*$ because $Z_0 = W_0 = \pi_0 V_0^*$. Since $Z_t - A_t = W_t \geq Y_t$ for all t, from (14) we also know that

$$V_t^\theta = \frac{Z_t}{\pi_t} \geq \frac{1}{\pi_t}(Y_t + A_t) = X_t + \frac{A_t}{\pi_t} \geq X_t, \qquad t \leq \overline{\tau},$$

the last inequality following from the fact that $A_t \geq 0$ for all t. Thus property (b) is also satisfied, and θ is indeed a super-replicating strategy with $V_0^\theta = V_0^*$. This proves the proposition and implies that unless there is an arbitrage, the initial price V_0 of the American security is equal to the market value V_0^* associated with a rational exercise policy.

The Snell envelope W is also the key to finding a rational exercise policy. As for the deterministic case illustrated in Figure 2.1, a rational exercise policy is given by $\tau^0 = \min\{t : W_t = Y_t\}$. We now show the optimality of τ^0. First, we know that if τ is a rational exercise policy, then $W_\tau = Y_\tau$. (This can be seen from the fact that $W_\tau \geq Y_\tau$, and if $W_\tau > Y_\tau$ then τ cannot be rational.) From this fact, any rational exercise policy τ has the property that $\tau \geq \tau^0$. For any such τ, we have

$$E_{\tau^0}[Y(\tau)] \leq W(\tau^0) = Y(\tau^0),$$

and the law of iterated expectations implies that $E[Y(\tau)] \leq E[Y(\tau^0)]$, so τ^0 is rational.

We have shown the following.

Theorem. *Given $(X, \overline{\tau}, \delta, S)$, suppose, for each $\tau \leq \overline{\tau}$, that $\delta^{X,\tau}$ is redundant. Suppose there is a state-price deflator π for (δ, S), and let W be the Snell envelope of $X\pi$ up to the expiration time $\overline{\tau}$. Then a rational exercise* *policy for* $(X, \overline{\tau}, \delta, S)$ *is given by* $\tau^0 = \min\{t : W_t = \pi_t X_t\}$. *The unique initial arbitrage-free price process of of the American security is*

$$V_0^* = \frac{1}{\pi_0} E\left[X(\tau^0)\pi(\tau^0)\right].$$

2J Is Early Exercise Optimal?

With the equivalent martingale measure Q defined in Section 2G, we can also write the optimal stopping problem (13) in the form

$$V_0^* = \max_{\tau \in \mathcal{T}(0)} E^Q\left(\frac{X_\tau}{R_{0,\tau}}\right). \tag{15}$$

This representation of the rational exercise problem is sometimes convenient. For example, let us consider the case of an American call option on a security with price process p. We have $X_t = (p_t - K)^+$ for some exercise price K. Suppose the underlying security has no dividends before or at the expiration time $\overline{\tau}$. We suppose positive interest rates, meaning that $R_{t,s} \geq 1$ for all t and $s \geq t$. With these assumptions, we will show that it is never optimal to exercise the call option before its expiration date $\overline{\tau}$. This property is sometimes called "no early exercise," or "better alive than dead."

We define the "discounted price process" p^* by $p_t^* = p_t / R_{0,t}$. The fact that the underlying security pays dividends only after the expiration time $\overline{\tau}$ implies, by Lemma 2G, that p^* is a Q-martingale at least up to the expiration time $\overline{\tau}$. That is, for $t \leq s \leq \overline{\tau}$, we have $E_t^Q(p_s^*) = p_t^*$.

Jensen's Inequality can be used to show the following fact about convex functions of martingales, which we will use to obtain conditions for the no-early-exercise result.

Lemma. *Suppose* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is convex with respect to its first argument,* Y *is a martingale,* $\tau(1)$ *and* $\tau(2)$ *are two stopping times with* $\tau(2) \geq \tau(1)$, *and* Z *is an adapted process. Then* $f(Y_{\tau(1)}, Z_{\tau(1)}) \leq E_{\tau(1)}[f(Y_{\tau(2)}, Z_{\tau(1)})]$. *Moreover, the law of iterated expectations implies that* $E[f(Y_{\tau(1)}, Z_{\tau(1)})] \leq E[f(Y_{\tau(2)}, Z_{\tau(1)})]$.

With the benefit of this lemma and positive interest rates, we have, for any stopping time $\tau \leq \overline{\tau}$,

$$E^Q\left[\frac{1}{R_{0,\tau}}(p_\tau - K)^+\right] \quad = \quad E^Q\left[\left(p_\tau^* - \frac{K}{R_{0,\tau}}\right)^+\right]$$

$$
\begin{aligned}
&\leq E^Q\left[\left(p^*_{\overline{\tau}} - \frac{K}{R_{0,\tau}}\right)^+\right] \\
&\leq E^Q\left[\left(p^*_{\overline{\tau}} - \frac{K}{R_{0,\overline{\tau}}}\right)^+\right] \\
&= E^Q\left[\frac{1}{R_{0,\overline{\tau}}}(p_{\overline{\tau}} - K)^+\right].
\end{aligned}
$$

It follows that $\overline{\tau}$ is a rational exercise policy. In typical cases, $\overline{\tau}$ is the unique rational exercise policy.

If the underlying security pays dividends before expiration, then early exercise of the American call is, in certain cases, optimal. From the fact that the put payoff is increasing in the strike price (as opposed to decreasing for the call option), the second inequality above is reversed for the case of a put option, and one can guess that early exercise of the American put is sometimes optimal.

Exercise 2.1 gives a simple example of American security valuation in a complete-markets setting. Chapter 3 presents the idea in a Markovian setting, which offers computational advantages in solving for the rational exercise policy and market value of American securities. In Chapter 3 we also consider the case of American securities that offer dividends before expiration.

The real difficulties with analyzing American securities begin with incomplete markets. In that case, the choice of exercise policy may play a role in determining the marketed subspace, and therefore a role in pricing securities. If the state-price deflator depends on the exercise policy, it could even turn out that the notion of a rational exercise policy is not well defined.

Exercises

Exercise 2.1 Suppose in the setting of Section 2B that S is the price process of a security with zero dividends before T. We assume that

$$S_{t+1} = S_t H_{t+1}; \qquad t \geq 0; \qquad S_0 > 0,$$

where H is an adapted process such that for all $t \geq 1$, H_t has only two possible outcomes $U > 0$ and $D > 0$, each with positive conditional probability given $\mathcal{F}_{t-1}$. Suppose β is the price process of a security, also with no dividends before T, such that

$$\beta_{t+1} = \beta_t R; \qquad t \geq 1; \qquad \beta_0 > 0,$$

where $R > 1$ is a constant. We can think of β as the price process of a riskless bond. Consider a third security, a *European call option* on S with *expiration* at some fixed date $\tau < T$ and exercise price $K \geq 0$. This means that the price process C^τ of the third security has expiration value

$$C_\tau^\tau = (S_\tau - K)^+ \equiv \max(S_\tau - K, 0),$$

with $C_t^\tau = 0$, $t > \tau$. That is, the option gives its holder the right, but not the obligation, to purchase the stock at time τ at price K.

The absence of arbitrage implies that $U = R = D$, or that $U > R > D$. We will assume the latter (non-degeneracy of retruns) for the remainder of the exercise.

(A) Assuming no arbitrage, show that for $0 \leq t < \tau$,

$$C_t^\tau = \frac{1}{R^{\tau-t}} \sum_{i=0}^{\tau-t} b(i; \tau - t, p)(U^i D^{\tau-t-i} S_t - K)^+, \tag{16}$$

where $p = (R - D)/(U - D)$ and where

$$b(i; n, p) = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \tag{17}$$

is the probability of i successes, each with probability p, out of n independent binomial trials. One can thus view (16) as the discounted expected exercise value of the option, with expectation under some probability measure constructed from the stock and bond returns. In order to model this viewpoint, let $\hat{S}$ be the process defined by

$$\hat{S}_{t+1} = \hat{S}_t \hat{H}_{t+1}; \qquad t \geq 0; \qquad \hat{S}_0 = S_0, \tag{18}$$

where $\{\hat{H}_0, \hat{H}_1, \ldots\}$ is a sequence of independent random variables with outcomes U and D of probability p and $1 - p$, respectively. Then (18) implies that

$$C_0^\tau = E\left[\frac{(\hat{S}_\tau - K)^+}{R^\tau}\right]. \tag{19}$$

(B) We take it that $\mathbb{F}$ is the filtration generated by the return process H, meaning that for all $t \geq 1$, $\mathcal{F}_t$ is the tribe generated by $\{H_1, \ldots, H_t\}$. We extend the definition of the option described in part (A) by allowing the expiry date τ to be a stopping time. Show that (19) is still implied by the absence of arbitrage.

(C) An *American call option* with expiration date $\overline{\tau} < T$ is merely an option of the form described in part (A), with the exception that the exercise date τ is a finite-valued stopping time selected by the holder of the option from the set $\mathcal{T}(0)$ of all stopping times bounded by $\overline{\tau}$. Show that the rational exercise problem

$$\sup_{\tau \in \mathcal{T}(0)} C_0^{\tau} \tag{20}$$

is solved by $\tau = \overline{\tau}$. In other words, the holder of the American call option maximizes its value by holding the option to expiration. Hint: Jensen's Inequality states that for f a convex function, X a random variable on $(\Omega, \mathcal{F}, P)$, and $\mathcal{G}$ a sub-tribe of $\mathcal{F}$, we have $E[f(X) \mid \mathcal{G}] \geq f[E(X \mid \mathcal{G})]$. (D) Show that the unique arbitrage-free price of the American call described in part (C) is at any time t equal to $C_t^{\overline{\tau}}$, which is the corresponding European call price. (E) Now let $\overline{\tau}$ be a stopping time. Show that the price of an American call option that expires at $\overline{\tau}$ is given by (19), for $\tau = \overline{\tau}$.

(F) A *European put option* is defined just as is the European call, with the exception that the exercise value is $(K - S_\tau)^+$ rather than $(S_\tau - K)^+$. That is, the put gives its holder the right, but not the obligation, to sell (rather than buy) the stock at τ for the exercise price K. Let F^τ denote the European put price process for expiration at τ. The *American put* with expiration $\overline{\tau}$, analogous to the case of calls, has an exercise date τ selected by the holder from the set $\mathcal{T}(0)$ of stopping times bounded by $\overline{\tau}$. Show by counterexample that the problem

$$\sup_{\tau \in \mathcal{T}(0)} F_0^{\tau} \tag{21}$$

is not, in general, solved by $\tau = \overline{\tau}$, and that the arbitrage-free American put price process is not generally the same as the corresponding arbitrage-free European put price process $F^{\overline{\tau}}$, contrary to the case of American call options on stocks with no dividends before expiration. An easy algorithm for computing the value of the American put in this setting is given in Chapter 3. (G) Show that markets are complete. (H) The *moneyness* of a call option with expiration at a deterministic time $\tau < T$ and with strike price K is $(S_0 R^\tau - K)/K$. If the moneyness is positive, the option is said to be *in the money*. If the moneyness is negative, the option is said to be *out of the money*. If the moneyness is zero, the option is said to be *at the money*. For a put option with same strike K and expiration time τ, the moneyness is $(K - S_0 R^\tau)/K$ (minus the call moneyness), and the same terms apply. Let $S_0 = 100$, $U = 1.028$, $R = 1.001$, and $D = 0.978$. Plot (or tabulate) the price of European call and American call and put options, with expiration at $\tau = 100$, against moneyness. (I) For the parameters in Part (H), suppose that up and down returns for the stock are equally likely in each period, and that one time period represents 0.01 years. Compute the annualized continuously compounding interest rate r, and the mean and standard deviation of the annual return $\rho = (\log S_{100} - \log S_0)/100$. Plot the likelihood of ρ as a frequency diagram, that is, showing the probability of each outcome of ρ above that outcome. (J) A *barrier* option is one that can be exercised or not depending on whether or not the underlying price has crossed a given level before expiration. For example, a *down-and-out call*, at barrier $\underline{S}$, exercise time τ (possibly a stopping time), and exercise price $K \geq \underline{S}$, is a security that pays $1_A(S_\tau - K)^+$ at τ, where A is the event $\{\omega : \min\{S_0(\omega), S_1(\omega), \ldots, S_{\tau(\omega)}(\omega)\} > \underline{S}\}$. That is, A is the event that the minimum price achieved through τ is larger than the barrier $\underline{S}$. Show that an American down-and-out call, exercisable at any time $\tau \leq \overline{\tau}$, is rationally exercised (if at all) only at $\overline{\tau}$. (K) For the parameters in Part (H), and $\overline{\tau} = 100$, price a down-and-out call with barrier $\underline{S} = 80$, for strikes K from 80 to 120. Plot (or, if you can't, tabulate) the prices for integer K in this range. (L) For the parameters in Part (H), price European and American up-and-out put options, with knock-out barrier $\overline{S} = 120$. Again, obtain the prices for strikes ranging from 80 to 120. Plot (or, if you can't, tabulate) the prices for integer K in this range. For the special case of $K = 100$, plot the optimal exercise region for the American up-and-out put. That is, for each $t \leq 100$, show the set of outcomes for S_t at which it is optimal to exercise at time t.

Exercise 2.2 Suppose in the context of problem (4) that (δ, S) admits no arbitrage and that U is continuous. Show the existence of a solution. Hint: A continuous function on a compact set has a maximum. In this setting, a set is compact if closed and bounded.

Exercise 2.3 Suppose in the context of problem (4) that $e \gg 0$ and that U has the additive form (5), where for each t, u_t is concave with an unbounded derivative. Show that any solution c^* is strictly positive. Show that the same conclusion follows if the assumption that $e \gg 0$ is replaced with the assumption that markets are complete and that e is not zero.

Exercise 2.4 Prove Lemma 2C. Hint: For any x and y in L, let

$$(x \mid y) = E\left(\sum_{t=0}^{T} x_t y_t\right).$$

Then follow the hint given for Exercise 1.17, remembering that we write $x = y$ whenever $x_t = y_t$ for all t almost surely.

Exercise 2.5 For U of the additive form (5), show that the gradient $\nabla U(c)$, if it exists, is represented as in (6).

Exercise 2.6 Suppose $(c^{(1)}, \ldots, c^{(m)})$ is a strictly positive equilibrium consumption allocation and that for all i, U_i is of the additive form: $U_i(c) = E[\sum_{t=0}^{T} u_{it}(c_t)]$. Assume there is a constant $\overline{c}$ larger than c_t^i for all i and t such that for all i and t, $u_{it}(x) = A_{it}x - B_{it}x^2$, $x \leq \overline{c}$, for some positive constants A_{it} and B_{it}. That is, utility is quadratic in the relevant range.
(A) In the context of Corollary 2 of Section 2F, show that for each t, there are some constants k_t and K_t such that $u'_{\lambda t}(e_t) = k_t + K_t e_t$. Suppose for a given trading strategy θ and time t that the following are well defined:

- $R_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t)/\theta_{t-1} \cdot S_{t-1}$, the return on θ at time t;
- R_t^M, the return at time t on a strategy φ maximizing $\mathrm{corr}_{t-1}(R_t^\varphi, e_t)$, where $\mathrm{corr}_t(\,\cdot\,)$ denotes $\mathcal{F}_t$-conditional correlation;
- $\beta_{t-1}^\theta = \mathrm{cov}_{t-1}(R_t^\theta, R_t^M)/\mathrm{var}_{t-1}(R_t^M)$, the conditional beta of the trading strategy θ with respect to the market return, where $\mathrm{cov}_t(\,\cdot\,)$ denotes $\mathcal{F}_t$-conditional covariance and $\mathrm{var}_t(\,\cdot\,)$ denotes $\mathcal{F}_t$-conditional variance;
- R_t^0, the return at time t on a strategy η with $\mathrm{corr}_{t-1}(R_t^\eta, e_t) = 0$.

Derive the following beta-form of the *consumption-based CAPM*:

$$E_{t-1}(R_t^\theta - R_t^0) = \beta_{t-1}^\theta E_{t-1}(R_t^M - R_t^0). \tag{22}$$

(B) Prove the same beta-form (22) of the CAPM holds in equilibrium even without assuming complete markets.

(C) Extend the state-price beta model of Section 1F to this setting, as follows, without using the assumptions of the CAPM. Let π be a state-price deflator. For each t, suppose R_t^* is the return on a trading strategy solving

$$\sup_{\theta} \operatorname{corr}_{t-1}(R_t^\theta, \pi_t).$$

Assume that $\operatorname{var}_{t-1}(R_t^*)$ is nonzero almost surely. Show that for any return R_t^θ,

$$E_{t-1}(R_t^\theta - R_t^0) = \beta_{t-1}^\theta E_{t-1}(R_t^* - R_t^0), \tag{23}$$

where $\beta_{t-1}^\theta = \operatorname{cov}_{t-1}(R_t^\theta, R_t^*)/\operatorname{var}_{t-1}(R_t^*)$ and $\operatorname{corr}_{t-1}(R_t^0, \pi_t) = 0$.

Exercise 2.7 Prove Proposition 2E.

Exercise 2.8 In the context of Section 2D, suppose that U is the *habit-formation* utility function defined by $U(c) = E[\sum_{t=0}^T u(c_t, h_t)]$, where $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable on the interior of its domain and, for any t, the "habit" level of consumption is defined by $h_t = \sum_{j=0}^t \alpha_j c_{t-j}$ for some $\alpha \in \mathbb{R}^T$. For example, we could take $\alpha_j = \gamma^j$ for $\gamma \in (0, 1)$, which gives geometrically declining weights on past consumption. Calculate the Riesz representation of the gradient of U at a strictly positive consumption process c.

Exercise 2.9 Consider a utility function U defined by $U(c) = V_0$, where the *utility process* V is defined recursively, backward from T in time, by $V_T = J(c_T, h(0))$ and, for $t < T$, by $V_t = J(c_t, E_t[h(V_{t+1})])$, where $J : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is increasing and continuously differentiable on the interior of its domain and $h : \mathbb{R} \to \mathbb{R}$ is increasing and continuously differentiable. This is a special case of what is known as *recursive utility*, and also a special case of what is known as *Kreps-Porteus* utility. Note that the utility function can depend nontrivially on the filtration $\mathbb{F}$, which is not true for additive or habit-formation utility functions. This utility model is reconsidered in an exercise in Chapter 3.

(A) Compute the Riesz Representation π of the gradient of U at a strictly positive consumption process c.

(B) Suppose that h and J are concave and increasing functions. Show that U is concave and increasing.

Exercise 2.10 In the setting of Section 2E, an Arrow-Debreu equilibrium is a feasible consumption allocation $(c^{(1)}, \ldots, c^{(m)})$ and a nonzero linear function $\Psi : L \to \mathbb{R}$ such that for all i, c^i solves $\max_{c \in L_+} U_i(c)$ subject to $\Psi(c^i) \leq \Psi(e^i)$. Suppose that $(c^{(1)}, \ldots, c^{(m)})$ and Ψ form an Arrow-Debreu equilibrium and that π is the Riesz representation of Ψ. Let S be defined by $S_T = 0$ and by taking π to be a state-price deflator. Suppose, given (δ, S), that markets are complete. Show the existence of trading strategies $\theta^{(1)}, \ldots, \theta^{(m)}$ such that $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ is an equilibrium with the same consumption allocation $(c^{(1)}, \ldots, c^{(m)})$.

Exercise 2.11 Given a finite set Ω of states, a *partition* of Ω is a collection of disjoint nonempty subsets of Ω whose union is Ω. For example, a partition of $\{1, 2, 3\}$ is given by $\{\{1\}, \{2, 3\}\}$. The tribe on a finite set Ω generated by a given partition p of Ω, denoted $\sigma(p)$, is the smallest tribe $\mathcal{F}$ on Ω such that $p \subset \mathcal{F}$. Conversely, for any tribe $\mathcal{F}$ on Ω, the partition $\mathcal{P}(\mathcal{F})$ generated by $\mathcal{F}$ is the smallest partition p of Ω such that $\mathcal{F} = \sigma(p)$. For instance, the tribe $\{\emptyset, \Omega, \{1\}, \{2, 3\}\}$ is generated by the partition in the above example. Since partitions and tribes on a given finite set Ω are in one-to-one correspondence, we could have developed the results of Chapter 2 in terms of an increasing sequence $p_0, p_1, \ldots, p_T$ of partitions of Ω rather than a filtration of tribes, $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T$. (In the infinite-state models of Part II, however, it is more convenient to use tribes than partitions.)

Given a subset B of Ω and a partition p of Ω, let $n(B, p)$ denote the minimum number of elements of p whose union contains B. In a sense, this is the number of distinct nonempty events that might occur if B is to occur. For $t < T$, let

$$n_t = \max_{B \in p_t} \; n(B, p_{t+1}).$$

Finally, the *spanning number* of the filtration $\mathbb{F}$ generated by $p_0, \ldots, p_T$ is $n(\mathbb{F}) \equiv \max_{t<T} n_t$. In a sense, $n(\mathbb{F})$ is the maximum number of distinct events that could be revealed between two periods.

Show that complete markets requires at least $n(\mathbb{F})$ securities, and that given the filtration $\mathbb{F}$, there exists a set of $n(\mathbb{F})$ dividend processes and associated arbitrage-free security-price processes such that markets are complete. This issue is further investigated in sources indicated in the Notes.

Exercise 2.12 Given securities with a dividend-price pair (δ, S), extend Theorem 2G to show, in the presence of riskless borrowing at a strictly positive discount at each date, the equivalence of these statements:

(a) There exists a state-price deflator.

(b) There exists a deflator π such that (3) holds for any trading strategy θ.

(c) $S_T = 0$ and there exists a deflator π such that the deflated gain process G^π is a martingale.

(d) There is no arbitrage.

(e) There is an equivalent martingale measure.

Exercise 2.13 Show, from (11) and (12), that (10) is indeed satisfied, confirming that Q is an equivalent martingale measure.

Exercise 2.14 Show, as claimed in Section 2I, that if τ^* is a rational exercise policy for the American security X and if V^* is the cum-dividend price process for the American security with this rational exercise policy, then $V^*_\tau \geq X_\tau$ for any stopping time $\tau \leq \tau^*$.

Exercise 2.15 (Aggregation Revisited) Suppose, in the context of the supremum (8), that $x \gg 0$ and, for all i, $U_i(c) = E[\sum_{t=0}^T u_t(c_t)]$, where, for all t, $u_t(x) = k_t x^{\gamma(t)}/\gamma(t)$, where k_t and $\gamma(t) < 1$, $\gamma(t) \neq 0$, are constants (depending on t).

(A) Show that U_λ is of the same utility form as U_i.

(B) Suppose that $\gamma(t)$ is a constant independent of t. Replace the assumption of complete markets in Proposition 2F with the assumption that $e \gg 0$ and that, for all i, there is a security whose dividend process is e^i. (This can easily be weakened.) Demonstrate the existence of an equilibrium with the same properties described in Proposition 2F, including a consumption allocation that is Pareto optimal.

Exercise 2.16 (Put-Call Parity) In the general setting explained in Section 2B, suppose there exist the following securities:

(a) a "stock," with price process X;

(b) a European call option on the stock with strike price K and expiration τ;

(c) a European put option on the stock with strike price K and expiration τ;

(d) a τ-period zero-coupon riskless bond.

Let X_0, C_0, P_0, and B_0 denote the initial respective prices of the securities. Suppose there is no arbitrage, and that the stock pays no dividends before time τ. Solve for C_0 explicitly in terms of X_0, P_0, and B_0.

Exercise 2.17 (Futures-Forward Price Equivalence) This exercise defines (in ideal terms) a *forward contract* and a *futures contract*, and gives simple conditions under which the *futures price* and the *forward price* coincide. We adopt the setting of Section 2B, in the absence of arbitrage. Fixed throughout are a *delivery date* τ and a settlement amount W_τ (an $\mathcal{F}_\tau$-measurable random variable).

Informally speaking, the associated forward contract made at time t is a commitment to pay an amount F_t (the forward price), which is agreed upon at time t and paid at time τ, in return for the amount W_τ at time τ. Formally speaking, the forward contract made at time t for delivery of W_τ at time τ for a forward price of F_t is a security whose price at time t is zero and whose dividend process δ is defined by $\delta_t = 0$, $t \neq \tau$, and $\delta_\tau = W_\tau - F_t$.
(A) Suppose that Q is an equivalent martingale measure and that there is riskless short-term borrowing at any date t at a discount d_t that is deterministic. Show that $\{F_0, F_1, \ldots, F_\tau\}$ is a Q-martingale, in that $F_t = E_t^Q(F_\tau)$ for all $t \leq \tau$.

A futures contract differs from a forward contract in several practical ways that depend on institutional details. One of the details that is particularly important for pricing purposes is *resettlement*. For theoretical modeling purposes, we can describe resettlement as follows: A futures-price process $\Phi = \{\Phi_0, \ldots, \Phi_\tau\}$ for delivery of W_τ at time τ is taken as given. At any time t, an investor can adopt a position of θ futures contracts by agreeing to accept the resettlement payment $\theta(\Phi_{t+1} - \Phi_t)$ at time $t+1$, $\theta(\Phi_{t+2} - \Phi_{t+1})$ at time $t+2$, and so on, until the position is changed (or eliminated). This process of paying or collecting any changes in the futures price, period by period, is called *marking to market*, and serves in practice to reduce the likelihood or magnitude of potential defaults. Formally, all of this means simply that the dividend process δ of the futures contract is defined by $\delta_t = \Phi_t - \Phi_{t-1}$, $1 \leq t \leq \tau$.

For our purposes, it is natural to assume that the delivery value Φ_τ is contractually equated with W_τ. (In a more detailed model, we could equate Φ_τ and W_τ by the absence of *delivery arbitrage*.)
(B) Suppose Q is an equivalent martingale measure and show that for all $t \leq \tau$, $\Phi_t = E_t^Q(W_\tau)$. It follows from parts (A) and (B) that with deterministic interest rates and the absence of arbitrage, futures and forward prices coincide. We now suppose that W_τ is the market value S_τ of a security with dividend process δ. (C) Suppose that δ and the discount process $d = \{d_1, \ldots, d_T\}$ on riskless borrowing are both deterministic. Calculate the futures and forward prices, Φ_t and F_t, explicitly in terms of S_t, d, and δ.

Exercise 2.18 Provide details fleshing out the following outline of a proof of the converse part of Proposition 2G.

Let $J = \{(x_1, \ldots, x_T) : x \in L\}$ and $H = \{(\delta_1^\theta, \ldots, \delta_T^\theta) : \theta \in \Theta\}$. Markets are complete if and only if $J = H$. By Theorem 2G, there is a unique equivalent martingale measure if and only if there is a unique state-price deflator π such that $\pi_0 = 1$. Suppose $H \neq J$. Since H is a linear subspace of J, there is some nonzero y in J "orthogonal" to H, in the sense that $E(\sum_{t=1}^T y_t h_t) = 0$ for all h in H. Let $\hat{\pi} \in L$ be defined by $\hat{\pi}_0 = 1$ and $\hat{\pi}_t = \pi_t + \alpha y_t$, $t \geq 1$, where $\alpha > 0$ is a scalar small enough that $\hat{\pi} \gg 0$. Then $\hat{\pi}$ is a distinct state-price deflator with $\hat{\pi}_0 = 1$. This shows that if there is a unique state-price deflator π with $\pi_0 = 1$, then markets must be complete. Hint: Let

$$(y \mid h) \equiv E\left(\sum_{t=1}^T y_t h_t\right), \qquad h \in H$$

define an inner $(\,\cdot \mid \cdot\,)$ for H in the sense of Exercise 1.17.

Exercise 2.19 It is asserted in Section 2I that if W is the Snell envelope of Y, then $W_t = \max[Y_t, E_t(W_{t+1})]$. Prove this natural property.

Exercise 2.20 Prove Lemma 2J.

Exercise 2.21 Consider the "tree" of prices for securities A and B shown in Figure 2.2. At each node in the tree, a pair (p_A, p_B) of prices is shown, the first of which is the price of A at that node, the second of which is the price of B.

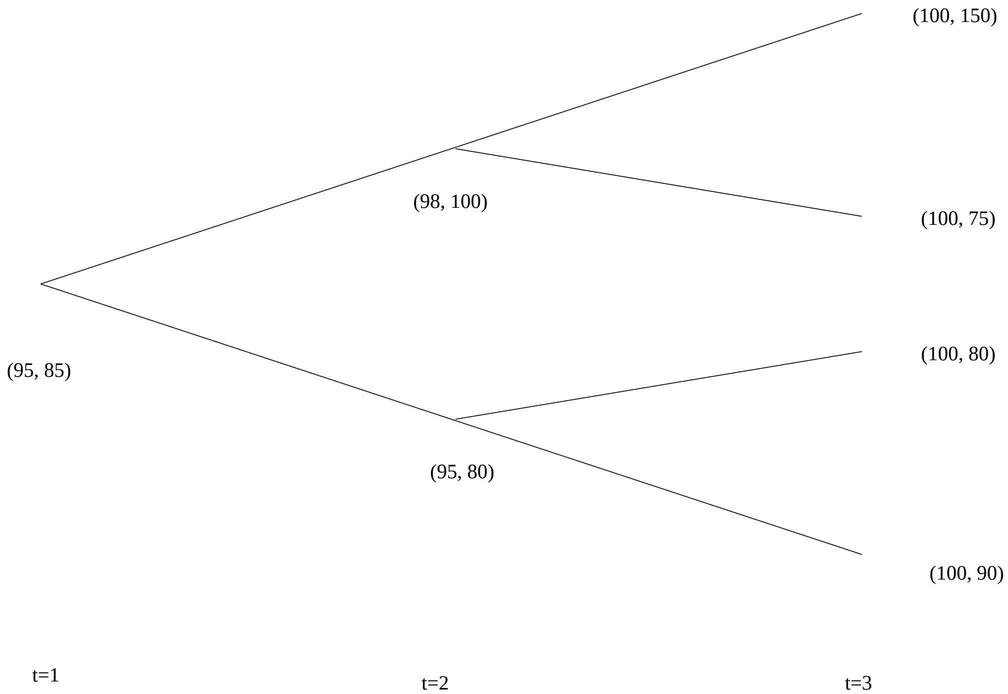

Figure2.2: An Event Tree With Prices

(A) Construct a probability space $(\Omega, \mathcal{F}, P)$, a filtration of tribes, $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$, and a vector security price process S, that formally encode the information in the figure. Please be explicit. Take the security price process to be *cum-dividend*, so that S_2 is both the price and the dividend payoff vector of the securities at time 2. There are no dividend payments in periods 0 and 1.

(B) Suppose there is no arbitrage. Find the price at time 0 of an American put option on asset B, with an exercise price of 95 and expiring at time 2. (Remember, this is an option to sell B for 95 at any of times 0, 1, or 2.) (C) Suppose the price at time 0 in the market of this put option is in fact

10 percent lower than the arbitrage-free price you arrived at in part B. Show explicitly how to create a riskless profit of 1 million dollars at time 0, with no cash flow after time 0. (D) Suppose the price in the market of this put

option is in fact 10 percent higher than the arbitrage-free price you arrived at in part B. Show explicitly how to create a riskless profit of 1 million dollars at time 0, with non-negative cash flow after time 0. *Hint:* If you decide to sell the option, you should not assume that the person to whom you sold it will exercise it in any particular fashion.

Exercise 2.22 Let $T = 1$, and suppose there are 3 equally likely states of world, ω_1, ω_2, and ω_3, one of which is revealed as true at time 1. A particular agent has utility function U and has equilibrium consumption choices $c_0 = 25$ and

$$c_1(\omega_1) = 9, \qquad c_1(\omega_2) = 16, \qquad c_1(\omega_3) = 4.$$

In each case below, compute the price of a security that pays 3 in state ω_1, 6 in state ω_2, and 5 in state ω_3. Show your work.

(A) Expected, but not time-additive, utility $U(c_0, c_1) = E[u(c_0, c_1)]$, with $u(x, y) = \sqrt{xy}$.

(B) Non-expected utility $U(c_0, c_1) = \sqrt{c_0\, c_1(\omega_1)\, c_1(\omega_2)\, c_1(\omega_3)}$.

Exercise 2.23 For concreteness, the length of one period is one year. There are two basic types of investments. The first is riskless borrowing or lending. The equilibrium one-year short rate is 25 percent (simple interest per year), each year. (So one can invest 1 at time zero and collect 1.25 at the end of the first year, or invest 1 at the end of the first year and collect 1.25 at the end of the second year.) At the end of each year a fair coin is flipped. A risky security has zero initial market value. Its market value goes up by one unit at the end of each year if the outcome of the coin flip for that year is Heads. Its market value goes down by one unit at the end of each year if the outcome of the coin flip for that year is Tails. For example, the price of the risky security at the end of the second year is -2, 0, or $+2$, with respective probabilities 0.25, 0.50, and 0.25.

There is also a European option to purchase the risky security above at the end of the second year (only) at an exercise price of 1 unit of account.

(A) Suppose there is no arbitrage. State the initial market price q of the option. (Show your reasoning.)

(B) Now suppose the option is actually selling for $q/2$. Construct a trading strategy that generates a net initial positive cash flow of 1000 units of account and no subsequent cash flows. (State a precise recipe for the quantities of each security to buy or sell, at each time, in each contingency.)

Exercise 2.24 Prove Corollary 2F.

Exercise 2.25 (Numeraire Invariance) Consider a dividend-price pair $(\delta, S) \in L^N \times L^N$, and a deflator γ. Let $\hat{S} = S\gamma$ and $\hat{\delta} = \delta\gamma$ denote the deflated price and dividend processes. Let θ be any given trading strategy. Show that the dividend process $\hat{\delta}^\theta$ generated by θ given $(\hat{\delta}, \hat{S})$ and the dividend process δ^θ generated by θ under (δ, S) are related by $\hat{\delta}^\theta = \gamma\delta^\theta$. Show that θ is an arbitrage with respect to (δ, S) if and only if θ is an arbitrage with respect to $(\hat{\delta}, \hat{S})$. If π is a state-price deflator for (δ, S), compute a state-price deflator $\hat{\pi}$ for $(\hat{\delta}, \hat{S})$ in terms of π and γ.

Notes

The model of uncertainty and information is standard. The model of uncertainty is equivalent to that originated in the general equilibrium model of Debreu [1953], which appears in Chapter 7 of Debreu [1959]. For more details in a finance setting, see Dothan [1990]. The connection between arbitrage and martingales given in Sections 2C and 2G is from the more general model of Harrison and Kreps [1979]. Girotto and Ortu [1993] present general results, in this finite-dimensional setting, on the equivalence between no arbitrage and the existence of an equivalent martingale measure. The spirit of the results on optimality and state prices is also from Harrison and Kreps [1979]. **?**], Girotto and Ortu [1995], and Girotto and Ortu [1997] full explore this equivalence in financial dimensional multi-period economies.

The habit-formation utility model was developed by Dunn and Singleton [1986] and in continuous time by Ryder and Heal [1973]. An application of habit formation to state-pricing in this setting appears in Chapman [1998]. The recursive-utility model, in various forms, is due to Selden [1978], Kreps and Porteus [1978], and Epstein and Zin [1989], and is surveyed by Epstein [1992]. Koopmans [1960] presented an early precursor. The recursive-utility model allows for preference for earlier or later resolution of uncertainty (which have no impact on additive utility). This is relevant, for example, in the context of the remarks by Ross [1989], as shown by Skiadas [991a] and Duffie, Schroder, and Skiadas [1997]. For a more general form of recursive utility than that appearing in Exercise 2.9, the von Neumann-Morgenstern function h can be replaced with a function of the conditional distribution of next-period utility. Examples are the local-expected-utility model of Machina [1982] and the *betweenness certainty equivalent* model of Chew [1983], Chew [1989], Dekel [1987], and Gul and Lantto [1992]. The equilibrium state-price associated with recursive utility is computed in a Markovian version of this setting by **?**]. For further justification and properties of recursive utility, see Chew and Epstein [1991], Skiadas [991a] and Skiadas [991b]. For further implications for asset pricing, see Epstein [1988], Epstein [1992], Epstein and Zin [1991], and Giovannini and Weil [1989]. Kakutani [1993] explored the utility gradient representation of recursive utility in this setting.

Radner [1967] and Radner [1972] originated the sort of dynamic equilibrium model treated in this chapter. The basic approach to existence given in Exercise 2.11 is suggested by Kreps [1982], and is shown to work for "generic" dividends and endowments, under technical regularity conditions, in McManus [1984], Repullo [1986], and Magill and Shafer [1990], provided the number of securities is at least as large as the spanning number of the filtration $\mathbb{F}$ (as suggested in Exercise 2.11). This literature is reviewed in depth by Geanakoplos [1990]. See Duffie and Huang [1985] for the definition of spanning number in more general settings and for a continuous-time version of a similar result. Duffie and Shafer [1985] and Duffie and Shafer [986a] show generic existence of equilibrium in incomplete markets; Hart [1975] gives a counterexample. Bottazzi [1995] has a somewhat more advanced version of this result in its single-period multiple-commodity version. See, also Won [1996] and Won [996c].

Related existence topics are studied by Bottazzi and Hens [1993], Hens [1991], and Zhou [1993]. Dispersed expectations, in a temporary-equilibrium variant of the model, is shown to lead to existence by Henrotte [1994] and by Honda [1992]. Alternative proofs of existence of equilibrium are given in the 2-period version of the model by Geanakoplos and Shafer [1990], Hirsch, Magill, and Mas-Colell [1990], and Husseini, Lasry, and Magill [1990]; and in a T-period version by Florenzano and Gourdel [1994]. If one defines security dividends in nominal terms, rather than in units of consumption, then equilibria always exist under standard technical conditions on preferences and endowments, as shown by Cass [1984], Werner [1985], Duffie [1987], and Gottardi and Hens [1994], although equilibrium may be indeterminate, as shown by Cass [1989] and Geanakoplos and Mas-Colell [1989]. On this point, see also Kydland and Prescott [1991], Mas-Colell [1991], and Cass [1991]. Likewise, one obtains existence in a one-period version of the model provided securities have payoffs in a single commodity (the framework of most of this book), as shown by Chae [1988] and Geanakoplos and Polemarchakis [1986]. Surveys of general equilibrium models in incomplete markets setting are given by Cass [1991], Duffie [1992], Geanakoplos [1990], **?**], and Magill and Shafer [1991]. In the presence of price-dependent options, existence can be more problematic, as shown by Polemarchakis and Ku [1990], but variants of the formulation will suffice for existence in many cases, as shown by Huang and Wu [1994] and Krasa and Werner [1991]. Detemple and Selden [1991] examine the implications of options for asset pricing in a general equilibrium model with incomplete markets. Bajeux-Besnainou and Rochet [1995] explore the dynamic spanning implications of options. The importance of the timing of information in this setting is described by Berk and Uhlig [1993]. Hindy and Huang [1993b] show the implications of linear collateral constraints on security valuation. Hara [1993] treats the role of "redundant" securities in the presence of transactions costs.

Hahn [1992] and Hahn [1994] raise some philosophical issues regarding the possibility of complete markets and efficiency. The Pareto inefficiency of incomplete markets equilibrium consumption allocations, and notions of constrained efficiency, are discussed by Hart [1975], Kreps [1979] (and references therein), Citanna, Kajii, and Villanacci [1994], Citanna and Villanacci [1993], Pan [1992], and Pan [1993].

The optimality of individual portfolio and consumption choices in incomplete markets in this setting is given a dual interpretation by He and es [1993]. (Girotto and Ortu [1994] offer related remarks.) Methods for computation of equilibrium with incomplete markets are developed by Brown, DeMarzo, and Eaves [993a], Brown, DeMarzo, and Eaves [993b] and DeMarzo and Eaves [1993].

The representative agent state-pricing model for this setting was shown by Constantinides [1982]. An extension of this notion to incomplete markets, where one cannot generally exploit Pareto optimality, is given by Cuoco and He [992a]. Kraus and Litzenberger [1975] and Stapleton and Subrahmanyam [1978] present parametric examples of equilibrium. Hansen and Richard [1987] explore the state-price beta model in a much more general multiperiod setting. Ross [1987] and Prisman [1985] show the impact of taxes and transactions costs on the state-pricing model. Hara [1993] discusses the role of redundant securities in the presence of transactions costs. The consumption-based CAPM of Exercise 2.6 is found, in a different form, in Rubinstein [1976]. The aggregation result of Exercise 2.15 is based on Rubinstein [974b]. Rubinstein [974a] has a detailed treatment of asset pricing results in the setting of this chapter. Rubinstein [1987] is a useful expository treatment of derivative asset pricing in this setting.

Cox, Ross, and Rubinstein [1979] developed the muli-period binomial option pricing model analyzed in Exercise 2.1, and further analyzed in terms of convergece to the Black-Scholes Formula in Chapter 11.

The role of production is considered by Duffie and Shafer [1986] and Naik [1993]. The Modigliani-Miller Theorems are reconsidered in this setting by DeMarzo [1988], Duffie and Shafer [1986], and Gottardi [1995].

The modeling of American security valuation given here is similar to the continuous-time treatments of Bensoussan [1984] and Karatzas [1988], who do not formally connect the valuation of American securities with the absence of arbitrage, but rather deal with the similar notion of "fair price." Merton [1973] was the first to attack American option valuation systematically using arbitrage-based methods and to point out the inoptimality of early exercise of certain American options in a Black-Scholes style setting. American option valuation is reconsidered in Chapters 3 and 8, whose literature notes cite many additional references.

Chapter 3

The Dynamic Programming Approach

THIS CHAPTER PRESENTS portfolio choice and asset pricing in the framework of dynamic programming, a technique for solving dynamic optimization problems with a recursive structure. The asset-pricing implications go little beyond those of the previous chapter, but there are computational advantages. After introducing the idea of dynamic programming in a deterministic setting, we review the basics of a finite-state Markov chain. The Bellman equation is shown to characterize optimality in a Markov setting. The first-order condition for the Bellman equation, often called the "stochastic Euler equation," is then shown to characterize equilibrium security prices. This is done with additive utility in the main body of the chapter, and extended to more general recursive forms of utility in the exercises. The last sections of the chapter show the computation of arbitrage-free derivative security values in a Markov setting, including an application of Bellman's equation for optimal stopping to the valuation of American securities such as the American put option. An exercise presents algorithms for the numerical solution of term-structure derivative securities in a simple binomial setting.

3A The Bellman Approach

To get the basic idea, we start in the T-period setting of the previous chapter, with no securities except those permitting short-term riskless borrowing at any time t at the discount $d_t > 0$. The endowment process of a given agent is e. Given a consumption process c, it is convenient to define the agent's *wealth process* W^c by $W_0^c = 0$ and

$$W_t^c = \frac{W_{t-1}^c + e_{t-1} - c_{t-1}}{d_{t-1}}, \qquad t \geq 1. \tag{1}$$

Given a utility function $U : L_+ \to \mathbb{R}$ on the set L of nonnegative adapted processes, the agent's problem can be rewritten as

$$\sup_{c \in L_+} U(c) \qquad \text{subject to (1) and } c_T \leq W_T^c + e_T. \tag{2}$$

Dynamic programming is only convenient with special types of utility functions. One example is an additive utility function U, defined by

$$U(c) = E\left[\sum_{t=0}^{T} u_t(c_t)\right], \tag{3}$$

with $u_t : \mathbb{R}_+ \to \mathbb{R}$ strictly increasing and continuous for each t. Given this utility function, it is natural to consider the problem at any time t of maximizing the "remaining utility," given current wealth $W_t^c = w$. In order to keep things simple at first, we take the case in which there is no uncertainty, meaning that $\mathcal{F}_t = \{\Omega, \emptyset\}$ for all t. The maximum remaining utility at time t is then written, for each w in $\mathbb{R}$, as

$$V_t(w) = \sup_{c \in L_+} \sum_{s=t}^{T} u_s(c_s),$$

subject to $W_t^c = w$, the wealth dynamic (1), and $c_T \leq W_T^c + e_T$. If there is no budget-feasible consumption choice (because w is excessively negative), we write $V_t(w) = -\infty$.

Clearly $V_T(w) = u_T(w + e_T)$ for $w \geq -e_T$, and it is shown as an exercise that for $t < T$,

$$V_t(w) = \sup_{\overline{c} \in \mathbb{R}_+} u_t(\overline{c}) + V_{t+1}\left(\frac{w + e_t - \overline{c}}{d_t}\right), \tag{4}$$

the *Bellman equation.* It is also left as an exercise to show that an optimal consumption policy c is defined inductively by $c_t = C_t(W_t^c)$, where $C_t(w)$ denotes a solution to (4) for $t < T$, and where $C_T(w) = w + e_T$. From (4), the *value function* V_{t+1} thus summarizes all information regarding the "future" of the problem that is required for choice at time t.

3B First-Order Conditions of the Bellman Equation

Throughout this section, we take the additive model (3) and assume in addition that for each t, u_t is strictly concave and differentiable on $(0, \infty)$. Extending Exercise 2.2, there exists an optimal consumption policy c^*. We assume that c^* is strictly positive. Let W^* denote the wealth process associated with c^* by (1).

Lemma. *For any t, V_t is strictly concave and continuously differentiable at W_t^*, with $V_t'(W_t^*) = u_t'(c_t^*)$.*

Proof is left as Exercise 3.3, which gives a broad hint. The first-order conditions for the Bellman equation (4) then imply, for any $t < T$, that the one-period discount is

$$d_t = \frac{u_{t+1}'(c_{t+1}^*)}{u_t'(c_t^*)}. \tag{5}$$

The same equation is easily derived from the general characterization of equilibrium security prices given by equation (2.9). More generally, the price $\Lambda_{t,\tau}$ at time t of a unit riskless bond maturing at any time $\tau > t$ is

$$\Lambda_{t,\tau} \equiv d_t\, d_{t+1} \cdots d_{\tau-1} = \frac{u_\tau'(c_\tau^*)}{u_t'(c_t^*)}, \tag{6}$$

which, naturally, is the marginal rate of substitution of consumption between the two dates.

Since the price of a coupon-bearing bond, the only kind of security in a deterministic setting, is merely the sum of the prices of its coupons and principal, (6) provides a complete characterization of security prices in this setting.

3C Markov Uncertainty

We take the easiest kind of Markov uncertainty, a *time-homogeneous Markov chain*. Let the elements of a fixed set $Z = \{1, \ldots, k\}$ be known as *shocks*. For any shocks i and j, let $q_{ij} \in [0, 1]$ be thought of as the probability, for any t, that shock j occurs in period $t+1$ given that shock i occurs in period t. Of course, for each i, $q_{i1} + \cdots + q_{ik} = 1$. The $k \times k$ *transition matrix* q is thus a complete characterization of transition probabilities. This idea is formalized with the following construction of a probability space and filtration of tribes. It is enough to consider a state of the world as some particular sequence $(z_0, \ldots, z_T)$ of shocks that might occur. We therefore let $\Omega = Z^{T+1}$ and let $\mathcal{F}$ be the set of all subsets of Ω. For each t, let $X_t : \Omega \to Z$ (the random shock at time t) be the random variable defined by $X_t(z_0, \ldots, z_T) = z_t$. Finally, for each i in Z, let P_i be the probability measure on $(\Omega, \mathcal{F})$ uniquely defined by two conditions:

$$P_i(X_0 = i) = 1 \tag{7}$$

and, for all $t < T$,

$$P_i\left[X(t+1) = j \mid X(0),\, X(1),\, X(2), \ldots, X(t)\right] = q_{X(t),j}. \tag{8}$$

Relations (7) and (8) mean that under probability measure P_i, X starts at i with probability 1 and has the transition probabilities previously described informally. In particular, (8) means that $X = \{X_0, \ldots, X_T\}$ is a *Markov process*: the conditional distribution of X_{t+1} given $X_0, \ldots, X_t$ depends only on X_t. To complete the formal picture, for each t, we let $\mathcal{F}_t$ be the tribe generated by $\{X_0, \ldots, X_t\}$, meaning that the information available at time t is that obtained by observing the shock process X until time t.

Lemma. *For any time t, let $f : Z^{T-t+1} \to \mathbb{R}$ be arbitrary. Then there exists a fixed function $g : Z \to \mathbb{R}$ such that for any i in Z,*

$$E^i\left[f(X_t, \ldots, X_T) \mid \mathcal{F}_t\right] = E^i[f(X_t, \ldots, X_T) \mid X_t] = g(X_t),$$

where E^i denotes expectation under P_i.

This lemma gives the complete flavor of the Markov property.

3D Markov Asset Pricing

Taking the particular Markov source of uncertainty described in Section 3C, we now consider the prices of securities in a single- or representative-agent setting with additive utility of the form (3), where, for all t, u_t has a strictly positive derivative on $(0, \infty)$. Suppose, moreover, that for each t there are functions $f_t : Z \to \mathbb{R}^N$ and $g_t : Z \to \mathbb{R}$ such that the dividend is $\delta_t = f_t(X_t)$ and the endowment is $e_t = g_t(X_t)$. Then Lemma 3C and the general gradient solution (2.9) for equilibrium security prices imply the following characterization of the equilibrium security price process S. For each t there is a function $\mathcal{S}_t : Z \to \mathbb{R}^N$ such that $S_t = \mathcal{S}_t(X_t)$. In particular, for any initial shock i and any time $t < T$,

$$\mathcal{S}_t(X_t) = \frac{1}{\pi_t} E^i \left(\pi_{t+1} \left[f_{t+1}(X_{t+1}) + \mathcal{S}_{t+1}(X_{t+1}) \right] \,\Big|\, X_t \right), \tag{9}$$

where π is the state-price deflator given by $\pi_t = u_t'[g_t(X_t)]$. This has been called the *stochastic Euler equation* for security prices.

3E Security Pricing by Markov Control

We will demonstrate (9) once again, under stronger conditions, using instead Markov dynamic programming methods. Suppose that X is the shock process already described. For notational simplicity, in this section we suppose that the transition matrix q is strictly positive and that for all t,

- u_t is continuous, strictly concave, increasing, and differentiable on $(0, \infty)$;
- $e_t = g_t(X_t)$ for some $g_t : Z \to \mathbb{R}_{++}$; and
- $\delta_t = f_t(X_t)$ for some $f_t : Z \to \mathbb{R}^N_{++}$.

We assume, naturally, that $\mathcal{S}_t : Z \to \mathbb{R}^N_{++}$, $t < T$, and that there is no arbitrage. We let Θ denote the space of trading strategies and L_+ the space of nonnegative adapted processes (for consumption). For each $t \le T$, consider the value function $V_t : Z \times \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$V_t(i, w) = \sup_{(c,\theta) \in L_+ \times \Theta} E \left[\sum_{j=t}^{T} u_j(c_j) \,\Big|\, X_t = i \right], \tag{10}$$

subject to

$$W_j^\theta = \theta_{j-1} \cdot \left[\mathcal{S}_j(X_j) + f_j(X_j) \right], \qquad j > t; \qquad W_t^\theta = w, \tag{11}$$

and

$$c_j + \theta_j \cdot \mathcal{S}_j(X_j) \le W_j^\theta + g_j(X_j), \qquad t \le j \le T.$$

One may think of $V_t(X_t, \cdot\,)$ as an indirect utility function for wealth at time t. The conditional expectation in (10) does not depend on the initial state

X_0 according to Lemma 3C, so we abuse the notation by simply ignoring the initial state in this sort of expression. For sufficiently negative w, there is no (θ, c) that is feasible for (10), in which case we take $V_t(i,w) = -\infty$. For initial wealth $w = 0$ and time $t = 0$, (10) is equivalent to problem (2.4) with $S_j = \mathcal{S}_j(X_j)$ for any time j.

We now define a sequence $F_0, \ldots, F_T$ of functions on $Z \times \mathbb{R}$ into $\overline{\mathbb{R}}$ that will eventually be shown to coincide with the value functions $V_0, \ldots, V_T$. We first define $F_{T+1} \equiv 0$. For $t \le T$, we let F_t be given by the Bellman equation

$$F_t(i,w) = \sup_{\left(\overline{\theta},\overline{c}\right) \in \mathbb{R}^N \times \mathbb{R}_+} G_{it}\left(\overline{\theta},\overline{c}\right) \qquad \text{subject to } \overline{c} + \overline{\theta} \cdot \mathcal{S}_t(i) \le w + g_t(i), \tag{12}$$

where

$$G_{it}(\overline{\theta},\overline{c}) = u_t(\overline{c}) + E\left[F_{t+1}\left(X_{t+1}, \overline{\theta} \cdot [\mathcal{S}_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1})]\right) \,\middle|\, X_t = i\right].$$

The following technical conditions extend those of Lemma 3B, and have essentially the same proof.

Proposition. *For any i in Z and $t \le T$, the function $F_t(i, \cdot\,) : \mathbb{R} \to \overline{\mathbb{R}}$, restricted to its domain of finiteness $\{w : F_t(i,w) > -\infty\}$, is strictly concave and increasing. If $\left(\overline{c}, \overline{\theta}\right)$ solves (12) and $\overline{c} > 0$, then $F_t(i, \cdot\,)$ is continuously differentiable at w with derivative $F_{tw}(i,w) = u_t'(\overline{c})$.*

It can be shown as an exercise that unless the constraint of (12) is infeasible, a solution to (12) always exists. In this case, for any i, t, and w, let $[\Phi_t(i,w), C_t(i,w)]$ denote a solution. We can then define the associated wealth process W^* recursively, for any initial condition w, by $W_0^* = w$ and

$$W_t^* = \Phi_{t-1}(X_{t-1}, W_{t-1}^*) \cdot [\mathcal{S}_t(X_t) + f_t(X_t)], \qquad t \ge 1.$$

Let (c^*, θ^*) be defined, at each t, by $c_t^* = C_t(X_t, W_t^*)$ and $\theta_t^* = \Phi_t(X_t, W_t^*)$. The fact that (c^*, θ^*) solves (10) for $t = 0$ can be shown as follows: Let (c, θ) be an arbitrary feasible policy. For each t, from the Bellman equation (12),

$$F_t(X_t, W_t^\theta) \ge u_t(c_t) + E\left[F_{t+1}\left(X_{t+1}, \theta_t \cdot [\mathcal{S}_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1})]\right) \,\middle|\, X_t\right].$$

Rearranging this inequality and applying the law of iterated expectations,

$$E[F_t(X_t, W_t^\theta)] - E\left[F_{t+1}\left(X_{t+1}, W_{t+1}^\theta\right)\right] \ge E[u_t(c_t)]. \tag{13}$$

Adding equation (13) from $t = 0$ to $t = T$ shows that $F_0(X_0, W_0) \geq U(c)$. Repeating the same calculations for the special policy $(c, \theta) = (c^*, \theta^*)$ allows us to replace the inequality in (13) with an equality, leaving $F_0(X_0, W_0) = U(c^*)$. This shows that $U(c^*) \geq U(c)$ for any feasible (θ, c), meaning that (θ^*, c^*) indeed solves equation (10) for $t = 0$. An optimal policy can thus be captured in *feedback-policy* form in terms of the functions C_t and Φ_t, $t \leq T$. We also see that for all $t \leq T$, $F_t = V_t$, so V_t inherits the properties of F given by the last proposition.

We can now recover the stochastic Euler equation (9) directly from the first-order conditions to (12), rather than from the more general first-order conditions developed in Chapter 2 based on the gradient of U.

Theorem. *Suppose c^* is a strictly positive consumption process and θ^* is a trading strategy θ^* such that*

$$c_t^* = W_t^{\theta^*} + e_t - \theta_t^* \cdot S_t.$$

Then (c^*, θ) solves (10) for $t = 0$ if and only if, for all $t < T$,

$$\mathcal{S}_t(X_t) = \frac{1}{u_t'(c_t^*)} E\left[u_{t+1}'\left(c_{t+1}^*\right)\left[\mathcal{S}_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1})\right] |X_t\right]. \qquad (14)$$

The theorem follows from the necessity and sufficiency of the first-order conditions for (12), relying on the last proposition for the fact that $F_{t+1,w}(X_{t+1}, W_{t+1}^*) = u_{t+1}'(c_{t+1}^*)$.

In a single-agent model, we define a sequence $\{\mathcal{S}_0, \ldots, \mathcal{S}_T\}$ of security-price functions to be a *single-agent equilibrium* if $(e, 0)$ (no trade) solves (10) for $t = 0$, $w = 0$, and any initial shock i.

Corollary. *$\{\mathcal{S}_0, \ldots, \mathcal{S}_T\}$ is a single-agent equilibrium if and only if $\mathcal{S}_T = 0$ and, for all $t < T$, the stochastic Euler equation (9) is satisfied taking $c^* = e$.*

3F Arbitrage-Free Valuation in a Markov Setting

Taking the setting of Markov uncertainty described in Section 3C, but assuming no particular optimality properties or equilibrium, suppose that security prices and dividends are given, at each t, by functions $\mathcal{S}_t$ and f_t on Z into

$\mathbb{R}^N$. It can be shown as an exercise that the absence of arbitrage is equivalent to the existence of a state-price deflator π given by $\pi_t = \psi_t(X_t)$ for some $\psi_t : Z \to (0, \infty)$. With this, we have, for $0 < t \leq T$,

$$\mathcal{S}_{t-1}(X_{t-1}) = \frac{1}{\psi_{t-1}(X_{t-1})} E\left(\psi_t(X_t)\left[f_t(X_t) + \mathcal{S}_t(X_t)\right] | X_{t-1}\right). \tag{15}$$

In the special setting of Section 3E, for example, (9) tells us that we can take $\psi_t(i) = u_t'[g(i)]$.

Since $Z = \{1, \ldots, k\}$ for some integer k, we can abuse the notation by treating any function such as $\psi_t : Z \to \mathbb{R}$ interchangeably as a vector in $\mathbb{R}^k$ denoted ψ_t, with i-th element $\psi_t(i)$. Likewise, $\mathcal{S}_t$ can be treated as a $k \times N$ matrix, and so on. In this sense, (15) can also be written

$$\mathcal{S}_{t-1} = \Pi_{t-1}(f_t + \mathcal{S}_t), \tag{16}$$

where Π_{t-1} is the $k \times k$ matrix with (i, j)-element $q_{ij}\psi_t(j)/\psi_{t-1}(i)$. For each t and $s > t$, we let $\Pi_{t,s} = \Pi_t \Pi_{t+1} \cdots \Pi_{s-1}$. Then (16) is equivalent to, for any t and $\tau > t$,

$$\mathcal{S}_t = \Pi_{t,\tau}\mathcal{S}_\tau + \sum_{s=t+1}^{\tau} \Pi_{t,s} f_s. \tag{17}$$

As an example, consider the "binomial" model of Exercise 2.1. We can let $Z = \{0, 1, \ldots, T\}$, with shock i having the interpretation: "There have so far occurred i 'up' returns on the stock." From the calculations in Exercise 2.1, it is apparent that for any t, we may choose $\Pi_t = \Pi$, where

$$\begin{aligned} \Pi_{ij} &= \frac{p}{R}, \qquad j = i + 1, \\ &= \frac{1-p}{R}, \qquad j = i, \\ &= 0, \qquad \text{otherwise}, \end{aligned}$$

where $p = (R - D)/(U - D)$, for constant coefficients R, U, and D, with $0 < D < R < U$. For a given initial stock price x and any $i \in Z$, the stock-price process S of Exercise 2.1 can indeed be represented at each time t by $\mathcal{S}_t : Z \to \mathbb{R}$, where $\mathcal{S}_t(i) = xU^i D^{t-i}$.

We can recover the "binomial" option-pricing formula (2.16) by noting that the European call option with strike price K and expiration time τ may be treated as a security with dividends only at time τ given by the function

$g : Z \to \mathbb{R}$, with $g(i) = [\mathcal{S}_\tau(i) - K]^+$. From (17), the arbitrage-free value of the option at time t is $C_t^\tau = \Pi^{\tau - t} g$, where Π^t denotes the t-th power of Π. This same valuation formula applies to an arbitrary security paying a dividend at time τ defined by some payoff function $g : Z \to \mathbb{R}$.

3G Early Exercise and Optimal Stopping

In the setting of Section 3F, consider an "American" security, defined by some payoff functions $g_t : Z \to \mathbb{R}_+$, $t \in \{0, \ldots, T\}$. As explained in Section 2I, the security is a claim to the dividend $g_\tau(X_\tau)$ at any stopping time τ selected by the owner. Expiration of the security at some time $\overline{\tau}$ is handled by defining g_t to be zero for $t > \overline{\tau}$. Given the state-price deflator π defined by $\pi_t = \psi_t(X_t)$, as outlined in the previous section, the rational exercise problem (2.13) for the American security, with initial shock i, is given by

$$J_0(i) \equiv \max_{\tau \in \mathcal{T}} \frac{1}{\psi_0(i)} E^i \left[\psi_\tau(X_\tau) g(X_\tau) \right], \tag{18}$$

where $\mathcal{T}$ is the set of stopping times bounded by T. As explained in Section 2I, if the American security is redundant and there is no arbitrage, then $J_0(i)$ is its cum-dividend value at time 0 with initial shock i. The Bellman equation for (18) is

$$J_t(X_t) \equiv \max \left(g_t(X_t), \; \frac{1}{\psi_t(X_t)} E \left[\psi_{t+1}(X_{t+1}) J_{t+1}(X_{t+1}) \,|\, X_t \right] \right). \tag{19}$$

It is left as an exercise to show that J_0 is indeed determined inductively, backward in time from T, by (19) and $J_T = g_T$. Moreover, as demonstrated in Section 2I, problem (18) is solved by the stopping time

$$\tau^* = \min \left[t : J_t(X_t) = g_t(X_t) \right]. \tag{20}$$

In our alternate notation that treats J_t as a vector in $\mathbb{R}^k$, we can rewrite the Bellman equation (19) in the form

$$J_t = \max \left(g_t, \Pi_t J_{t+1} \right), \tag{21}$$

where, for any x and y in $\mathbb{R}^k$, $\max(x, y)$ denotes the vector in $\mathbb{R}^k$ that has $\max(x_i, y_i)$ as its i-th element.

The Bellman equation (21) leads to a simple recursive solution algorithm for the American put valuation problem of Exercise 2.1. Given an expiration time $\overline{\tau} < T$ and exercise price K, we have $J_{\tau+1} = 0$ and

$$J_t = \max\left[(K - \mathcal{S}_t)^+,\ \Pi J_{t+1}\right], \qquad t \leq \tau. \tag{22}$$

More explicitly: For any t and $i \leq t$,

$$J_t(i) = \max\left([K - \mathcal{S}_t(i)]^+,\ \frac{pJ_{t+1}(i+1) + (1-p)J_{t+1}(i)}{R}\right), \tag{23}$$

where $\mathcal{S}_t(i) = xU^i D^{t-i}$ and $p = (R-D)/(U-D)$, for constant coefficients R, U, and D, with $0 < D < R < U$.

More generally, consider an *American security* defined by dividend functions $h_0, \ldots, h_T$ and exercise payoff functions $g_0, \ldots, g_T$. For a given expiration time $\overline{\tau}$, we have $h_t = g_t = 0$, $t > \overline{\tau}$. The owner of the security chooses a stopping time τ at which to exercise, generating the dividend process δ^τ defined by

$$\begin{aligned} \delta_t^\tau &= h_t(X_t), \qquad t < \tau, \\ &= g_t(X_t), \qquad t = \tau, \\ &= 0, \qquad t > \tau. \end{aligned}$$

Assuming that δ^τ is redundant for any exercise policy τ, the security's arbitrage-free cum-dividend value is defined recursively by $J_{\tau+1} = 0$ and the extension of (21):

$$J_t = \max\left(g_t,\ h_t + \Pi_t J_{t+1}\right). \tag{24}$$

Exercises

Exercise 3.1 Prove the Bellman equation (4).

Exercise 3.2 For each t and each w such that there exists a feasible policy, let $C_t(w)$ solve equation (4). Let W^* be determined by equation (1) with $c_{t-1} = C_{t-1}(W^*_{t-1})$ for $t > 0$. Show that an optimal policy c^* is given by $c^*_t = C_t(W^*_t)$, $t < T$, and $c^*_T = e_T + W^*_T$.

Exercise 3.3 Prove Lemma 3B. Hint: If $f : \mathbb{R} \to \mathbb{R}$ is concave, then for each x there is a number β such that $\beta(x-y) \leq f(x) - f(y)$ for all y. If f is also differentiable at x, then $\beta = f'(x)$. If f is differentiable and strictly concave, then f is continuously differentiable. Let $w^* = W_t^*$. If $c_t^* > 0$, there is an interval $I = (\underline{w}, \overline{w}) \subset \mathbb{R}$ with $w^* \in I$ such that $v : I \to \mathbb{R}$ is well defined by

$$v(w) = u_t(c_t^* + w - w^*) + V_{t+1}(W_{t+1}^*).$$

Now use the differentiability of v, the definition of a derivative, and the fact that $v(w) \leq V_t(w)$ for all $w \in I$.

Exercise 3.4 Prove equation (9).

Exercise 3.5 Prove Proposition 3E.

Exercise 3.6 Prove Theorem 3E and its corollary.

Exercise 3.7 Consider the case of securities in positive supply, which can be taken without loss of generality to be a supply of 1 each. Equilibrium in the context of Section 3E is thus redefined by the following: $\{\mathcal{S}_0, \ldots, \mathcal{S}_T\}$ is an equilibrium if (c^*, θ^*) solves (10) at $t = 0$ and $w = \mathbf{1} \cdot [\mathcal{S}_0(X_0) + f_0(X_0)]$, where $\mathbf{1} = (1, \ldots, 1)$ and, for all t, $\theta_t^* = \mathbf{1}$, and $c_t^* = g_t(X_t) + \mathbf{1} \cdot f_t(X_t)$. Demonstrate a new version of the stochastic Euler equation (9) that characterizes equilibrium in this case.

Exercise 3.8 (Recursive Utility Revisited) The objective in this exercise is to extend the basic results of the chapter to the case of a recursive-utility function that generalizes additive utility. Rather than assuming a typical additive-utility function U of the form

$$U(c) = E\left[\sum_{t=0}^{T} \rho^t u(c_t)\right], \tag{25}$$

we adopt instead the more general recursive definition of utility given by $U(c) = Y_0$, where Y is a process defined by $Y_{T+1} = 0$ and, for any $t \leq T$,

$$Y_t = J\left(c_t, E_t[h(Y_{t+1})]\right), \tag{26}$$

where $J : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. This is the special case treated in Exercise 2.9 of what is known as recursive utility. (In an even more general recursive-utility model, the von Neumann-Morgenstern criterion $E[h(\,\cdot\,)]$ is replaced by a general functional on distributions, but we do not deal with this further generalization.) Note that the special case $J(q, w) = u(q) + \rho w$ and $h(y) = y$ gives us the additively separable criterion (25). The conventional additive utility has the disadvantage that the elasticity of intertemporal substitution (as measured in a deterministic setting) and relative risk aversion are fixed in terms of one another. The recursive criterion, however, allows one to examine the effects of varying risk aversion while holding fixed the utility's elasticity of intertemporal substitution in a deterministic setting.

(A) (Dynamic Programming) Provide an extension of the Bellman equation (12) for optimal portfolio and consumption choice, substituting the recursive utility for the additive utility. That is, state a revised Bellman equation and regularity conditions on the utility primitives (J, h) under which a solution to the Bellman equation implies that the associated feedback policies solving the Bellman equation generate optimal consumption and portfolio choice. (State a theorem with proof.) Also, include conditions under which there exists a solution to the Bellman equation. For simplicity, among your conditions you may wish to impose the assumptions that J and h are continuous and strictly increasing.

(B) (Asset Pricing Theory) Suppose that J and h are differentiable, increasing, and concave, with either h or J (or both) strictly concave. Provide any additional regularity conditions that you feel are called for in order to derive an analogue to the stochastic Euler equation (9) for security prices.

(C) (An Investment Problem) Let $G : Z \times \mathbb{R}_+ \to \mathbb{R}_+$ and consider the capital-stock investment problem defined by

$$\sup_{c \in L_+} U(c) \tag{27}$$

subject to $0 \leq c_t \leq K_t$ for all t, where $K_0, K_1, \ldots,$ is a capital-stock process defined by $K_t = G(X_t, K_{t-1} - c_{t-1})$, and where $X_0, \ldots, X_T$ is the Markov process defined in Section 3C. The utility function U is the recursive function defined above in terms of (J, h). Provide reasonable conditions on (J, h, G) under which there exists a solution. State the Bellman equation.

(D) (Parametric Example) For this part, in order to obtain closed-form solutions, we depart from the assumption that the shock takes only a finite number of possible values, and replace this with a normality assumption. Solve the problem of part (C) in the following case:

(a) X is the real-valued shock process defined by $X_{t+1} = A + BX_t + \epsilon_{t+1}$, where A and B are scalars and $\epsilon_1, \epsilon_2, \ldots$ is an *i.i.d.* sequence of normally distributed random variables with $E(\epsilon_t) = 0$ and $\text{var}(\epsilon_t) = \sigma^2$.

(b) $G(x, a) = a^\gamma e^x$ for some $\gamma \in (0, 1)$.

(c) $J(q, w) = \log(q) + \rho \log(w^{1/\alpha})$ for some $\alpha \in (0, 1)$.

(d) $h(v) = e^{\alpha v}$ for $v \geq 0$.

Hint: You may wish to conjecture a solution to the value function of the form $V_t(x, k) = A_1(t) \log(k) + A_2(t)x + A_3(t)$, for time-dependent coefficients A_1, A_2, and A_3. This example is unlikely to satisfy the regularity conditions that you imposed in part (C).

(E) (Term Structure) For the consumption endowment process e defined by the solution to part (D), return to the setting of part (B), and calculate the price $\Lambda_{t,s}$ at time t of a pure discount bond paying one unit of consumption at time $s > t$. Note that α is a measure of risk tolerance that can be studied independently of the effects of intertemporal substitution in this model, since, for deterministic consumption processes, utility is independent of α, with $J[q, h(v)] = \log(q) + \rho \log(v)$.

Exercise 3.9 Show equation (5) directly from equation (2.9).

Exercise 3.10 Consider, as in the setup described in Section 3F, securities defined by the dividend-price pair (δ, S), where, for all t, there are functions f_t and $\mathcal{S}_t$ on Z into $\mathbb{R}^N$ such that $\delta_t = f_t(X_t)$ and $S_t = \mathcal{S}_t(X_t)$. Show that there is no arbitrage if and only if there is a state-price deflator π such that, for each time t, $\pi_t = \psi_t(X_t)$ for some function $\psi_t : Z \to (0, \infty)$.

Exercise 3.11 (Binomial Term-Structure Algorithms) This exercise asks for a series of numerical solutions of term-structure valuation problems in a setting with binomial changes in short-term interest rates. In the setting of Section 3F, under the absence of arbitrage, suppose that short-term riskless borrowing is possible at any time t at the discount d_t. The one-period interest rate at time t is denoted r_t, and is given by its definition:

$$d_t = \frac{1}{1 + r_t}.$$

The underlying shock process X has the property that either $X_t = X_{t-1} + 1$ or $X_t = X_{t-1}$. That is, in each period, the new shock is the old shock plus a 0-1 binomial trial. An example is the binomial stock-option pricing model of Exercise 2.1, which is reconsidered in Section 3F. As opposed to that example, we do not necesarily assume here that interest rates are constant. Rather, we allow, at each time t, a function $\rho_t : Z \to \mathbb{R}$ such that $r_t = \rho_t(X_t)$. For simplicity, however, we take it that at any time t the pricing matrix Π_t defined in Section 3F is of the form

$$\begin{aligned}(\Pi_t)_{ij} &= \frac{p}{1+\rho_t(i)}, \qquad j = i+1, \\ &= \frac{1-p}{1+\rho_t(i)}, \qquad j = i, \\ &= 0, \qquad \text{otherwise},\end{aligned}$$

where $p \in (0,1)$ is the "risk-neutral" probability that $X_{t+1} - X_t = 1$. Literally, there is an equivalent martingale measure Q under which, for all t, we have

$$Q(X_{t+1} - X_t = 1 \mid X_0, \ldots, X_t) = p.$$

It may help to imagine the calculation of security prices at the nodes of the "tree" illustrated in Figure 3.1. The horizontal axis indicates the time periods; the vertical axis corresponds to the possible levels of the shock, assuming that $X_0 = 0$. At each time t and at each shock level i, the price of a given security at the (i,t)-node of the tree is given by a weighted sum of its value at the two successor nodes $(i+1, t+1)$ and $(i, t+1)$. Specifically,

$$\mathcal{S}_t(i) = \frac{1}{1+\rho_t(i)} \left[p(\mathcal{S}_{t+1}(i+1) + f_{t+1}(i+1)) + (1-p)(\mathcal{S}_{t+1}(i) + f_{t+1}(i)) \right].$$

Two typical models for the short rate are obtained by taking $p = 1/2$ and either

(a) the *Ho-Lee model*: For each $t < T$, $\rho_t(i) = a_t + b_t i$ for some constants a_t and b_t; or

(b) the *Black-Derman-Toy model*: For each t, $\rho_t(i) = a_t \exp(b_t i)$ for some constants a_t and b_t.

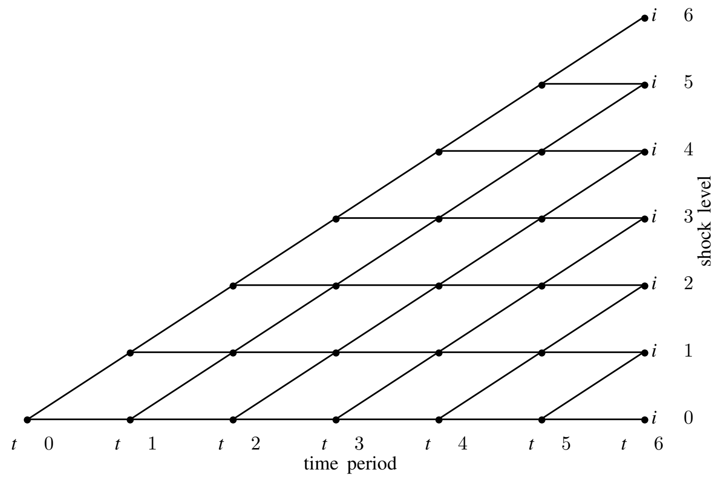

Figure3.1: A Binomial Tree

(A) For case (b), prepare computer code to calculate the arbitrage-free price $\Lambda_{0,t}$ of a zero-coupon bond of any given maturity t, given the coefficients a_t and b_t for each t. Prepare an example taking $b_t = 0.01$ for all t and $a_0, a_1, \ldots, a_T$ such that $E^Q(r_t) = 0.01$ for all t. (These parameters are of a typical order of magnitude for monthly periods.) Solve for the price $\Lambda_{0,t}$ of a unit zero-coupon riskless bond maturing at time t, for all t in $\{1, \ldots, 50\}$.

(B) Consider, for any i and t, the price $\psi(i,t)$ at time 0 of a security that pays one unit of account at time t if and only if $X_t = i$.

Show that ψ can be calculated recursively by the difference equation

$$\begin{aligned}
\psi(i,t+1) &= \frac{\psi(i,t)}{2[1+\rho_t(i)]} + \frac{\psi(i-1,t)}{2[1+\rho_t(i-1)]}, \qquad 0 < i < t+1, \\
&= \frac{\psi(i-1,t)}{2[1+\rho_t(i-1)]}, \qquad i = t+1, \\
&= \frac{\psi(i,t)}{2[1+\rho_t(i)]}, \qquad i = 0.
\end{aligned} \tag{28}$$

The initial condition is $\psi(0,0) = 1$ and $\psi(i,0) = 0$ for $i > 0$. Knowledge of this "shock-price" function ψ is useful. For example, the arbitrage-free price at time 0 of a security that pays the dividend $f(X_t)$ at time t (and nothing otherwise) is given by $\sum_{i=0}^{t} \psi_t(i) f(i)$.

(C) In practice, the coefficients a_t and b_t are often fitted to match the initial term structure $\Lambda_{0,1}, \ldots, \Lambda_{0,T}$, given the "volatility" coefficients $b_0, \ldots, b_T$. The following algorithm has been suggested for this purpose, using the fact that $\Lambda_{0,t} = \sum_{i=0}^{t} \psi_t(i)$.

(a) Let $\psi(0,0) = 1$ and let $t = 1$.

(b) Fixing ψ_{t-1} and b_t, let $\lambda_t(a_{t-1}) = \sum_{i=0}^{t} \psi_t(i)$, where ψ_t is given by the forward difference equation (28). Only the dependence of the t-maturity zero-coupon bond price $\lambda_t(a_{t-1})$ on a_{t-1} is notationally explicit. Since $\lambda_t(a_{t-1})$ is strictly monotone in a_{t-1}, we can solve numerically for that coefficient a_{t-1} such that $\Lambda_{0,t} = \lambda_t(a_{t-1})$. (A Newton-Raphson search will suffice.)

(c) Let t be increased by 1. Return to step (b) if $t \leq T$. Otherwise, stop.

Prepare computer code for this algorithm (a)–(b)–(c). Given $b_t = 0.01$ for all t, solve for a_t for all t, using the Black-Derman-Toy model, given an initial term structure that is given by $\Lambda_{0,t} = \alpha^t$, where $\alpha = 0.99$.

(D) Extend your code as necessary to give the price of American call options on coupon bonds of any given maturity. For the coefficients $a_0, \ldots, a_{T-1}$ that you determined from part (C), calculate the initial price of an American option on a bond that pays coupons of 0.011 each period until its maturity at time 20, at which time it pays 1 unit of account in addition to its coupon. The option has an exercise price of 1.00, ex dividend, and expiration at time 10. Do this for the Black-Derman-Toy model only.

Notes

Bellman's principle of optimality is due to Bellman [1957]. Freedman [1983] covers the theory of Markov chains. For general treatments of dynamic programming in a discrete-time Markov setting, see Bertsekas [1976] and Bertsekas and Shreve [1978]. The proof for Lemma 3B that is sketched in Exercise 3.3, on the differentiability of the value function, is from Benveniste and Scheinkman [1979], and easily extends to general state spaces; see, for example, Duffie [1988b] and Stokey and Lucas [1989]. The *semi-group* pricing approach implicit in equation (17) is from Duffie and Garman [1991]. Exercise 3.8, treating asset pricing with the recursive utility of Exercise 2.9, is extended to the infinite-horizon setting of Epstein and Zin [1989] in Exercise 4.12. See the Notes of Chapter 2 for additional references on recursive utility and Streufert [1991a], Streufert [1991b] and Streufert [1991c] for more on dynamic programming with a recursive-utility function. For additional work on recursive utility and asset pricing in a discrete-time Markovian setting, see Kan [1995], Ma [991a] and Ma [1994].

The extensive exercise on binomial term-structure models is based almost entirely on Jamshidian [1991], who emphasizes the connection between the solution ψ of the difference equation (28) and state pricing of contingent claims. This connection is reconsidered in Chapters 7 and 11 for continuous-time applications. The two particular term-structure models appearing in this exercise are based, respectively, on Ho and Lee [1986] and Black, Derman, and Toy [1990]. The parametric form shown here for the Ho-Lee model is slightly more general than the form actually appearing in Ho and Lee (1986). Most authors take the convention that X_{t+1} is X_t+1 or X_t-1, which generates a slightly different form for the same model. The two forms are equivalent after a change of the parameters. Pye [1966], whose work predates the notion of "risk-neutral valuation" provides a remarkably early precursor to these discrete-time Markovian models of the term structure. Continuous-time versions of these models are considered in Chapter 7. Chapter 11 also deals in more detail with algorithms designed to match the initial term structure. Exercise 11.5 demonstrates convergence, with a decreasing length of time period, of the discrete-time Black-Derman-Toy model to its continuous-time version. Jamshidian [1991] considers a larger class of examples.

Derman and Kani [1994], Dupire [1994], Rubinstein [1994] and Rubinstein [1995] provide various methods for calibrating a "binomial" Markov stock price process, similar to that of Section 3F, to the available prices of options of various strike prices and, in some cases, of various maturities. This is part of a literature devoted to the *smile curve*, which refers to the shape often found for a plot of the "implied volatilities" of option prices derived from the Black-Scholes formula against the exercise prices of the respective options. If the assumptions underlying the Black-Scholes formula are correct, the implied volatility does not depend on the strike price. The smile curve is discussed once again in Chapter 8, under the topic of stochastic volatility. Related literature is cited in the Notes of Chapter 8. A related approach to calibration of the stochastic behavior of the underlying asset price to the prices of options is considered by Shimko [1993].

Bossaerts, Ghysels, and Gouriéroux [1996] analyze arbitrage-based pricing with stochastic volatility. Jouini, Koehl, and Touzi [1995] explore the effects of transactions costs and incomplete markets in this setting.

Chapter 4

The Infinite-Horizon Setting

THIS CHAPTER PRESENTS infinite-period analogues of the results of Chapters 2 and 3. Although it requires additional technicalities and produces few new insights, this setting is often deemed important for reasons of elegance or for serving the large-sample theory of econometrics, which calls for an unbounded number of observations. We start directly with a Markov dynamic programming extension of the finite-horizon results of Chapter 3, and only later consider the implications of no arbitrage or optimality for security prices without using the Markov assumption. Finally, we return to the stationary Markov setting to review briefly the large-sample approach to estimating asset pricing models. Only Sections 4A and 4B are essential; the remainder could be skipped on a first reading.

4A Markov Dynamic Programming

Suppose $X = \{X_0, X_1, X_2, \dots\}$ is a time-homogeneous Markov chain of shocks valued in a finite set $Z = \{1, \dots, k\}$, defined exactly as in Section 3C, with the exception that there is an infinite number of time periods. Given a $k \times k$ nonnegative matrix whose rows sum to 1, sources given in the Notes explain the existence of a probability space $(\Omega, \mathcal{F}, P_i)$, for each initial shock i, satisfying the defining properties $P_i(X_0 = i) = 1$ and

$$P_i(X_{t+1} = j \mid X_0, \dots, X_t) = q_{X(t),j}.$$

As in Chapter 3, $\mathcal{F}_t$ denotes the tribe generated by $\{X_0, \dots, X_t\}$. That is, the source of information is the Markov chain $\{X_t\}$. This is the first appearance in the book of a set Ω of states that need not be finite, but because there is only a finite number of events in $\mathcal{F}_t$ for each t, most of this chapter can be easily understood without referring to Appendix C for a review of general probability spaces.

Let L denote the space of sequences of random variables of the form $c = \{c_0, c_1, c_2, \ldots\}$ such that there is a constant k with the property that for all t, c_t is $\mathcal{F}_t$-measurable with $|c_t| \leq k$. In other words, L is the space of bounded adapted processes. Agents choose a consumption process from the set L_+ of nonnegative processes in L. There are N securities; security n is defined by a dividend process δ^n in L and has a price process S^n in L. A trading strategy is some $\theta = (\theta^1, \ldots, \theta^N) \in \Theta \equiv L^N$. Each strategy θ in Θ generates a dividend process δ^θ in L defined, just as in Chapter 2, by

$$\delta^\theta_t = \theta_{t-1} \cdot (S_t + \delta_t) - \theta_t \cdot S_t, \qquad t \geq 0,$$

with "θ_{-1}" $= 0$ by convention. A given agent has an endowment process e in L_+ and, given a particular initial shock i, a utility function $U^i : L_+ \to \mathbb{R}$. The agent's problem is

$$\sup_{\theta \in \Theta(e)} \quad U^i(e + \delta^\theta),$$

where $\Theta(e) = \{\theta \in \Theta : e + \delta^\theta \geq 0\}$.

In order to develop a time-homogeneous Markov model, we restrict ourselves initially to utility functions, endowments, and security dividends with special time-homogeneous properties. Given an initial shock i, consider the utility function $U^i : L_+ \to \mathbb{R}$ defined by a discount $\rho \in (0,1)$ and a strictly increasing, bounded, concave, and continuous $u : \mathbb{R}_+ \to \mathbb{R}$ according to

$$U^i(c) = E^i \left[\sum_{t=0}^{\infty} \rho^t \, u(c_t) \right], \tag{1}$$

where E^i denotes expectation under the probability measure P_i associated with the initial shock $X_0 = i$. Suppose that $g : Z \to \mathbb{R}_{++}$ and $f : Z \to \mathbb{R}^N_{++}$ are such that for all t, the endowment is $e_t = g(X_t)$ and the dividend vector is $\delta_t = f(X_t)$. Finally, suppose that security prices are given by some fixed $\mathcal{S} : Z \to \mathbb{R}^N_{++}$ so that for all t, $S_t = \mathcal{S}(X_t)$.

We fix a portfolio b in $\mathbb{R}^N_{++}$ and think of $-b$ as a lower bound on short positions. This restriction will later be removed. For now, however, wealth is bounded below by

$$\underline{w} = \min_{i \in Z} \; -b \cdot [\mathcal{S}(i) + f(i)].$$

Let $D = Z \times [\underline{w}, \infty)$. A function $F : D \to \mathbb{R}$ is defined to be in the space denoted $B(D)$ if, for each i in Z, $F(i, \cdot\,) : [\underline{w}, \infty) \to \mathbb{R}$ is bounded, continuous, and concave. We are looking for some V in $B(D)$ as the *value* of the agent's control problem. That is, we want some V in $B(D)$ with

$$V(i, w) = \sup_{(c,\theta) \in L_+ \times \Theta} U^i(c), \tag{2}$$

subject to

$$W_0^\theta = w, \tag{3}$$

$$W_t^\theta = \theta_{t-1} \cdot [\mathcal{S}(X_t) + f(X_t)], \qquad t \geq 1, \tag{4}$$

$$c_t + \theta_t \cdot \mathcal{S}(X_t) \leq W_t^\theta + g(X_t), \qquad t \geq 0, \tag{5}$$

$$\theta_t \geq -b, \qquad t \geq 0. \tag{6}$$

We will solve for the value function V by taking an arbitrary F in $B(D)$ and, from this candidate, construct a new candidate denoted $\mathcal{U}F$ that is described below. Our method will show that if $F = \mathcal{U}F$, then $F = V$. In fact, this approach also leads to an algorithm, called *value iteration*, for calculating V. This algorithm is laid out below. For any F in $B(D)$, let $\mathcal{U}F : D \to \mathbb{R}$ be defined by

$$\mathcal{U}F(i, w) = \sup_{(\overline{\theta}, \overline{c}) \in \mathbb{R}^N \times \mathbb{R}_+} u(\overline{c}) + \rho\, E^i \left[F\left(X_1, \overline{\theta} \cdot [\mathcal{S}(X_1) + f(X_1)] \right) \right], \tag{7}$$

subject to

$$\overline{c} + \overline{\theta} \cdot \mathcal{S}(i) \leq w + g(i), \tag{8}$$

$$\overline{\theta} \geq -b. \tag{9}$$

In other words, $\mathcal{U}F(i, w)$ is the supremum utility that can be achieved at (i, w), assuming that the value function in the next period is F.

Proofs of the next three results are left as exercises.

Fact. *If F is in $B(D)$, then $\mathcal{U}F$ is in $B(D)$.*

For any F and G in $B(D)$, let

$$d(F, G) = \sup\{|F(i, w) - G(i, w)| : (i, w) \in D\},$$

giving a notion of the *distance* between any two such functions. Clearly $F = G$ if and only if $d(F, G) = 0$.

Lemma. *For any F and G in $B(D)$, $d(\mathcal{U}F, \mathcal{U}G) \leq \rho\, d(F, G)$.*

Using this lemma, we can construct the unique solution F to the equation $\mathcal{U}F = F$, which is known as the *Bellman equation* for problem (2)–(6). The solution F is called the *fixed point* of $\mathcal{U}$ and, as shown by the following result, can be constructed as the limit of the finite-horizon versions of the value functions as the horizon goes to infinity.

Proposition. *Let $F_{-1}(i, w) = 0$ for all (i, w) in D, and let $F_t = \mathcal{U}F_{t-1}$, $t \geq 0$. Then $F(i, w) \equiv \lim_{t\to\infty} F_t(i, w)$ exists for all (i, w) in D and defines the unique function F in $B(D)$ satisfying $F = \mathcal{U}F$.*

We take F to be the unique fixed point of $\mathcal{U}$. We will show that F is the value function V of problem (2)–(6). Let $C : D \to \mathbb{R}_+$ and $\Phi : D \to \mathbb{R}^N$ be functions defined by letting $[C(i, w), \Phi(i, w)]$ solve (7)–(9). Given the initial conditions (i, w) of (2)–(6), let W^* be defined by $W_0^* = w$ and $W_t^* = \Phi(X_{t-1}, W_{t-1}^*) \cdot [\mathcal{S}(X_t) + f(X_t)]$, $t \geq 1$. Then let (c^*, θ^*) be defined by $c_t^* = C(X_t, W_t^*)$ and $\theta_t^* = \Phi(X_t, W_t^*)$, $t \geq 0$. We refer to any such (θ^*, c^*) as an *optimal feedback control.* The functions Φ and C are the associated feedback policy functions.

Theorem. *The value function V of (2)–(6) is the unique fixed point of $\mathcal{U}$. Any optimal feedback control (c^*, θ^*) solves (2)–(6).*

Proof: Let F be the unique solution of the Bellman equation $\mathcal{U}F = F$. Fix any initial shock i in Z and initial wealth w in $[\underline{w}, \infty)$. Let (θ, c) be an arbitrary feasible control. For any time t, by the Bellman equation (7)–(9),

$$F(X_t, W_t^\theta) \geq u(c_t) + \rho E^i \left[F(X_{t+1}, W_{t+1}^\theta) \mid X_t \right].$$

Multiplying through by ρ^t and rearranging,

$$\rho^t F(X_t, W_t^\theta) - \rho^{t+1} E^i \left[F(X_{t+1}, W_{t+1}^\theta) \mid X_t \right] \geq \rho^t u(c_t). \tag{10}$$

Taking expectations on each side, and using the law of iterated expectations,

$$E^i \left[\rho^t F(X_t, W_t^\theta) \right] - \rho^{t+1} E^i \left[F(X_{t+1}, W_{t+1}^\theta) \right] \geq E^i[\rho^t u(c_t)].$$

Calculating the sum of this expression from $t = 0$ to $t = T$, for any time $T \geq 1$, causes telescopic cancellation on the left-hand side, leaving only

$$E^i \left[F(X_0, W_0^\theta) \right] - \rho^{T+1} E^i \left[F(X_{T+1}, W_{T+1}^\theta) \right] \geq E^i \left[\sum_{t=0}^{T} \rho^t u(c_t) \right].$$

Since F is a bounded function and $\rho \in (0,1)$, the limit of the left-hand side as $T \to \infty$ is $F(i,w)$. By the Dominated Convergence Theorem (Appendix C), the limit of the right-hand side is $U^i(c)$. Thus $F(i,w) \geq U^i(c)$, implying that $F(i,w) \geq V(i,w)$ for any i and w. All of the above calculations apply for the given optimal feedback control (c^*, θ^*), for which we can replace the inequality in (10) with an equality, using the definition of C and Φ. This leaves $F(i,w) = U^i(c^*)$. It follows, since (i,w) is arbitrary, that V is indeed the value function and that (c^*, θ^*) is optimal, in that it solves (2)-(6), proving the result. ∎

4B Dynamic Programming and Equilibrium

Section 4A shows the existence of optimal control in feedback form, given by *policy functions* C and Φ that specify optimal consumption and portfolio choices in terms of the current shock–wealth pair (i,w). In order to characterize an equilibrium by the same approach, we adopt stronger utility conditions for this section. In addition to our standing assumption that u is strictly increasing, bounded, concave, and continuous, we add the following regularity condition.

Assumption A. *The function u is strictly concave and differentiable on $(0,\infty)$.*

We define $\mathcal{S}$ to be a single-agent *Markov equilibrium* if associated optimal feedback policy functions C and Φ can be chosen so that for any shock i, $C(i,0) = g(i)$ and $\Phi(i,0) = 0$. With this, the consumption and security markets always clear if the agent is originally endowed with no wealth beyond that of his or her private endowment. The short sales restriction on portfolios is superfluous in equilibrium since this short sales constraint is not binding at the solution $(e,0)$, and since the equilibrium shown (which is the unique equilibrium) does not depend on the particular lower bound $-b$ chosen. (It is assigned as an exercise to verify this fact.) Our main objective is to demonstrate the following characterization of equilibrium.

Proposition. *$\mathcal{S}$ is a Markov equilibrium if and only if, for all i,*

$$\mathcal{S}(i) = \frac{1}{u'[g(i)]} E^i \left(\sum_{t=1}^{\infty} \rho^t \, u'[g(X_t)] \, f(X_t) \right). \tag{11}$$

The law of iterated expectations implies the following equivalent form of (11), sometimes called the stochastic Euler equation.

Corollary. *$\mathcal{S}$ is a Markov equilibrium if and only if, for any time t and any initial shock i,*

$$\mathcal{S}(X_t) = \frac{1}{u'[g(X_t)]} E^i \left(\rho u'[g(X_{t+1})] \left[\mathcal{S}(X_{t+1}) + f(X_{t+1}) \right] \,\middle|\, X_t \right). \tag{12}$$

We will demonstrate these results by exploiting the properties of the value function V.

Fact 1. *For each i, $V(i,\,\cdot\,) : [\underline{w}, \infty) \to \mathbb{R}$ is increasing and strictly concave.*

Fact 2. *Fixing $\mathcal{S}$ arbitrarily, let (C, Φ) be optimal feedback policy functions, as above. Suppose, at a given i and $\hat{w} > \underline{w}$, that $\hat{c} = C(i, \hat{w}) > 0$. Then $V(i,\,\cdot\,)$ is continuously differentiable at $\hat{w}$ with derivative $V_w(i, \hat{w}) = u'(\hat{c})$.*

These two facts, proved in a manner similar to their analogues in Chapter 3, imply, from the first-order conditions of the Bellman equation (7) and the fact that V solves the Bellman equation, that C and Φ can be chosen with $C(i, 0) = g(i)$ and $\Phi(i, 0) = 0$ for all i if and only if

$$\mathcal{S}(i) = \frac{1}{u'[g(i)]}\, E^i \left(\rho\, u'[g(X_1)] \left[\mathcal{S}(X_1) + f(X_1) \right] \right), \qquad i \in Z. \tag{13}$$

Then (13) is equivalent to (11) and (12), proving the Proposition and Corollary.

4C Arbitrage and State Prices

We turn away from the special case of Markov uncertainty in order to investigate the implications of lack of arbitrage and of optimality for security prices in an abstract infinite-horizon setting. Suppose Ω is a set, $\mathcal{F}$ is a tribe on Ω, and, for each nonnegative integer t, $\mathcal{F}_t$ is a finite sub-tribe with $\mathcal{F}_t \subset \mathcal{F}_s$ for $s \geq t$. We also fix a probability measure P on $(\Omega, \mathcal{F})$. As usual, we assume that $\mathcal{F}_0$ includes only events of probability 0 or 1. We again denote by L the space of bounded adapted processes. There are N securities; security n is defined by a dividend process δ^n in L and has a price process S^n in L. A trading strategy is some $\theta = (\theta^1, \ldots, \theta^N) \in \Theta \equiv L^N$.

An *arbitrage* is a trading strategy θ with $\delta^\theta > 0$. If there is no arbitrage, then for any T, there is no *T-period arbitrage*, meaning an arbitrage θ with $\theta_t = 0$, $t \geq T$. Fixing T momentarily, if there is no T-period arbitrage, then the results of Chapter 2 imply that there is a *T-period state-price deflator*, a strictly positive process π^T in L with $\pi_0^T = 1$ such that for any trading strategy θ with $\theta_t = 0$, $t \geq T$, we have $E(\sum_{t=0}^T \pi_t^T \, \delta_t^\theta) = 0$. Likewise, there is a $(T+1)$-period state-price deflator π^{T+1}. It can be checked that the process $\hat{\pi}$ defined by $\hat{\pi}_t = \pi_t^T$, $t \leq T$, and $\hat{\pi}_t = \pi_t^{T+1}$, $t > T$, is also a $(T+1)$-period state-price deflator. By induction in T, this means that there is a strictly positive adapted process π such that for any trading strategy θ with $\theta_t = 0$ for all t larger than some T, we have $E(\sum_{t=0}^\infty \pi_t \, \delta_t^\theta) = 0$. In particular, π has the property that for any times t and $\tau \geq t$, we have the now-familiar state-pricing relationship

$$S_t = \frac{1}{\pi_t} E_t \left(\pi_\tau \, S_\tau + \sum_{j=t+1}^{\tau} \pi_j \, \delta_j \right). \tag{14}$$

Equation (14) even holds when τ is a bounded stopping time. Unfortunately, there is no reason (yet) to believe that there is a state-price deflator, a strictly positive adapted process π such that (14) holds for τ an unbounded stopping time, or that for any t,

$$S_t = \frac{1}{\pi_t} E_t \left(\sum_{j=t+1}^{\infty} \pi_j \, \delta_j \right). \tag{15}$$

Indeed, the right-hand side of (15) may not even be well defined. We need some restriction on π!

We call an adapted process x *mean-summable* if $E\left(\sum_{t=0}^\infty |x_t|\right) < \infty$, and let L^* denote the space of mean-summable processes. If $\pi \in L^*$ and $c \in L$, then the Dominated Convergence Theorem (Appendix C) implies that $E\left(\sum_{t=0}^\infty \pi_t \, c_t\right)$ is well defined and finite, so L^* may be a natural space of candidate state-price deflators if (15) is to work.

4D Optimality and State Prices

An agent is defined by an endowment process e in the space L_+ of nonnegative processes in L, and by a strictly increasing utility function $U : L_+ \to \mathbb{R}$.

Given the dividend-price pair $(\delta,\, S) \in L^N \times L^N$, the agent faces the problem

$$\sup_{\theta} U(e + \delta^{\theta}). \tag{16}$$

We say that the utility function U is *L^*-smooth* at c if the gradient $\nabla U(c)$ exists and moreover has a unique Riesz representation π in L^* defined by

$$\nabla U(c; x) = E\left(\sum_{t=0}^{\infty} \pi_t\, x_t\right),$$

for any feasible direction x in L. (See Appendix B for the definition of the gradient and feasible directions.) For example, suppose that U is defined by $U(c) = E\left[\sum_{t=0}^{\infty} \rho^t u(c_t)\right]$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and continuously differentiable on $(0, \infty)$, and where $\rho \in (0, 1)$. Then, for any c in L_+ that is bounded away from zero, U is L^*-smooth at c, any x in L is a feasible direction at c, and

$$\nabla U(c; x) = E\left[\sum_{t=0}^{\infty} \rho^t\, u'(c_t)\, x_t\right], \qquad x \in L, \tag{17}$$

implying that the Riesz representation of the utility gradient is in this case the process π defined by $\pi_t = \rho^t u_t'(c_t)$.

More generally, we have the following characterization of state-price deflators.

Proposition. *Suppose c^* solves (16), c^* is bounded away from zero, and U is L^*-smooth at c^*. Then the Riesz representation π of $\nabla U(c^*)$ is a state-price deflator.*

Corollary. *Suppose, moreover, that U is defined by*

$$U(c) = E\left[\sum_{t=0}^{\infty} \rho^t u(c_t)\right],$$

where $\rho \in (0, 1)$ and u has a strictly positive derivative on $(0, \infty)$. Then π, defined by $\pi_t = \rho^t\, u'(c_t^*)$, is a state-price deflator and, for any time t and stopping time $\tau > t$,

$$S_t = \frac{1}{u'(c_t^*)}\, E_t\left[\rho^{\tau - t} u'(c_\tau^*)\, S_\tau + \sum_{j=t+1}^{\tau} \rho^{j-t}\, u'(c_j^*) \delta_j\right].$$

This corollary gives a necessary condition for optimality that, when specialized to the case of equilibrium, recovers the stochastic Euler equation (12) as a necessary condition on equilibrium without relying on Markov uncertainty or dynamic programming. For sufficiency, we should give conditions under which the stochastic Euler equation implies that S is an equilibrium. For this, we define S to be a single-agent equilibrium if $\theta^* = 0$ solves (16) given S.

Theorem. *Suppose that U is strictly increasing, concave, and L^*-smooth at the endowment process e. Suppose that the endowment process e is bounded away from zero. Let $\pi \in L^*$ be the Riesz representation of $\nabla U(e)$. It is necessary and sufficient for S to be a single-agent equilibrium that π is a state-price deflator.*

The assumption that e is bounded away from zero is automatically satisfied in the Markovian example of Section 4A. Proof of the theorem is assigned as an exercise.

4E Method-of-Moments Estimation

Although it is not our main purpose to delve into econometrics, it seems worthwhile to illustrate here why the infinite-horizon setting is deemed useful for empirical modeling.

Suppose, for some integer $m \geq 1$, that $B \subset \mathbb{R}^m$ is a set of parameters. Each b in B corresponds to a different Markov economy with the same state space Z. In particular, the transition matrix $q(b)$ of the Markov process X may vary with b. For instance, we could take a single agent with utility given by a discount factor $\rho \in (0,1)$ and a reward function $u_\alpha(x) = x^\alpha/\alpha$ for $\alpha < 1$ (with $u_0(x) = \log x$). We could then take $m = 2$ and $b = (\rho, \alpha) \in B = (0, \infty) \times (-\infty, 1)$. In this example, the transition matrix $q(b)$ does not depend on b.

We fix some b_0 in B, to be thought of as the "true" parameter vector governing the economy. Our goal is to estimate the unknown parameter vector b_0.

For simplicity, we will assume that the transition matrix $q(b_0)$ of X is strictly positive. With this, a result known as the *Frobenius-Perron Theorem* implies that there is a unique vector $p \in \mathbb{R}^k_{++}$ whose elements sum to 1 with the property that $q(b_0)^\top p = p$. Letting $q(b_0)^t$ denote the t-fold product of $q(b_0)$, we see that $P_i(X_t = j) = q(b_0)^t_{ij}$, so that $q(b_0)^t$ is the t-period transition matrix. It can be shown that p is given by any row of $\lim_t q(b_0)^t$. Thus, regardless of the initial shock i, $\lim_t P_i(X_t = j) = p_j$. Indeed, the convergence to the "steady-state" probability vector p is exponentially fast, in the sense that there is a constant $\beta > 1$ such that for any i and j,

$$\beta^t [p_j - P_i(X_t = j)] \to 0. \tag{18}$$

From this, it follows immediately that for any $H : Z \to \mathbb{R}$ and any initial condition $j \in Z$, we have $E^j[H(X_t)] \to \sum_{i=1}^k p_i H(i)$, and again convergence is exponentially fast. The *empirical distribution vector* p_T of X at time T is defined by

$$p_{Ti} = \frac{1}{T} \#\{t < T : X_t = i\},$$

where $\#A$ denotes the number of elements in a finite set A. That is, p_{Ti} is the average fraction of time, up to T, spent in state i. From the law of large numbers for *i.i.d.* sequences of random variables, it is not hard to show that p_T converges almost surely to the steady-state distribution vector p. Proof of this fact is assigned as Exercise 4.14, which includes a broad hint. From this, we have the following form of the *law of large numbers for Markov chains.*

The Strong Law of Large Numbers for Markov Chains. *For any $H : Z \to \mathbb{R}$, the empirical average $\sum_{t=0}^T H(X_t)/T$ converges almost surely to the steady-state mean $\sum_{i=1}^k p_i H(i)$.*

Proof: Since $\sum_{t=0}^T H(X_t)/T = \sum_{i=1}^k p_{Ti} H(i)$, the result follows from the fact that $p_T \to p$ almost surely. ∎

Suppose that there is some integer $\ell \geq 0$ such that for each time t, the econometrician observes at time $t+\ell$ the data $h(Y_t)$, where $Y_t = (X_t, X_{t+1}, \dots, X_{t+\ell})$ and $h : Z^{\ell+1} \to \mathbb{R}^n$. For example, the data could be in the form of security prices, dividends, endowments, or functions of these. It is easy to check that the strong law of large numbers would apply even if $q(b_0)$ were not strictly positive, provided the t-period transition matrix $q(b_0)^t$ is strictly positive for some t. From this fact, Y also satisfies the strong law of large numbers, since Y can be treated as a Markov process whose $(\ell+1)$-period transition matrix is strictly positive. In particular, for any $G : Z^{\ell+1} \to \mathbb{R}$, the empirical average $\sum_{t=1}^T G(Y_t)/T$ converges almost surely to the corresponding steady-state mean, which is also equal to $\lim_t E^i[G(Y_t)]$, a quantity that is independent of the initial shock i.

We now specify some *test moment function* $K : \mathbb{R}^n \times B \to \mathbb{R}^M$, for some integer M, with the property that for all t, $E_t\left(K[h(Y_t), b_0]\right) = 0$. For a simple example, we could take the single-agent Markov equilibrium described by the stochastic Euler equation (13), where the utility function is specified as above by the unknown parameter vector $b_0 = (\rho_0, \alpha_0)$. For this example, we can let $Y_t = (X_t, X_{t+1})$ and let $h(Y_t) = (R_{t+1}, e_{t+1}, e_t)$, where $e_t = g(X_t)$ is the current endowment and R_t is the $\mathbb{R}^N$-valued return vector defined by

$$R_{it} = \frac{\mathcal{S}_i(X_t) + f_i(X_t)}{\mathcal{S}_i(X_{t-1})}, \qquad i \in \{1, \ldots, N\}.$$

With $M = N$ and $b = (\rho, \alpha)$, we can let

$$K_i[h(Y_t), b] = \frac{\rho e_{t+1}^{\alpha-1} R_{i,t+1}}{e_t^{\alpha-1}} - 1. \tag{19}$$

From (13), we confirm that $E_t[K(Y_t, b_0)] = 0$.

We know from the strong law of large numbers that for each b in B, the empirical average $\overline{K}_T(b) \equiv \sum_{t=1}^T K(Y_t, b)/T$ converges almost surely to a limit denoted $\overline{K}_\infty(b)$. By the law of iterated expectations, for any initial state i,

$$E^i[K(Y_t, b_0)] = E^i(E_t[K(Y_t, b_0)]) = 0.$$

From this, we know that $\overline{K}_\infty(b_0) = 0$ almost surely. A natural estimator of b_0 at time t is then given by a solution $\hat{b}_t$ to the problem

$$\inf_{b \in B} \ \| \overline{K}_t(b) \| . \tag{20}$$

Any such sequence $\{\hat{b}_t\}$ of solutions to (20) is called a *generalized-method-of-moments*, or *GMM*, estimator of b_0. Under conditions, one can show that a GMM estimator is *consistent*, in the sense that $\hat{b}_t \to b_0$ almost surely. A sufficient set of technical conditions is as follows.

GMM Regularity Conditions. *The parameter set B is compact. For any b in B other than b_0, $\overline{K}_\infty(b) \neq 0$. The function K is Lipschitz with respect to b, in the sense that there is a constant k such that for any y in $Z^{\ell+1}$ and any b_1 and b_2 in B, we have*

$$\| K(y, b_1) - K(y, b_2) \| \leq \ k \| b_1 - b_2 \| .$$

Theorem. *Under the GMM regularity conditions, a GMM estimator exists and any GMM estimator is consistent.*

The proof follows immediately from the following proposition.

Uniform Strong Law of Large Numbers. *Under the GMM regularity conditions,*

$$\sup_{b \in B} |\overline{K}_T(b) - \overline{K}_\infty(b)| \to 0 \qquad \text{almost surely.}$$

Proof: The following proof is adapted from a source indicated in the Notes. Without loss of generality for the following arguments, we can take $M = 1$. Since B is a compact set and K is Lipschitz with respect to b, for each $\epsilon \in (0, \infty)$ there is a finite set $B_\epsilon \subset B$ with the following property: For any b in B there is some b_ϵ and b^ϵ in B_ϵ satisfying, for all y,

$$K(y, b_\epsilon) \le K(y, b) \le K(y, b^\epsilon), \qquad |K(y, b_\epsilon) - K(y, b^\epsilon)| \le \epsilon. \tag{21}$$

As is customary, for any sequence $\{x_n\}$ of numbers we let

$$\underline{\lim}_n\, x_n = \sup_n \inf_{k \ge n} x_k.$$

For a given $\epsilon > 0$,

$$\begin{aligned}
\underline{\lim}_t \inf_b \left[\overline{K}_t(b) - \overline{K}_\infty(b)\right] &\ge \underline{\lim}_t \inf_b \left[\overline{K}_t(b_\epsilon) - \overline{K}_\infty(b)\right] \\
&\ge \underline{\lim}_t \inf_b \left[\overline{K}_t(b_\epsilon) - \overline{K}_\infty(b_\epsilon)\right] \\
&\qquad + \inf_b \left[\overline{K}_\infty(b_\epsilon) - \overline{K}_\infty(b)\right] \\
&\ge -\epsilon \qquad \text{almost surely,}
\end{aligned}$$

by the strong law of large numbers, (21), and the fact that B_ϵ is finite. Let $A_\epsilon \subset \Omega$ be the event of probability 1 on which this inequality holds, and let $A = A_1 \cap A_{1/2} \cap A_{1/3} \cdots$. Then A also has probability 1, and on A we have

$$\underline{\lim}_t \inf_b \left[\overline{K}_t(b) - \overline{K}_\infty(b)\right] \ge 0. \tag{22}$$

Likewise, by using b^ϵ in place of b_ϵ and $-\overline{K}$ in place of $\overline{K}$, we have

$$\underline{\lim}_t \inf_b \left[-\overline{K}_t(b) + \overline{K}_\infty(b)\right] \ge 0 \qquad \text{almost surely.} \tag{23}$$

The claim follows from (22) and (23). ∎

The Notes cite papers that prove the consistency of GMM estimators under weaker conditions and analyze the theoretical properties of this estimator. Included in these are technical conditions implying the normality of the limit of the distribution of $(b_t - b_0)/\sqrt{t}$, as well as the form of covariance matrix Σ of this asymptotic distribution. As shown in these references, the efficiency properties of the GMM estimator, in terms of this asymptotic covariance matrix Σ, can be improved by replacing the criterion function $b \mapsto \| \overline{K}_t(b) \|$ in (20) with the criterion function $b \mapsto \overline{K}_t(b)^\top W_t \overline{K}_t(b)$, for a particular adapted sequence $\{W_t\}$ of positive semi-definite "weighting" matrices. Other papers cited in the Notes apply GMM estimators in a financial setting.

Exercises

Exercise 4.1 Prove Fact 4A.

Exercise 4.2 Prove Lemma 4A.

Exercise 4.3 Prove Proposition 4A.

Exercise 4.4 Prove Fact 1 of Section 4B.

Exercise 4.5 Prove Fact 2 of Section 4B.

Exercise 4.6 Show that (13) is necessary and sufficient for optimality of $C(i,0) = g(i)$ and $\Phi(i,0) = 0$, that is, for equilibrium.

Exercise 4.7 Show that (11), (12), and (13) are equivalent.

Exercise 4.8 Show that the constraint (9), placing a lower bound on portfolios, is not binding in a Markov equilibrium.

Exercise 4.9 Suppose there is a single security with price process $S \equiv 1$ and with dividend process δ satisfying $\delta_t > -1$ for all t. The utility function U is defined by (1), where $u(x) = x^\alpha/\alpha$ for $\alpha < 1$ and $\alpha \neq 0$. The endowment process e is given by $e_t = 0$, $t > 1$, and $e_0 = w > 0$. Let $\mathcal{L}_+$ denote the space of nonnegative adapted processes. With a nonnegative wealth constraint and no other bounding restrictions, the agent's problem is modified to

$$\sup_{c \in \mathcal{L}_+} U(c) \qquad \text{subject to } W_t^c \geq 0,\ t \geq 0, \tag{24}$$

where $W_0^c = w$ and $W_t^c = (W_{t-1}^c - c_{t-1})(1 + \delta_t)$, $t > 1$.

(A) Suppose $\delta_t = \epsilon$ for all t, where $\epsilon > -1$ is a constant. Provide regularity conditions on α, ρ, and ϵ under which there exists a solution to (24). Solve for the value function and the optimal consumption control. Hint: Use dynamic programming and conjecture that the value function is of the form $V(w) = kw^\alpha/\alpha$ for some constant k. Solve the Bellman equation explicitly for V, and then show that the Bellman equation characterizes optimality by showing that $V(w) \geq U(c)$ for any feasible c, and that $V(w) = U(c^*)$, where c^* is your candidate control. Note that this will require a demonstration that $\rho^t V(W_t^c) \to 0$ for any feasible c. (B) Suppose that δ is an *i.i.d.* process. Provide regularity conditions on $\beta = \rho E[(1 + \delta_1)^\alpha]$, ρ, and α under which there exists a solution to (24). Solve for the value function and the optimal consumption control. (C) Solve parts (A) and (B) once again for $u(x) = \log(x)$, $x > 0$, and $u(0) = -\infty$. The utility function U may now take $-\infty$ as a value.

Exercise 4.10 Extend the solutions to parts (D) and (E) of Exercise 3.8 to the infinite-horizon case, adding any additional regularity conditions on the parameters $(\gamma, \alpha, \rho, A, B, \sigma^2)$ that you feel are called for.

Exercise 4.11 Demonstrate the Riesz representation (17) of the gradient of the additive discounted utility function. Hint: Use the Dominated Convergence Theorem.

Exercise 4.12 Prove Theorem 4D.

Exercise 4.13 Prove relation (18), showing exponential convergence of probabilities to their steady-state counterparts.

Exercise 4.14 Prove the version of the strong law of large numbers shown in Section 4E. Hint: Prove the almost sure convergence of the empirical distribution vector p_T to p by using the strong law of large numbers for *i.i.d.* random variables with finite expectations. For this, given any $l \in Z$, let $\tau_n(l)$ be the n-th time $t \geq 0$ that $X_t = l$. Note that

$$Q_{nlj} \equiv \#\{t : X_t = j,\ \tau_n(l) \leq t < \tau_{n+1}\}$$

has a distribution that does not depend on n or the initial state i, and that for each l and j, the sequences $\{Q_{1lj}, Q_{2lj}, \ldots\}$ and $\{t_n\}$, with $t_n = \tau_{n+1}(l) - \tau_n(l)$, are each *i.i.d.* with distributions that do not depend on the initial state i. Complete the proof from this point, considering the properties, for each l and j in Z, of

$$\frac{N^{-1}\sum_{n=1}^{N} Q_{nlj}}{N^{-1}\sum_{n=1}^{N} t_n}.$$

Notes

Freedman [1983] covers the theory of Markov chains. Revuz [1975] is a treatment of Markov processes on a general state space. Sections 4A and 4B are based on Lucas [1978], although the details here are different. LeRoy [1973] gives a precursor of this model. The probability space, on which the Markov process X is defined, is constructed in Bertsekas and Shreve [1978]. The fixed-point approach of Section 4A is based on Blackwell [1965]; Lemma 4A, in more general guises, is called Blackwell's Theorem. The results extend easily to a general compact metric space Z of shocks, as, for example, in Lucas [1978], Duffie [1988b], or Stokey and Lucas [1989]. Smoothness of the policy function or the value function is addressed by Benveniste and Scheinkman [1979]. Santos [1991] and Santos [1994] have recent results on smoothness and provide references to the extensive related literature. Versions of some of the results for this chapter that include production are found in Brock [1979], Brock [1982], Duffie [1988b] and Stokey and Lucas [1989]. The recursive-utility model was introduced into this setting by Epstein and Zin [1989]. See also Becker and Boyd [1992], Hong and Epstein [1989], ?], ?], Ma [991a], ?], Ma [1996], Streufert [1991a], Streufert [1991b] and Streufert [1991c]. Wang [1991] and Wang [993b] show the generic ability to distinguish between additive and nonadditive recursive utility from security-price data. Sections 4C and 4D are slightly unconventional, and are designed merely to bridge the gap from the finite-dimensional results of Chapter 2 to this infinite-dimensional setting. Strong assumptions are adopted here in order to guarantee the "transversality" conditions. Much weaker conditions suffice. See, for example, Kocherlakota [1990]. Schachermayer [1994] and Santos and Woodford [] give conditions for the existence of a state-price deflator in this and more general settings. Yu [1997] treats arbitrage valuation with frictions in this setting.

Kandori [1988] gives a proof of Pareto optimality and a representative agent in a complete-markets general equilibrium model. Examples are given by Abel [1986], Campbell [1984], Donaldson, Johnson, and Mehra [1987], and

Dumas and Luciano [1989]. Further characterization of equilibrium is given by Prescott and Mehra [1980] and Donaldson and Mehra [1984]. Hernandez and Santos [1994], Levine [1989], Levine and Zame [1996], **?**], Lucas [1995], Magill and Quinzii [1996], and Magill and Quinzii [1994]. Araujo, Monteiro, and Páscoa [1996] show conditions for the existence of equilibrium in an incomplete market with incomplete markets. Conditions for the existence of a stationary Markov equilibrium (with incomplete markets and heterogeneous agents) are given by Duffie, Geanakoplos, Mas-Colell, and McLennan [1994].

The role of debt constraints in promoting existence of equilibria in this setting is developed by Florenzano and Gourdel [1993], Kehoe and Levine [1993], Levine and Zame [1996], Magill and Quinzii [1996] and Magill and Quinzii [1994]. The related issue of speculative bubbles is addressed by Gilles and LeRoy [992a], Gilles and LeRoy [992b], Magill and Quinzii [1996] and Santos and Woodford []. Kurz [1992], and Kurz [1993], Kurz [1997], and Kurz and Beltratti [1996] develop the implications of stationarity and rationality in this setting. propose a rational-beliefs model that allows individual probability assessments by agents to be restricted only by absence of conflict with long-run empirical behavior. Shannon [1996] gives conditions for determinacy. Hansen and Sargent [1990] have worked out extensive examples for equilibrium in this setting with quadratic utilities and linear dynamics.

A recent spate of literature has addressed the issue of asset pricing with heterogeneous agents and incomplete markets, partly spurred by the *equity premium puzzle* pointed out by Mehra and Prescott [1985], showing the difference in expected returns between equity and riskless bonds to be far in excess of what one would find from a typical representative-agent model. Bewley [1982] and Mankiw [1986] have seminal examples of the effects of incomplete markets. The more recent literature includes Acharya and Madan [1993a], Acharya and Madan [1993b], Aiyagari and Gertler [1990], Calvet [1999], Constantinides and Duffie [1996], Duffie [1992], Haan [1994], Heaton and Lucas [1996] Judd [1997], **?**], Lucas [1991], Marcet and Singleton [1991], Mehrling [1990], Mehrling [1994], Sandroni [1995], Scheinkman [1989], Scheinkman and Weiss [1986], Svensson and Werner [1993] Telmer [1993] and Weil [1992]. Others have attempted to resolve the perceived equity premium puzzle by turning to more general utility functions, such as the habit-formation model (see, for example, Constantinides [1990] and Hansen and Jaganathan [1990]) or the recursive model (see Epstein and Zin [1989] and Epstein and Zin [1991]). For the effect of first-order risk aversion or Knightian uncertainty, see Epstein and Wang [1994]. Judd, Kubler, and Schmedders [1997] and Santos and Vigo [1995] treat the computation of equilibria in this setting.

Dumas and Luciano [1989] treat optimal portfolio selection with transactions costs in this setting. Cover and Ordentlich [1996] is a recent example of the literature on optimal long-run returns, sometimes called log-optimal growth-rate models.

Section 4E gives a "baby version" of the estimation technique used in Hansen and Singleton [1982] and Hansen and Singleton [1983]. Brown and Gibbons [1985] give an alternative exposition of this model. The generalized method of moments, in a much more general setting than that of Section 4E, is shown by Hansen [1982] to be consistent. We have used the exponential convergence of probabilities given by equation (18) to avoid the assumption that the shock process X is stationary. This extends to a more general Markov setting under regularity conditions. The proof given for the uniform strong law of large numbers is based on Pollard [1984]. A general treatment of method-of-moments estimation can be found in Gallant and White [1988]. Duffie and Singleton [1993], Lee and Ingram [1991], McFadden [1986] and Pakes and Pollard [1986] extend the GMM to a setting with simulated estimation of moments. General treatments of dynamic programming are given by Bertsekas and Shreve [1978] and Dynkin and Yushkevich [1979]. Exercise 4.11 is based on Samuelson [1969] and Levhari and Srinivasan [1969], and is extended by Hakansson [1970], Blume, Easley, and O'Hara [1982] and others. For a related *turnpike* theorem, see Hakansson [1974]. Many further results in the vein of Chapter 4 are summarized in Duffie [1988b] uffie and, especially, Stokey and Lucas [1989].

Barberis, Huang, and Santos [1999] proposed a limited-rationality asset pricing model that stressed the role of investors' aversion to negative asset returns.

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Chapter 5

The Black-Scholes Model

THIS CHAPTER PRESENTS the basic Black-Scholes model of arbitrage pricing in continuous time, as well as extensions to a nonparametric multivariate Markov setting. We first introduce the Brownian model of uncertainty and continuous security trading, and then derive partial differential equations for the arbitrage-free prices of derivative securities. The classic example is the Black-Scholes option-pricing formula. Chapter 6 extends to a non-Markovian setting using more general techniques.

5A Trading Gains for Brownian Prices

We fix a probability space $(\Omega, \mathcal{F}, P)$. A *process* is a measurable function on $\Omega \times [0, \infty)$ into $\mathbb{R}$. (For a definition of measurability with respect to a product space of this variety, see Appendix 6M.) The value of a process X at time t is the random variable variously written as X_t, $X(t)$, or $X(\,\cdot\,, t) : \Omega \to \mathbb{R}$. A *standard Brownian motion* is a process B defined by the following properties:

(a) $B_0 = 0$ almost surely;

(b) for any times t and $s > t$, $B_s - B_t$ is normally distributed with mean zero and variance $s - t$;

(c) for any times $t_0, \ldots, t_n$ such that $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the random variables $B(t_0)$, $B(t_1) - B(t_0)$, $\ldots$, $B(t_n) - B(t_{n-1})$ are independently distributed; and

(d) for each ω in Ω, the *sample path* $t \mapsto B(\omega, t)$ is continuous.

It is a nontrivial fact, whose proof has a colorful history, that the probability space $(\Omega, \mathcal{F}, P)$ can be constructed so that there exist standard Brownian motions. By 1900, in perhaps the first scientific work involving Brownian motion, Louis Bachelier proposed Brownian motion as a model of stock prices. We will follow his lead for the time being and suppose that a given standard Brownian motion B is the price process of a security. Later we consider more general classes of price processes.

The tribe $\mathcal{F}_t^B$ generated by $\{B_s : 0 \leq s \leq t\}$ is, on intuitive grounds, a reasonable model of the information available at time t for trading the security, since $\mathcal{F}_t^B$ includes every event based on the history of the price process B up to that time. For technical reasons, however, one must be able to assign probabilities to the *null sets* of Ω, the subsets of events of zero probability. For this reason, we will fix instead the *standard filtration* $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ of B, with $\mathcal{F}_t$ defined as the tribe generated by the union of $\mathcal{F}_t^B$ and the null sets. The probability measure P is also extended by letting $P(A) = 0$ for any null set A. This *completion* of the probability space is defined in more detail in Appendix C.

A *trading strategy* is an adapted process θ specifying at each state ω and time t the number $\theta_t(\omega)$ of units of the security to hold. If a strategy θ is a constant, say $\overline{\theta}$, between two dates t and $s > t$, then the total gain between those two dates is $\overline{\theta}(B_s - B_t)$, the quantity held multiplied by the price change. So long as the strategy is piecewise constant, we would have no difficulty in defining the total gain between any two times. In order to make for a good model of trading gains when we do not necessarily require piecewise constant trading, a trading strategy θ is required to satisfy $\int_0^T \theta_t^2 \, dt < \infty$ almost surely for each T. Let $\mathcal{L}^2$ denote the space of adapted processes satisfying this integrability restriction. For each θ in $\mathcal{L}^2$ there is an adapted process with continuous sample paths, denoted $\int \theta \, dB$, that is called the *stochastic integral* of θ with respect to B. The definition of $\int \theta \, dB$ is outlined in Appendix D. The value of the process $\int \theta \, dB$ at time T is usually denoted $\int_0^T \theta_t \, dB_t$, and represents the total gain generated up to time T by trading the security with price process B according to the trading strategy θ.

An interpretation of $\int_0^T \theta_t \, dB_t$ can be drawn from the discrete-time analogue $\sum_{t=0}^T \theta_t \, \Delta^1 B_t$, where $\Delta^1 B_t \equiv B_{t+1} - B_t$, that is, the sum (over t) of the shares held at t multiplied by the change in price between t and $t+1$. More generally, let $\Delta^n B_t = B_{(t+1)/n} - B_{t/n}$. In a sense that we shall not make precise, $\int_0^T \theta_t \, dB_t$ can be thought of as the limit of $\sum_{t=0}^{Tn} \theta_{t/n} \, \Delta^n B_t$, as the number n of trading intervals per unit of time goes to infinity. This statement is literally true, for example, if θ has continuous sample paths, taking "limit" to mean limit in probability. The definition of $\int_0^T \theta_t \, dB_t$ as a limit in probability of the discrete-time analogue extends to a larger class of θ, but not large enough to capture some of the applications in later chapters. The definition of $\int_0^T \theta_t \, dB_t$ given in Appendix D therefore admits any θ in $\mathcal{L}^2$.

The stochastic integral has some of the properties that one would expect from the fact that it is a good model of trading gains. For example, suppose a trading strategy θ is piecewise constant on $[0, T\,]$ in that for some stopping times $t_0, \ldots, t_N$ with $0 = t_0 < t_1 < \cdots < t_N = T$, and for any n, we have $\theta(t) = \theta(t_{n-1})$ for all $t \in [t_{n-1}, t_n)$. Then

$$\int_0^T \theta_t \, dB_t = \sum_{n=1}^{N} \theta(t_{n-1})[B(t_n) - B(t_{n-1})].$$

A second natural property of stochastic integration as a model for trading gains is linearity: For any θ and φ in $\mathcal{L}^2$ and any scalars a and b, the process $a\theta + b\varphi$ is also in $\mathcal{L}^2$, and, for any time $T > 0$,

$$\int_0^T (a\theta_t + b\varphi_t)\, dB_t = a \int_0^T \theta_t \, dB_t + b \int_0^T \varphi_t \, dB_t.$$

5B Martingale Trading Gains

The properties of standard Brownian motion imply that B is a martingale. (This follows basically from the property that its increments are independent and of zero expectation.) A process θ is bounded if there is a fixed constant k such that $|\theta(\omega, t)| \leq k$ for all (ω, t). For any bounded θ in $\mathcal{L}^2$, the law of iterated expectations and the "martingality" of B imply, for any integer times t and $\tau > t$, that $E_t(\sum_{s=t}^{\tau} \theta_s \, \Delta^1 B_s) = 0$. This means that the discrete-time gain process X, defined by $X_0 = 0$ and $X_t = \sum_{s=0}^{t-1} \theta_s \, \Delta^1 B_s$, is itself a martingale with respect to the discrete-time filtration $\{\mathcal{F}_0, \mathcal{F}_1, \ldots\}$, an exercise for the reader. The same is also true in continuous time: For any bounded θ in $\mathcal{L}^2$, $\int \theta \, dB$ is a martingale. This is natural; it should be impossible to generate an expected profit by trading a security that never experiences an expected price change. If one places no bound or other restriction on θ, however, the expectation of $\int_0^T \theta_t \, dB_t$ may not even exist. Even if $\int_0^T \theta_t \, dB_t$ and its expectation exist, we may not have a reasonable model of trading gains without some restriction on θ, as shown by example in Chapter 6. The following proposition assists in determining whether the expectation or the variance of $\int_0^T \theta_t \, dB_t$ is finite, and whether $\int \theta \, dB$ is indeed a martingale. Consider the spaces

$$\mathcal{H}^1 = \left\{ \theta \in \mathcal{L}^2 : E\left[\left(\int_0^T \theta_t^2 \, dt \right)^{1/2} \right] < \infty, \qquad T > 0 \right\}$$

$$\mathcal{H}^2 = \left\{ \theta \in \mathcal{L}^2 : E\left(\int_0^T \theta_t^2 \, dt \right) < \infty, \qquad T > 0 \right\}.$$

Of course, $\mathcal{H}^2$ is contained by $\mathcal{H}^1$.

Proposition. *If θ is in $\mathcal{H}^1$, then $\int \theta \, dB$ is a martingale. If $\int \theta \, dB$ is a martingale, then*

$$\operatorname{var}\left(\int_0^T \theta_t \, dB_t \right) = E\left(\int_0^T \theta_t^2 \, dt \right). \tag{1}$$

In particular, (1) applies to any θ in $\mathcal{H}^2$. A proof of the proposition is cited in the Notes.

5C Ito Prices and Gains

As a model of security-price processes, standard Brownian motion is too restrictive for most purposes. Consider, instead, a process of the form

$$S_t = x + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s, \qquad t \geq 0, \tag{2}$$

where x is a real number, σ is in $\mathcal{L}^2$, and μ is in $\mathcal{L}^1$, meaning that μ is an adapted process such that $\int_0^t |\mu_s| \, ds < \infty$ almost surely for all t. We call a process S of this form (2) an *Ito process*. It is common to write (2) in the informal "differential" form

$$dS_t = \mu_t \, dt + \sigma_t \, dB_t; \qquad S_0 = x.$$

One often thinks intuitively of dS_t as the "increment" of S at time t, made up of two parts, the "dt" part and the "dB_t" part. In order to further interpret this differential representation of an Ito process, suppose that σ and μ have continuous sample paths and are in $\mathcal{H}^2$. It is then literally the case that for any time t,

$$\frac{d}{d\tau} E_t\left(S_\tau\right)\Big|_{\tau=t} = \mu_t \qquad \text{almost surely} \tag{3}$$

and

$$\frac{d}{d\tau} \operatorname{var}_t\left(S_\tau\right)\Big|_{\tau=t} = \sigma_t^2 \qquad \text{almost surely,} \tag{4}$$

where the derivatives are taken from the right, and where, for any random variable X with finite variance, $\operatorname{var}_t(X) \equiv E_t(X^2) - [E_t(X)]^2$ is the $\mathcal{F}_t$-conditional variance of X. In this sense of (3) and (4), we can interpret μ_t as the rate of change of the expectation of S, conditional on information available at time t, and likewise interpret σ_t^2 as the rate of change of the conditional variance of S at time t. One sometimes reads the associated abuses of notation "$E_t(dS_t) = \mu_t\, dt$" and "$\operatorname{var}_t(dS_t) = \sigma_t^2\, dt$." Of course, dS_t is not even a random variable, so this sort of characterization is not rigorously justified and is used purely for its intuitive content. We will refer to μ and σ as the *drift* and *diffusion* processes of S, respectively. Many authors reserve the term "diffusion" for σ_t^2 or other related quantities.

For an Ito process S of the form (2), let $\mathcal{L}(S)$ denote the space consisting of any adapted process θ with $\{\theta_t\, \mu_t : t \geq 0\}$ in $\mathcal{L}^1$ and $\{\theta_t\, \sigma_t : t \geq 0\}$ in $\mathcal{L}^2$. For θ in $\mathcal{L}(S)$, we define the stochastic integral $\int \theta\, dS$ as the Ito process given by

$$\int_0^T \theta_t\, dS_t = \int_0^T \theta_t \mu_t\, dt + \int_0^T \theta_t \sigma_t\, dB_t, \qquad T \geq 0. \tag{5}$$

We also refer to $\int \theta\, dS$ as the *gain process generated by* θ, given the price process S. If $\theta \in \mathcal{L}(S)$ is such that $\{\theta_t \sigma_t : t \geq 0\}$ is in $\mathcal{H}^2$ and $E\left[\left(\int_0^T \theta_t \mu_t\, dt\right)^2\right] < \infty$, then we write that θ is in $\mathcal{H}^2(S)$. By Proposition 5B, if θ is in $\mathcal{H}^2(S)$ then $\int \theta\, dS$ is a finite-variance process.

We will have occasion to refer to adapted processes θ and φ that are equal *almost everywhere*, by which we mean that $E(\int_0^\infty |\theta_t - \varphi_t|\, dt) = 0$. In fact, we shall write "$\theta = \varphi$" whenever $\theta = \varphi$ almost everywhere. This is a natural convention, for suppose that X and Y are Ito processes with $X_0 = Y_0$ and with $dX_t = \mu_t\, dt + \sigma_t\, dB_t$ and $dY_t = a_t\, dt + b_t\, dB_t$. Since stochastic integrals are defined for our purposes as continuous sample path processes, it turns out that $X_t = Y_t$ for all t almost surely if and only if $\mu = a$ almost everywhere and $\sigma = b$ almost everywhere. We call this the *unique decomposition property* of Ito processes.

5D Ito's Formula

More than any other result, *Ito's Formula* is the basis for explicit solutions to asset-pricing problems in a continuous-time setting.

Ito's Formula. *Suppose X is an Ito process with $dX_t = \mu_t\, dt + \sigma_t\, dB_t$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable. Then the process Y, defined by $Y_t = f(X_t, t)$, is an Ito process with*

$$dY_t = \left[f_x(X_t, t)\mu_t + f_t(X_t, t) + \frac{1}{2} f_{xx}(X_t, t)\sigma_t^2 \right] dt + f_x(X_t, t)\sigma_t\, dB_t. \tag{6}$$

A generalization of Ito's Formula (6) appears later in the chapter.

5E The Black-Scholes Option-Pricing Formula

Consider a security, to be called a *stock*, with price process

$$S_t = x \exp(\alpha t + \sigma B_t), \qquad t \geq 0, \tag{7}$$

where $x > 0$, α, and σ are constants. Such a process, called a *geometric Brownian motion*, is often called *log-normal* because, for any t, $\log(S_t) = \log(x) + \alpha t + \sigma B_t$ is normally distributed. Moreover, since $X_t \equiv \alpha t + \sigma B_t = \int_0^t \alpha\, ds + \int_0^t \sigma\, dB_s$ defines an Ito process X with constant drift α and diffusion σ, and since $y \mapsto x\, e^y$ is a C^2 function, Ito's Formula implies that S is an Ito process and that

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t; \qquad S_0 = x,$$

where $\mu = \alpha + \sigma^2/2$. From (3) and (4), at any time t, the rate of change of the conditional mean of S_t is μS_t, and the rate of change of the conditional variance is $\sigma^2 S_t^2$, so that per dollar invested in this security at time t, one may think of μ as the "instantaneous" expected rate of return, and σ as the "instantaneous" standard deviation of the rate of return. This sort of characterization abounds in the literature, and one often reads the associated abuses of notation "$E(dS_t/S_t) = \mu\, dt$" and "$\mathrm{var}(dS_t/S_t) = \sigma^2\, dt$." The coefficient σ is also known as the *volatility* of S. In any case, a geometric Brownian motion is a natural two-parameter model of a security-price process because of these simple interpretations of μ and σ.

Consider a second security, to be called a *bond*, with the price process β defined by

$$\beta_t = \beta_0 \, e^{rt}, \qquad t \geq 0, \tag{8}$$

for some constants $\beta_0 > 0$ and r. We have the obvious interpretation of r as the *continually compounding interest rate*, that is, the exponential rate at which riskless deposits accumulate with interest. Throughout, we will also refer to r as the *short rate*. Since $\{rt : t \geq 0\}$ is trivially an Ito process, β is also an Ito process with

$$d\beta_t = r\beta_t \, dt. \tag{9}$$

We can also view (9) as an ordinary differential equation with initial condition β_0 and solution (8).

We allow any trading strategies a in $\mathcal{H}^2(S)$ for the stock and b in $\mathcal{H}^2(\beta)$ for the bond. Such a trading strategy (a, b) is said to be *self-financing* if it generates no dividends (either positive or negative), meaning that for all t,

$$a_t S_t + b_t \beta_t = a_0 S_0 + b_0 \beta_0 + \int_0^t a_u \, dS_u + \int_0^t b_u \, d\beta_u. \tag{10}$$

The self-financing condition (10) is merely a statement that the current portfolio value (on the left-hand side) is precisely the initial investment plus any trading gains, and therefore that no dividend "inflow" or "outflow" is generated.

Now consider a third security, an *option*. We begin with the case of a European call option on the stock, giving its owner the right, but not the obligation, to buy the stock at a given exercise price K on a given exercise date T. The option's price process Y is as yet unknown except for the fact that $Y_T = (S_T - K)^+ \equiv \max(S_T - K, 0)$, which follows from the fact that the option is rationally exercised if and only if $S_T > K$. (See Exercise 2.1 for a discrete-time analogue.)

Suppose there exists a self-financing trading strategy (a, b) in the stock and bond with $a_T S_T + b_T \beta_T = Y_T$. If $a_0 S_0 + b_0 \beta_0 < Y_0$, then one could sell the option for Y_0, make an initial investment of $a_0 S_0 + b_0 \beta_0$ in the trading strategy (a, b), and at time T liquidate the entire portfolio $(-1, a_T, b_T)$ of option, stock, and bond with payoff $-Y_T + a_T S_T + b_T \beta_T = 0$. The initial profit $Y_0 - a_0 S_0 - b_0 \beta_0 > 0$ is thus riskless, so the trading strategy $(-1, a, b)$ would be an arbitrage. Likewise, if $a_0 S_0 + b_0 \beta_0 > Y_0$, the strategy $(1, -a, -b)$ is an arbitrage. Thus, if there is no arbitrage, $Y_0 = a_0 S_0 + b_0 \beta_0$. The same arguments applied at each date t imply that in the absence of arbitrage, $Y_t = a_t S_t + b_t \beta_t$. A full definition of continuous-time arbitrage is given in Chapter 6, but for now we can proceed without much ambiguity at this informal level. Our objective now is to show the following.

The Black-Scholes Formula. *If there is no arbitrage, then, for all* $t < T$, $Y_t = C(S_t, t)$, *where*

$$C(x, t) = x\Phi(z) - e^{-r(T-t)} K \Phi\left(z - \sigma\sqrt{T-t}\right), \tag{11}$$

with

$$z = \frac{\log(x/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \tag{12}$$

where Φ *is the cumulative standard normal distribution function.*

5F Black-Scholes Formula: First Try

We will eventually see many different ways to arrive at the Black-Scholes formula (11). Although not the shortest argument, the following is perhaps the most obvious and constructive. We start by assuming that $Y_t = C(S_t, t)$, $t < T$, without knowledge of the function C aside from the assumption that it is twice continuously differentiable on $(0, \infty) \times [0, T)$ (allowing an application of Ito's Formula). This will lead us to deduce (11), justifying the assumption and proving the result at the same time.

Based on our assumption that $Y_t = C(S_t, t)$ and Ito's Formula,

$$dY_t = \mu_Y(t)\, dt + C_x(S_t, t)\sigma S_t\, dB_t, \qquad t < T, \tag{13}$$

where

$$\mu_Y(t) = C_x(S_t, t)\mu S_t + C_t(S_t, t) + \frac{1}{2} C_{xx}(S_t, t)\sigma^2 S_t^2.$$

Now suppose there is a self-financing trading strategy (a, b) with

$$a_t S_t + b_t \beta_t = Y_t, \qquad t \in [0, T], \tag{14}$$

as outlined in Section 5E. This assumption will also be justified shortly. Equations (10) and (14), along with the linearity of stochastic integration, imply that

$$dY_t = a_t\, dS_t + b_t\, d\beta_t = (a_t \mu S_t + b_t \beta_t r)\, dt + a_t \sigma S_t\, dB_t. \tag{15}$$

One way to choose the trading strategy (a, b) so that both (13) and (15) are satisfied is to "match coefficients separately in both dB_t and dt." In fact, the unique decomposition property of Ito processes explained at the end of Section 5C implies that this is the only way to ensure that (13) and (15) are consistent. Specifically, we choose a_t so that $a_t \sigma S_t = C_x(S_t, t)\sigma S_t$; for this, we let $a_t = C_x(S_t, t)$. From (14) and $Y_t = C(S_t, t)$, we then have $C_x(S_t, t)S_t + b_t \beta_t = C(S_t, t)$, or

$$b_t = \frac{1}{\beta_t}\left[C(S_t, t) - C_x(S_t, t)S_t\right]. \tag{16}$$

Finally, "matching coefficients in dt" from (13) and (15) leaves, for $t < T$,

$$-rC(S_t, t) + C_t(S_t, t) + rS_t C_x(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 C_{xx}(S_t, t) = 0. \tag{17}$$

In order for (17) to hold, it is enough that C satisfies the partial differential equation (PDE)

$$-rC(x, t) + C_t(x, t) + rxC_x(x, t) + \frac{1}{2}\sigma^2 x^2 C_{xx}(x, t) = 0, \tag{18}$$

for $(x, t) \in (0, \infty) \times [0, T)$. The fact that $Y_T = C(S_T, T) = (S_T - K)^+$ supplies the *boundary condition*:

$$C(x, T) = (x - K)^+, \qquad x \in (0, \infty). \tag{19}$$

By direct calculation of derivatives, one can show as an exercise that (11) is a solution to (18)–(19). All of this seems to confirm that $C(S_0, 0)$, with C defined by the Black-Scholes formula (11), is a good candidate for the initial price of the option. In order to make this solid, suppose that $Y_0 > C(S_0, 0)$, where C is defined by (11). Consider the strategy $(-1, a, b)$ in the option, stock, and bond, with $a_t = C_x(S_t, t)$ and b_t given by (16) for $t < T$. We can choose a_T and b_T arbitrarily so that (14) is satisfied; this does not affect the self-financing condition (10) because the value of the trading strategy at a single point in time has no effect on the stochastic integral. (For this, see the implications of equality "almost everywhere" at the end of Section 5C.) The result is that (a, b) is self-financing by construction and that $a_T S_T + b_T \beta_T = Y_T = (S_T - K)^+$. This strategy therefore nets an initial riskless profit of

$$Y_0 - a_0 S_0 - b_0 \beta_0 = Y_0 - C(S_0, 0) > 0,$$

which defines an arbitrage. Likewise, if $Y_0 < C(S_0, 0)$, the trading strategy $(+1, -a, -b)$ is an arbitrage. Thus, it is indeed a necessary condition for the absence of arbitrage that $Y_0 = C(S_0, 0)$. Sufficiency is a more delicate matter. We will see in Chapter 6 that under mild technical conditions on trading strategies, the Black-Scholes formula for the option price is also sufficient for the absence of arbitrage. One last piece of business is to show that the "option-hedging" strategy (a, b) is such that a is in $\mathcal{H}^2(S)$ and b is in $\mathcal{H}^2(\beta)$. This is true, and is left to show as an exercise.

Transactions costs play havoc with the sort of reasoning just applied. For example, if brokerage fees are any positive fixed fraction of the market value of stock trades, the stock-trading strategy a constructed above would call for infinite total brokerage fees, since, in effect, the number of shares traded is infinite! This fact and the literature on transactions costs in this setting is reviewed in the Notes of Chapters 6 and 9.

5G The PDE for Arbitrage-Free Derivative Security Prices

The expression $dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$ for the log-normal stock-price process S of Section 5E is a special case of a *stochastic differential equation* (SDE) of the form

$$dS_t = \mu(S_t, t) \, dt + \sigma(S_t, t) \, dB_t; \qquad S_0 = x, \tag{20}$$

where $\mu : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $\sigma : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ are given functions. Under regularity conditions on μ and σ reviewed in Appendix E, there is a unique Ito process S solving (20) for each starting point x in $\mathbb{R}$. Assuming that such a solution S defines a stock-price process, consider the price process β defined by

$$\beta_t = \beta_0 \exp\left[\int_0^t r(S_u, u) \, du\right], \tag{21}$$

where $r : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is well enough behaved for the existence of the integral in (21). We may view β_t as the market value at time t of an investment account that is continuously re-invested at the short rate $r(S_t, t)$. This is consistent with a trivial application of Ito's Formula, which implies that

$$d\beta_t = \beta_t r(S_t, t)\, dt; \qquad \beta_0 > 0. \tag{22}$$

Rather than restricting attention to the option payoff $Y_T = (S_T - K)^+$, consider a *derivative security* defined by the payoff $Y_T = g(S_T)$ at time T, for some continuous $g : \mathbb{R} \to \mathbb{R}$. Arguments like those in Section 5F lead one to formulate the arbitrage-free price process Y of the derivative security as $Y_t = C(S_t, t)$, $t \in [0, T]$, where C solves the PDE

$$-r(x,t)C(x,t) + C_t(x,t) + r(x,t)xC_x(x,t) + \frac{1}{2}\sigma(x,t)^2 C_{xx}(x,t) = 0, \tag{23}$$

for $(x, t) \in \mathbb{R} \times [0, T)$, with the boundary condition

$$C(x, T) = g(x), \qquad x \in \mathbb{R}. \tag{24}$$

In order to tie things together, suppose that C solves (23)–(24). If $Y_0 \neq C(S_0, 0)$, then an obvious extension of our earlier arguments implies that there is an arbitrage. (This extension is left as an exercise.) This is true even if C is not twice continuously differentiable, but merely $C^{2,1}(\mathbb{R} \times [0, T))$, meaning that the derivatives C_x, C_t, and C_{xx} exist and are continuous in $\mathbb{R} \times (0, T)$, and extend continuously to $\mathbb{R} \times [0, T)$. (Ito's Formula also applies to any function in this class.)

This PDE characterization of the arbitrage-free price of derivative securities is useful if there are convenient methods for solving PDEs of the form (23)–(24). Numerical solution techniques are discussed in Chapter 11. One of these techniques is based on a probabilistic representation of solutions given in the next section.

5H The Feynman-Kac Solution

A potential simplification of the PDE problem (23)–(24) is obtained as follows. For each (x, t) in $\mathbb{R} \times [0, T]$, let $Z^{x,t}$ be the Ito process defined by $Z_s^{x,t} = x$, $s \leq t$, and

$$dZ_s^{x,t} = r(Z_s^{x,t}, s) Z_s^{x,t}\, ds + \sigma(Z_s^{x,t}, s)\, dB_s, \qquad s > t. \tag{25}$$

That is, $Z^{x,t}$ starts at x at time t and continues from there by following the SDE (25).

Condition FK. *The functions σ, r, and g satisfy one of the technical sufficient conditions given in Appendix E for Feynman-Kac solutions.*

The FK (for "Feynman-Kac") condition is indeed only technical, and limits how quickly the functions σ, r, and g can grow or change. Referring to Appendix E, we have the following solution to the PDE (23)–(24) as an expectation of the discounted payoff of the derivative security, modified by replacing the original price process S with a *pseudo-price process* $Z^{x,t}$ whose expected rate of return is the riskless interest rate. This is sometimes known as *risk-neutral valuation.* This is not to say that agents are risk-neutral, but rather that risk-neutrality is (in this setting) without loss of generality for purposes of pricing derivative securities.

The Feynman-Kac Solution. *Under Condition FK, if there is no arbitrage, then the derivative security defined by the payoff $g(S_T)$ at time T has the price process Y with $Y_t = C(S_t, t)$, where C is the solution to (23)–(24) given by*

$$C(x,t) = E\left(\exp\left[-\int_t^T r(Z_s^{x,t}, s)\, ds\right] g\left(Z_T^{x,t}\right)\right), \qquad (x,t) \in \mathbb{R} \times [0,T]. \quad (26)$$

It can be checked as an exercise that (26) recovers the Black-Scholes option-pricing formula (11). Calculating this expectation directly is a simpler way to solve the corresponding PDE (18)–(19) than is the method originally used to discover the Black-Scholes formula. Chapter 11 presents numerical methods for solving (23)–(24), one of which involves Monte Carlo simulation of the Feynman-Kac solution (26), which bears a close resemblance to the discrete-time equivalent-martingale-measure arbitrage-free price representation of Chapter 2. This is more than a coincidence, as we shall see in Chapter 6.

5I The Multidimensional Case

Suppose that $B^1, \ldots, B^d$ are d independent standard Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$. The process $B = (B^1, \ldots, B^d)$ is known as a *standard Brownian motion in* $\mathbb{R}^d$. The standard filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ of B is defined just as in the one-dimensional case. Given $\mathbb{F}$, the subsets $\mathcal{L}^1$, $\mathcal{L}^2$, $\mathcal{H}^1$, and $\mathcal{H}^2$ of adapted processes are also as defined in Sections 5A and 5B.

In this setting, X is an Ito process if, for some x in $\mathbb{R}$, some μ in $\mathcal{L}^1$, and some $\theta^1, \ldots, \theta^d$ in $\mathcal{L}^2$,

$$X_t = x + \int_0^t \mu_s \, ds + \sum_{i=1}^{d} \int_0^t \theta_s^i \, dB_s^i, \qquad t \geq 0. \tag{27}$$

For convenience, (27) is also written

$$X_t = x + \int_0^t \mu_s \, ds + \int_0^t \theta_s \, dB_s, \qquad t \geq 0, \tag{28}$$

or in the convenient stochastic differential form

$$dX_t = \mu_t \, dt + \theta_t \, dB_t; \qquad X_0 = x. \tag{29}$$

If $X^1, \ldots, X^N$ are Ito processes, then we call $X = (X^1, \ldots, X^N)$ an *Ito process in* $\mathbb{R}^N$, which can be written

$$X_t = x + \int_0^t \mu_s \, ds + \int_0^t \theta_s \, dB_s, \qquad t \geq 0, \tag{30}$$

or

$$dX_t = \mu_t \, dt + \theta_t \, dB_t; \qquad X_0 = x \in \mathbb{R}^N, \tag{31}$$

where μ and θ are valued in $\mathbb{R}^N$ and $\mathbb{R}^{N \times d}$, respectively. (Here, $\mathbb{R}^{N \times d}$ denotes the space of real matrices with N rows and d columns.) Ito's Formula extends as follows.

Ito's Formula. *Suppose X is the Ito process in $\mathbb{R}^N$ given by (30) and f is in $C^{2,1}(\mathbb{R}^N \times [0, \infty))$. Then $\{f(X_t, t) : t \geq 0\}$ is an Ito process and, for any time t,*

$$f(X_t, t) = f(X_0, 0) + \int_0^t \mathcal{D}_X f(X_s, s) \, ds + \int_0^t f_x(X_s, s) \theta_s \, dB_s,$$

where

$$\mathcal{D}_X f(X_t, t) = f_x(X_t, t)\mu_t + f_t(X_t, t) + \frac{1}{2} \mathrm{tr} \left[\theta_t \theta_t^\top f_{xx}(X_t, t) \right].$$

Here, f_x, f_t, and f_{xx} denote the obvious partial derivatives of f valued in $\mathbb{R}^N$, $\mathbb{R}$, and $\mathbb{R}^{N \times N}$ respectively, and $\mathrm{tr}(A)$ denotes the *trace* of a square matrix A (the sum of its diagonal elements).

If X and Y are real-valued Ito processes with $dX_t = \mu_X(t)\,dt + \sigma_X(t)\,dB_t$ and $dY_t = \mu_Y(t)\,dt + \sigma_Y(t)\,dB_t$, then Ito's Formula (for $N = 2$) implies that the product $Z = XY$ is an Ito process, with drift μ_Z given by

$$\mu_Z(t) = X_t\,\mu_Y(t) + Y_t\mu_X(t) + \sigma_X(t)\cdot\sigma_Y(t). \tag{32}$$

If μ_X, μ_Y, σ_X, and σ_Y are all in $\mathcal{H}^2$ and have continuous sample paths, then an application of Fubini's Theorem (Appendix C) implies that

$$\frac{d}{ds}\operatorname{cov}_t(X_s, Y_s)\Big|_{s=t} = \sigma_X(t)\cdot\sigma_Y(t) \qquad \text{almost surely,} \tag{33}$$

where $\operatorname{cov}_t(X_s, Y_s) = E_t(X_sY_s) - E_t(X_s)E_t(Y_s)$ and where the derivative is taken from the right, extending the intuition developed with (3) and (4).

If X is an Ito process in $\mathbb{R}^N$ with $dX_t = \mu_t\,dt + \sigma_t\,dB_t$ and $\theta = (\theta^1, \ldots, \theta^N)$ is a vector of adapted processes such that $\theta\cdot\mu$ is in $\mathcal{L}^1$ and, for each i, $\theta\cdot\sigma^i$ is in $\mathcal{L}^2$, then we say that θ is in $\mathcal{L}(X)$, which implies that

$$\int_0^T \theta_t\,dX_t \equiv \int_0^T \theta_t\cdot\mu_t\,dt + \int_0^T \theta_t^\top\sigma_t\,dB_t, \qquad T \geq 0$$

is well defined as an Ito process. If $E[(\int_0^T \theta_t\cdot\mu_t\,dt)^2] < \infty$ and, for each i, $\theta\cdot\sigma^i$ is also in $\mathcal{H}^2$, then we say that θ is in $\mathcal{H}^2(X)$, which implies that $\int\theta\,dX$ is a finite-variance process.

Suppose that $S = (S^1, \ldots, S^N)$ is an Ito process in $\mathbb{R}^N$ specifying the prices of N given securities, and that S satisfies the stochastic differential equation:

$$dS_t = \mu(S_t, t)\,dt + \sigma(S_t, t)\,dB_t; \qquad S_0 = x \in \mathbb{R}^N, \tag{34}$$

where $\mu : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}^N$ and $\sigma : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}^{N\times d}$ satisfy enough regularity (conditions are given in Appendix E) for existence and uniqueness of a solution to (34). Let

$$\beta_t = \beta_0 \exp\left[\int_0^t r(S_u, u)\,du\right], \qquad \beta_0 > 0, \tag{35}$$

define the price process of a bond, where $r : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ defines a continuously compounding short rate, sufficiently well behaved that (35) is a well-defined Ito process. We can also use Ito's Formula to write

$$d\beta_t = \beta_t r(S_t, t)\,dt; \qquad \beta_0 > 0. \tag{36}$$

Finally, let some continuous $g : \mathbb{R}^N \to \mathbb{R}$ define the payoff $g(S_T)$ at time T of a derivative security whose price at time zero is to be determined.

Once again, the arguments of Section 5F can be extended to show that under technical regularity conditions and in the absence of arbitrage, the price process Y of the derivative security is given by $Y_t = C(S_t, t)$, where C solves the PDE:

$$\mathcal{D}_Z C(x,t) - r(x,t)C(x,t) = 0, \qquad (x,t) \in \mathbb{R}^N \times [0,T), \tag{37}$$

with boundary condition

$$C(x,T) = g(x), \qquad x \in \mathbb{R}^N, \tag{38}$$

where

$$\mathcal{D}_Z C(x,t) = C_x(x,t)r(x,t)x + C_t(x,t) + \frac{1}{2}\operatorname{tr}\left[\sigma(x,t)\sigma(x,t)^\top C_{xx}(x,t)\right]. \tag{39}$$

We exploit once again the technical condition FK on (r, σ, g) reviewed in Appendix E for existence of a probabilistic representation of solutions to the PDE (37)–(38).

The Feynman-Kac Solution. *Under Condition FK, if there is no arbitrage, then the derivative security with payoff $g(S_T)$ at time T has the price process Y given by $Y_t = C(S_t, t)$, where C is the solution to the PDE (37)–(38) given by*

$$C(x,t) = E\left[\exp\left(-\int_t^T r(Z_s^{x,t}, s)\, ds\right) g\left(Z_T^{x,t}\right)\right], \quad (x,t) \in \mathbb{R}^N \times [0,T], \tag{40}$$

where $Z^{x,t}$ is the Ito process defined by $Z_s^{x,t} = x$, $s \le t$, and

$$dZ_s^{x,t} = r(Z_s^{x,t}, s)Z_s^{x,t}\, ds + \sigma(Z_s^{x,t}, s)\, dB_s, \qquad s \ge t. \tag{41}$$

The exercises provide applications and additional extensions of this approach to the arbitrage-free valuation of derivative securities, allowing for intermediate dividends and for an underlying Markov-state process. Chapter 6 further extends arbitrage-free pricing to a non-Markovian setting using more general methods. Chapters 7 and 8 give further applications, including futures, forwards, American options, and the term structure of interest rates.

Exercises

Exercise 5.1 Fixing a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t : t \geq 0\}$, a process X is *Markov* if, for any time t and any integrable random variable Y that is measurable with respect to the tribe generated by $\{X_s : s \geq t\}$, we have $E(Y \mid \mathcal{F}_t) = E(Y \mid X_t)$ almost surely. In particular, for any measurable $f : \mathbb{R} \to \mathbb{R}$ such that $f(X_t)$ has finite expectation, we have $E[f(X_t) \mid \mathcal{F}_s] = E[f(X_t) \mid X_s]$ for $s \leq t$. It is a fact, which we shall not prove, that standard Brownian motion B is a Markov process with respect to its standard filtration. Use this fact to show that B is a martingale with respect to its standard filtration. Suppose that θ is a bounded adapted process. Show, as stated in Section 5B, that the discrete-time process X defined by $X_0 = 0$ and $X_t = \sum_{s=0}^{t-1} \theta_s \, \Delta^1 B_s$, $t \geq 1$, is a martingale with respect to $\{\mathcal{F}_0, \mathcal{F}_1, \ldots\}$.

Exercise 5.2 Suppose that S is defined by (7). Use Ito's Formula to show that, as claimed, $dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$, where $\mu = \alpha + \sigma^2/2$.

Exercise 5.3 Verify that the ordinary differential equation (9), with initial condition β_0, is solved by (8).

Exercise 5.4 Verify by direct calculation of the derivatives that the PDE (18)–(19) is solved by the Black-Scholes formula (11).

Exercise 5.5 Derive the PDE (23) for the arbitrage-free value of the derivative security. Hint: Use arguments analogous to those used to derive the PDE (18) for the Black-Scholes formula.

Exercise 5.6 Suppose the PDE (37) for the arbitrage-free value of the derivative security is not satisfied, in that the initial price Y_0 of the security is not equal to $C(S_0, 0)$, where C solves (37)–(38). Construct an arbitrage that nets an initial risk-free profit of m units of account, where m is an arbitrary number chosen by you.

Exercise 5.7 Suppose that the stock, whose price process S is given by (20), pays dividends at a rate $\delta(S_t, t)$ at time t, where a continuous function $\delta : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ defines a *cumulative dividend process* D by $D_t = \int_0^t \delta(S_u, u) \, du$. The total gain process G for the security is defined by $G_t = S_t + D_t$, and a trading strategy θ in $\mathcal{L}(G)$ generates the gain process $\int \theta \, dG$, the sum of capital and dividend gains. Derive a new PDE generalizing (23) for the arbitrage-free value of the derivative security defined by

g. Provide regularity conditions for the associated Feynman-Kac solution, extending (25)–(26).

Exercise 5.8 Suppose that S is a stock-price process defined by (20), β is a bond-price process defined by (21), and a derivative security is defined by the lump-sum payoff $g(S_T)$ at time T, as in Section 5G, and also by the cumulative dividend process H defined by $H_t = \int_0^t h(S_\tau, \tau)\, d\tau$, for some continuous $h : \mathbb{R} \times [0, T] \to \mathbb{R}$. By definition, a trading strategy (a, b) in $\mathcal{H}^2(S, \beta)$ *finances* this derivative security if

$$a_t S_t + b_t \beta_t = a_0 S_0 + b_0 \beta_0 + \int_0^t a_u \, dS_u + \int_0^t b_u \, d\beta_u - H_t, \qquad t \leq T, \quad (42)$$

and

$$a_T S_T + b_T \beta_T = g(S_T). \quad (43)$$

Relation (42) means that the current value of the portfolio is, at any time, the initial value, plus trading gains to date, less the payout to date of the derivative dividends. If (a, b) finances the derivative security in this sense and the derivative security's initial price Y_0 is not equal to $a_0 S_0 + b_0 \beta_0$, then there is an arbitrage. For example, if $Y_0 > a_0 S_0 + b_0 \beta_0$, then the strategy $(-1, a, b)$ in derivative security, stock, and bond generates the cumulative dividend process $-H + H = 0$ and the final payoff $-g(S_T) + g(S_T) = 0$, with the initial riskless profit $Y_0 - a_0 S_0 - b_0 \beta_0 > 0$. Derive an extension of the PDE (23)–(24) for the derivative security price, as well as an extension of the Feynman-Kac solution (25)–(26).

Exercise 5.9 Suppose that X is the Ito process in $\mathbb{R}^K$ solving the SDE

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dB_t; \qquad X_0 = x.$$

We could refer to X, by analogy with Chapter 3, as the "shock process." Suppose the price process S for the N "stocks" is defined by $S_t = \mathcal{S}(X_t, t)$, where $\mathcal{S}$ is in $C^{2,1}(\mathbb{R}^K \times [0, \infty))$, and that the bond-price process β is the Ito process defined by $d\beta_t = \beta_t r(X_t, t)\, dt$, $\beta_0 > 0$, where $r : \mathbb{R}^K \times [0, \infty) \to \mathbb{R}$ is bounded and continuous.

(A) State regularity conditions that you find appropriate in order to derive a PDE analogous to (37)–(38) for the price of an additional security defined by the payoff $g(X_T)$ at time T, where $g : \mathbb{R}^K \to \mathbb{R}$. Then provide the Feynman-Kac solution, analogous to (40)–(41), including a sufficient set of technical conditions based on Appendix E. (B) Extend part (A) to the case in which the stocks pay a cumulative dividend process D that is an Ito process in $\mathbb{R}^N$ well defined by $D_t = \int_0^t \delta(X_s, s)\, ds$, where $\delta : \mathbb{R}^K \times [0, \infty) \to \mathbb{R}^N$, and in which the additional security has the lump-sum payoff of $g(X_T)$ at time T, as well as a cumulative dividend Ito process H well defined by $H_t = \int_0^t h(X_s, s)\, ds$, where $h : \mathbb{R}^K \times [0, T] \to \mathbb{R}$.

Exercise 5.10 Suppose the short-rate process r is given by a bounded continuous function $r : [0, T\,] \to \mathbb{R}$. Consider a security with price process S defined by

$$dS_t = \mu(S_t)\, dt + \sigma(S_t)\, dB_t,$$

where μ and σ satisfy a Lipschitz condition. Suppose this security has the cumulative dividend process D defined by $D_t = \int_0^t \delta_u S_u\, du$, where $\delta : [0, T] \to \mathbb{R}$ is a continuous function. (Such a function δ is often called the "dividend yield.")

(A) (Put-Call Parity) Suppose there are markets for European call and put options on the above security with exercise price K and exercise date T. Let C_0 and P_0 denote the call and put prices at time zero. Give an explicit expression for P_0 in terms of C_0, in the absence of arbitrage and transactions costs.

(B) Suppose, for all t, that $r_t = 0.10$ and that $\delta_t = 0.08$. Consider a European option expiring in $T = 0.25$ years. Suppose that $K = 50$ and $S_0 = 45$. If the call sells for 3.75, what is the put price? (Give a specific dollar price, to the nearest penny, showing how you calculated it.)

(C) Suppose, instead, that the dividend process D is defined by $D_t = \int_0^t \delta_\tau \log(S_\tau) S_\tau\, d\tau$. Suppose $\sigma(x) = \epsilon x$, for some constant $\epsilon > 0$. Solve part (A) again. Then calculate the price of a European call with exercise price $K = 35$ given initial stock price $S_0 = 40$, assuming, for all t, that $\delta_t = 0.08$, $r_t = 0.10$, and $\epsilon = 0.20$. Assume expiration in 0.25 years. Justify your answer.

Exercise 5.11 Suppose the price of haggis (an unusually nasty food served in Scotland) follows the process H defined by

$$dH_t = H_t \mu_H\, dt + H_t \sigma_H\, dB_t; \qquad H_0 > 0,$$

in British pounds per pint, where μ_H is a constant, σ_H is a constant vector in $\mathbb{R}^2$, and B is a standard Brownian motion in $\mathbb{R}^2$. A trader at a Wall

Street investment bank, Gold in Sacks, Incorporated, has decided that since there are options on almost everything else, there may as well be options on haggis. Of course, there is the matter of selling the options in the United States, denominated in U.S. dollars. It has been noted that the price of the U.S. dollar, in British pounds per dollar, follows the process

$$dD_t = D_t \mu_D \, dt + D_t \sigma_D \, dB_t; \qquad D_0 > 0,$$

where μ_D is a constant and σ_D is a constant vector in $\mathbb{R}^2$. The continuously compounding short rate in U.S. funds is r_D, a constant. Although there are liquid markets in Edinburgh for haggis and for U.S. dollars, there is not a liquid market for haggis options. Gold in Sacks has therefore decided to sell call options on haggis at a U.S. dollar strike price of 6.50 per pint expiring in 3 months, and cover its option position with a replicating strategy in the other instruments, so as to earn a riskless profit equal to the markup in the sale price of the options over the initial investment cost to Gold in Sacks for the replicating strategy.

(A) What replicating strategy would you recommend?

(B) If the options are sold at a 10 percent profit markup, give an explicit formula for the option price Gold in Sacks should charge its customers.

(C) Suppose borrowing in U.S. funds is too clumsy, since the other two parts of the strategy (dollar and haggis trading) are done at Gold in Sacks's Edinburgh office. If the British pound borrowing rate is r_P, a constant, can you still answer parts (A) and (B), using British pound borrowing (and lending) rather than U.S. dollar borrowing (and lending)? If so, do so. If not, say why not. If you find it useful, you may use any arbitrage conditions relating the various coefficients $(\mu_H, \mu_D, \sigma_H, \sigma_D, r_D, r_P)$, if indeed there are any such coefficients precluding arbitrage.

Exercise 5.12 Show, in the setting of Section 5E, that (26) recovers the Black-Scholes formula (11).

Exercise 5.13 Show that the Black-Scholes option-hedging strategy (a, b) of Section 5F is such that $a \in \mathcal{H}^2(S)$ and $b \in \mathcal{H}^2(\beta)$, as assumed.

Notes

The material in this chapter is standard. Proposition 5B is from Protter [1990]. The Brownian model was introduced to the study of option pricing by Bachelier [1900]. The Black and Scholes [1973] formula was extended by Merton [1973] and Merton [1977] and subsequently given literally hundreds of further extensions and applications. Andreasen, Jensen, and Poulsen [1996] provide numerous alternative methods of deriving thge Black-Scholes Formula. Cox and Rubinstein [1985] is a standard reference on options, while Hull [1993] has further applications and references. The basic approach of using continuous-time self-financing strategies as the basis for making arbitrage arguments is due to Merton [1977] and Harrison and Kreps [1979]. The basic idea of risk-neutral valuation, via adjustment of the underlying stock-price process, is due to Cox and Ross [1976]. This is extended to the notion of equivalent martingale measures, found in Chapter 6, by Harrison and Kreps [1979]. The impact of variations in the "volatility" on the Black-Scholes option-pricing formula is shown, in two different senses, by El Karoui, Jeanblanc, and Shreve [1991], Grundy and Wiener [1995], Johnson and Shanno [1987] and Reisman [1986]. For "stochastic volatility" models, see Section 8E and references cited in the Notes to Chapter 8.

Alternative approaches to the standard methods of stochastic calculus have been developed by Cutland, Kopp, and Willinger [1991], Cutland, Kopp, and Willinger [1993] and Cutland, Kopp, and Willinger [993b], who apply nonstandard analysis; by Bick and Willinger [1994] and Willinger and Tacqu [1989], who use a path-wise integral established by Föllmer [1981]; and by Kunitomo [1993], Cutland, Kopp, and Willinger [1991], and Cutland, Kopp, and Willinger [993b], who exploit fractional Brownian motion. See Rogers [1998] for a discussion of the use of fractional Brownian motion in this setting, and Lacoste [1995] proposes a Wiener-Chaos

Part (C) of Exercise 5.10 was related to the author by Bruce Grundy. The line of exposition in this chapter is based on Gabay [1982] and Duffie [1988a]. For the case of transactions costs and other market "imperfections," see the Notes of Chapter 6.

Chapter 6

State Prices and Equivalent Martingale Measures

THIS CHAPTER SUMMARIZES arbitrage-free security pricing theory in the continuous-time setting introduced in Chapter 5. The main idea is the equivalence between no arbitrage, the existence of state prices, and the existence of an equivalent martingale measure, paralleling the discrete-state theory of Chapter 2. This extends the Markovian results of Chapter 5, which are based on PDE methods. For those interested mainly in applications, the first sections of Chapters 7 and 8 summarize the major conclusions of this chapter as a "black box," making it possible to skip this chapter on a first reading.

The existence of a state-price deflator is shown to imply the absence of arbitrage. Then a state-price "beta" model of expected returns is derived. Turning to equivalent martingale measures, we begin with the sufficiency of an equivalent martingale measure for the absence of arbitrage. Girsanov's Theorem (Appendix D) gives conditions under which there exists an equivalent martingale measure. This approach generates yet another proof of the Black-Scholes formula. State prices are then connected with equivalent martingale measures; the two concepts are more or less the same. They are literally equivalent in the analogous finite-state model of Chapter 2, and we will see that the distinction here is purely technical.

6A Arbitrage

We fix a standard Brownian motion $B = (B^1, \ldots, B^d)$ in $\mathbb{R}^d$, restricted to some time interval $[0, T]$, on a given probability space $(\Omega, \mathcal{F}, P)$. We also fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ of B, as defined in Section 5I. For simplicity, we take $\mathcal{F}$ to be $\mathcal{F}_T$. Suppose the price processes of N given securities form an Ito process $X = (X^{(1)}, \ldots, X^{(N)})$ in $\mathbb{R}^N$. We suppose that each security price process is in the space H^2 containing any Ito process Y with $dY_t = a(t)\,dt + b(t)\,dB(t)$ for which

$$E\left[\left(\int_0^t a(s)\,ds\right)^2\right] < \infty \quad \text{and} \quad E\left[\int_0^t b(s) \cdot b(s)\,ds\right] < \infty.$$

Until later in the chapter, we will suppose that the securities pay no dividends during the time interval $[0, T)$, and that X_T is the vector of cum-dividend security prices at time T.

A trading strategy θ, as we recall from Chapter 5, is an $\mathbb{R}^N$-valued process θ in $\mathcal{L}(X)$, as defined in Section 5I. This means simply that the stochastic integral $\int \theta\,dX$ defining trading gains is well defined. A trading strategy θ is *self-financing* if

$$\theta_t \cdot X_t = \theta_0 \cdot X_0 + \int_0^t \theta_s\,dX_s, \qquad t \leq T. \tag{1}$$

If there is some process r with the property that $\int_0^T |r_t|\,dt$ is finite almost surely and, for some security with strictly positive price process β we have

$$\beta_t = \beta_0 \exp\left(\int_0^t r_s\,ds\right), \qquad t \in [0, T], \tag{2}$$

then we call r the *short-rate process*. In this case, $d\beta_t = r_t \beta_t\,dt$, allowing us to view r_t as the riskless short-term continuously compounding rate of interest, in an instantaneous sense, and to view β_t as the market value of an account that is continually reinvested at the short-term interest rate r.

A self-financing strategy θ is an *arbitrage* if $\theta_0 \cdot X_0 < 0$ and $\theta_T \cdot X_T \geq 0$, or $\theta_0 \cdot X_0 \leq 0$ and $\theta_T \cdot X_T > 0$. Our main goal in this chapter is to characterize the properties of a price process X that admits no arbitrage, at least after placing some reasonable restrictions on trading strategies.

6B Numeraire Invariance

It is often convenient to renormalize all security prices, sometimes relative to a particular price process. This section shows that such a renormalization has essentially no economic effects. A *deflator* is a strictly positive Ito process. We can deflate the previously given security price process X by a deflator Y to get the new price process X^Y defined by $X_t^Y = X_t Y_t$.

Numeraire Invariance Theorem. *Suppose Y is a deflator. Then a trading strategy θ is self-financing with respect to X if and only if θ is self-financing with respect to X^Y.*

Proof: Let $W_t = \theta_0 \cdot X_0 + \int_0^t \theta_s \, dX_s$, $t \in [0,T]$. Let W^Y be the process defined by $W_t^Y = W_t Y_t$. Because W and Y are Ito processes, Ito's Formula implies, letting σ_X, σ_W, and σ_Y denote the respective diffusions of X, W, and Y, that

$$\begin{aligned}
dW_t^Y &= Y_t \, dW_t + W_t \, dY_t + \sigma_W(t) \cdot \sigma_Y(t) \, dt \\
&= Y_t \theta_t \, dX_t + (\theta_t \cdot X_t) \, dY_t + [\theta_t^\top \sigma_X(t)] \sigma_Y(t) \, dt \\
&= \theta_t \cdot [Y_t \, dX_t + X_t \, dY_t + \sigma_X(t) \sigma_Y(t) \, dt] \\
&= \theta_t \, dX_t^Y.
\end{aligned}$$

Thus, $\theta_t \cdot X_t^Y = \theta_0 \cdot X_0^Y + \int_0^t \theta_s \, dX_s^Y$ if and only if $\theta_t \cdot X_t = \theta_0 \cdot X_0 + \int_0^t \theta_s \, dX_s$, completing the proof. ∎

We have the following corollary, which is immediate from the Numeraire Invariance Theorem, the strict positivity of Y, and the definition of an arbitrage.

Corollary. *Suppose Y is a deflator. A trading strategy is an arbitrage with respect to X if and only if it is an arbitrage with respect to the deflated price process X^Y.*

6C State Prices and Doubling Strategies

Paralleling the terminology of Section 2C, a *state-price deflator* is a deflator π with the property that the deflated price process X^π is a martingale. Other terms used for this concept in the literature are *state-price density*, *marginal rates of substitution*, and *pricing kernel*. In the discrete-state discrete-time setting of Chapter 2, we found that there is a state-price deflator if and only if there is no arbitrage. In this chapter, we will see a sense in which this result is "almost" true, up to some technical issues. First, however, we need to establish some mild restrictions on trading strategies, as the following example shows.

Suppose we take B to be a single Brownian motion ($d = 1$) and take the price process $X = (S, \beta)$, where $\beta_t = 1$ for all t and $dS_t = S_t\, dB_t$, with $S_0 = 1$. Even though X is a martingale before deflation and S is a log-normal process as typically used in the Black-Scholes option pricing model, we are able to construct an arbitrage that produces any desired constant payoff $\alpha > 0$ at T with no initial investment. For this, consider the stopping time

$$\tau = \inf\left\{ t : \int_0^t (T - s)^{-1/2}\, dB_s = \alpha \right\}.$$

A source cited in the Notes shows that $0 < \tau < T$ almost surely. This is not surprising given the rate at which $(T - t)^{-1/2}$ "explodes" as t approaches T. Now consider the trading strategy θ defined by $\theta_t = (a_t, b_t)$, where

$$\begin{aligned} a_t &= \frac{1}{S_t \sqrt{T - t}}, \qquad t \le \tau \\ &= 0, \qquad t > \tau, \end{aligned} \tag{3}$$

and

$$b_t = -a_t S_t + \int_0^t a_u\, dS_u. \tag{4}$$

In effect, θ places a larger and larger "bet" on the risky asset as T approaches. It is immediate from (4) that $\theta = (a, b)$ is self-financing. It is also clear that $\theta_0 \cdot X_0 = 0$ and that $\theta_T \cdot X_T = \alpha$ almost surely. Thus θ is indeed an arbitrage, despite the natural assumptions on security prices. Some technical restrictions on trading strategies must be added if we are to expect some solid relationship between the existence of a state-price deflator and the absence of arbitrage.

For intuition, one may think of an analogy between the above example and a series of bets on fair and independent coin tosses at times 1/2, 3/4, 7/8, and so on. Suppose one's goal is to earn a riskless profit of α by time 1, where α is some arbitrarily large number. One can bet α on heads for the first coin toss at time 1/2. If the first toss comes up heads, one stops. Otherwise, one owes α to one's opponent. A bet of 2α on heads for the second toss at time $3/4$ produces the desired profit if heads comes up at that time. In that case, one stops. Otherwise, one is down 3α and bets 4α on the third toss, and so on. Because there is an infinite number of potential tosses, one will eventually stop with a riskless profit of α (almost surely), because the probability of losing on every one of an infinite number of tosses is $(1/2)\cdot(1/2)\cdot(1/2)\cdots = 0$. This is a classic "doubling strategy" that can be ruled out either by a technical limitation, such as limiting the total number of coin tosses, or by a credit restriction limiting the total amount that one is allowed to owe one's opponent.

For the case of continuous-time trading strategies, we will eliminate the possibility of "doubling strategies" by either of two approaches. One approach requires some extra integrability condition on θ, the other requires a credit constraint, limiting the extent to which the market value $\theta_t \cdot X_t$ of the portfolio may become negative. Given a state-price deflator π, a sufficient integrability condition is that $\theta \in \mathcal{H}^2(X^\pi)$, as defined in Section 5I. The credit constraint is that $\theta_t \cdot X_t^\pi$, the deflated market value of the trading strategy, is bounded below, in that there is some constant k with $\theta_t \cdot X_t^\pi \geq k$ for all t almost surely. We let $\underline{\Theta}(X^\pi)$ denote the space of such credit-constrained trading strategies. In the above example (3)–(4) of an arbitrage, $\pi \equiv 1$ defines a state-price deflator because X is itself a martingale, and the trading strategy θ defined by (3)–(4) is neither in $\mathcal{H}^2(X)$ nor in $\underline{\Theta}(X)$.

Proposition. *For any state-price deflator π, there is no arbitrage in $\mathcal{H}^2(X^\pi)$ or in $\underline{\Theta}(X^\pi)$.*

Proof: Suppose π is a state-price deflator. Let θ be any self-financing trading strategy.

Suppose, to begin, that θ is in $\mathcal{H}^2(X^\pi)$. Because X^π is a martingale, Proposition 5B implies that $E\left(\int_0^T \theta_t \, dX_t^\pi\right) = 0$. By numeraire invariance, θ is self-financing with respect to X^π, and we have

$$\theta_0 \cdot X_0^\pi = E\left(\theta_T \cdot X_T^\pi - \int_0^T \theta_t \, dX_t^\pi\right) = E\left(\theta_T \cdot X_T^\pi\right).$$

If $\theta_T \cdot X_T^\pi \geq 0$, then $\theta_0 \cdot X_0^\pi \geq 0$. Likewise, if $\theta_T \cdot X_T^\pi > 0$, then $\theta_0 \cdot X_0^\pi > 0$. It follows that θ cannot be an arbitrage for X^π. The Corollary to Theorem 6B implies that θ is not an arbitrage for X.

Now suppose that θ is in $\underline{\Theta}(X^\pi)$. Because X^π is a martingale, $\int \theta \, dX^\pi$ is a local martingale (as defined in Appendix D). By the Numeraire Invariance

Theorem, because θ is self-financing with respect to X, θ is also self-financing with respect to X^π. From this and the self-financing condition (1), we see that the deflated wealth process W, defined by $W_t = \theta_t \cdot X_t^\pi$, is a local martingale. Because $\theta \in \underline{\Theta}(X^\pi)$, we know that W is also bounded below. Because a local martingale that is bounded below is a supermartingale (Appendix D), we know that $E(W_T) \leq W_0$. From this, if $\theta_T \cdot X_T > 0$, then $W_T > 0$, so $W_0 > 0$ and thus $\theta_0 \cdot X_0 > 0$. Likewise, if $\theta_T \cdot X_T \geq 0$, then $W_T \geq 0$, so $W_0 \geq 0$ and thus $\theta_0 \cdot X_0 \geq 0$. This implies that θ is not an arbitrage. ▮

6D Expected Rates of Return

Suppose that π is a state-price deflator for X, and consider an arbitrary security with price process S. Because a state-price deflator is an Ito process, we can write

$$d\pi_t = \mu_\pi(t)\,dt + \sigma_\pi(t)\,dB_t$$

for appropriate μ_π and σ_π. Because S is an Ito process, we can also write $dS_t = \mu_S(t)\,dt + \sigma_S(t)\,dB_t$ for some μ_S and σ_S. As S^π is a martingale, its drift is zero. It follows from Ito's Formula that almost everywhere,

$$0 = \mu_\pi(t)S_t + \mu_S(t)\pi_t + \sigma_S(t) \cdot \sigma_\pi(t).$$

We suppose that S is a strictly positive process, and can therefore rearrange to get

$$\frac{\mu_S(t)}{S_t} = \frac{-\mu_\pi(t)}{\pi_t} - \frac{\sigma_S(t) \cdot \sigma_\pi(t)}{\pi_t S_t}. \tag{5}$$

The *cumulative-return process* of this security is the Ito process R defined by $R_0 = 0$ and

$$dR_t = \mu_R(t)\,dt + \sigma_R(t)\,dB_t \equiv \frac{\mu_S(t)}{S_t}\,dt + \frac{\sigma_S(t)}{S_t}\,dB_t.$$

We can now write $dS_t = S_t\,dR_t$. Looking back at equations (5.3) and (5.4), μ_R may be viewed as the conditional expected rate of return, and $\sigma_R(t) \cdot \sigma_R(t)$ as the rate of change in the conditional variance of the return. We re-express (5) as

$$\mu_R(t) - \widehat{r}_t = -\frac{1}{\pi_t}\sigma_R(t) \cdot \sigma_\pi(t), \tag{6}$$

where $\widehat{r}_t = -\mu_\pi(t)/\pi_t$. In the sense of equation (5.33), $\sigma_R(t)\cdot\sigma_\pi(t)$ is a notion of "instantaneous covariance" of the increments of R and π. Thus (6) is reminiscent of the results of Section 1F. If $\sigma_R(t) = 0$, then $\mu_R(t) = \widehat{r}_t$, so the short-term riskless rate process, if there is one, must be $\widehat{r}$. In summary, (6) implies a sense in which excess expected rates of return are proportional to the "instantaneous" conditional covariance between returns and state prices. The constant of proportionality, $-1/\pi_t$, does not depend on the security. This interpretation is a bit loose, but (6) itself is unambiguous.

We have been unnecessarily restrictive in deriving (6) only for a particular security. The same formula applies in principle to the return on an arbitrary self-financing trading strategy θ. In order to define this return, let W^θ denote the associated *market-value process*, defined by $W_t^\theta = \theta_t \cdot X_t$. If W^θ is strictly positive, then the cumulative-return process R^θ for θ is defined by

$$R_t^\theta = \int_0^t \frac{1}{W_s^\theta}\, dW_s^\theta, \qquad t \in [0, T\,].$$

It can be verified as an exercise that the drift μ_θ and diffusion σ_θ of R^θ satisfy the return restriction extending (6), given by

$$\mu_\theta(t) - \widehat{r}_t = -\frac{1}{\pi_t}\sigma_\theta(t)\cdot\sigma_\pi(t). \tag{7}$$

We will now see that (7) leads to a "beta model" for expected returns, analogous to that of Chapter 2. We can always find adapted processes φ and ϵ valued in $\mathbb{R}^N$ and $\mathbb{R}^d$ respectively such that

$$\sigma_\pi(t) = \sigma_X(t)^\top\varphi_t + \epsilon_t \qquad \text{and} \qquad \sigma_X(t)\epsilon_t = 0, \qquad t \in [0, T],$$

where σ_X is the $\mathbb{R}^{N\times d}$-valued diffusion of the price process X. For each (ω, t) in $\Omega\times[0,T]$, the vector $\sigma_X(\omega,t)^\top\varphi(\omega,t)$ is the orthogonal projection in $\mathbb{R}^N$ of $\sigma_\pi(\omega,t)$ onto the span of the rows of the matrix $\sigma_X(\omega,t)$. Suppose $\theta = (\theta^{(1)},\ldots,\theta^{(N)})$ is a self-financing trading strategy with $\sigma_X^\top\theta = \sigma_X^\top\varphi$. (For example, if $X_t^{(1)} = \exp(\int_0^t r_s\, ds)$ for a short-rate process r, we can construct θ by letting $\theta_t^{(j)} = \varphi_t^{(j)}$, $j > 1$, and by choosing $\theta^{(1)}$ so that the self-financing condition is met.) The market-value process W^θ of θ is an Ito process because θ is self-financing. We suppose that θ_0 can be chosen so that W^θ is also strictly positive, implying that the associated return process $R^* \equiv R^\theta$ is well defined. Because the diffusion of W^θ is $\sigma_X^\top \varphi$, the diffusion of R^* is $\sigma^* \equiv \sigma_X^\top \varphi / W^\theta$. For an arbitrary Ito return process R, (6) implies that

$$\begin{aligned}
\mu_R(t) - \widehat{r}_t &= -\frac{1}{\pi_t}\,\sigma_R(t)\cdot\sigma_\pi(t) \\
&= -\frac{1}{\pi_t}\,\sigma_R(t)\cdot[\sigma_X(t)^\top \varphi_t + \epsilon_t] \\
&= -\frac{W_t^\theta}{\pi_t}\,\sigma_R(t)\cdot\sigma_t^*,
\end{aligned}$$

using the fact that $\sigma_R(\omega, t)$ is (in each state ω) a linear combination of the rows of $\sigma_X(\omega, t)$. This in turn implies that $\sigma_R(t)\cdot\epsilon_t = 0$. In particular, for the return process R^*, we have

$$\mu_t^* - \widehat{r}_t = \frac{-W_t^\theta}{\pi_t}\,\sigma_t^*\cdot\sigma_t^*,$$

where μ^* is the drift (expected rate of return) of R^*. Substituting back into (6) the resulting expression for W_t^θ/π_t leaves the *state-price beta model of returns* given by

$$\mu_R(t) - \widehat{r}_t = \beta_R(t)\,(\mu_t^* - \hat{r}_t), \tag{8}$$

where

$$\beta_R(t) = \frac{\sigma_R(t)\cdot\sigma_t^*}{\sigma_t^*\cdot\sigma_t^*}.$$

In the "instantaneous sense" in which $\sigma_t^*\cdot\sigma_t^*$ stands for the conditional variance for dR_t^* and $\sigma_R(t)\cdot\sigma_t^*$ stands for the conditional covariance between dR_t and dR_t^*, we can view (8) as the continuous-time analogue to the state-price beta models of Section 1F and Exercise 2.6(C). Likewise, we can loosely think of R^* as a return process whose increments have maximal conditional correlation with the increments of the state-price deflator π.

6E Equivalent Martingale Measures

A probability measure Q on $(\Omega, \mathcal{F})$ is said to be *equivalent* to P provided, for any event A, we have $Q(A) > 0$ if and only if $P(A) > 0$. An equivalent probability measure Q is an *equivalent martingale measure* for the price process X of the N securities if X is a martingale with respect to Q, and if the Radon-Nikodym derivative $\frac{dQ}{dP}$ (defined in Appendix C) has finite variance. The finite-variance condition is a technical convenience that is not uniformly adopted in the literature cited in the Notes on equivalent martingale measures. An equivalent martingale measure is sometimes referred to as a "risk-neutral" measure.

In the finite-state setting of Chapter 2, it was shown that the existence of a state-price deflator is equivalent to the existence of an equivalent martingale measure (after some deflation). Later in this chapter, we will see technical conditions sustaining that equivalence in this continuous-time setting. Aside from offering a conceptual simplification of some asset-pricing and investment problems, the use of equivalent martingale measures is justified by the large body of useful properties of martingales that can be applied to simplify reasoning and calculations.

First, we establish the sufficiency of an equivalent martingale measure for the absence of arbitrage. We later show that a technical strengthening of the no-arbitrage condition of the following Theorem implies the existence of an equivalent martingale measure. Aside from technical issues, the arguments are the same as those used to show this equivalence in Chapter 2. As in Section 6C, we need to apply an integrability condition or a credit constraint to trading strategies.

Theorem. *If the price process X admits an equivalent martingale measure, then there is no arbitrage in $\mathcal{H}^2(X)$ or in $\underline{\Theta}(X)$.*

Proof: The proof is quite similar to that of Proposition 6C. Let Q be an equivalent martingale measure. Let θ be any self-financing trading strategy.

The idea of the proof is based on the case in which θ is bounded, which we assume for the moment. The fact that X is a martingale under Q implies that $E^Q(\int_0^T \theta_t \, dX_t) = 0$. The self-financing condition (1) therefore implies that

$$\theta_0 \cdot X_0 = E^Q\left(\theta_T \cdot X_T - \int_0^T \theta_t \, dX_t\right) = E^Q(\theta_T X_T).$$

Thus, if $\theta_T \cdot X_T \geq 0$, then $\theta_0 \cdot X_0 \geq 0$. Likewise, if $\theta_T \cdot X_T > 0$, then $\theta_0 \cdot X_0 > 0$. An arbitrage is therefore impossible using bounded trading strategies.

For the case of any self-financing trading strategy $\theta \in \mathcal{H}^2(X)$, additional technical arguments are needed to show that $E^Q(\int_0^T \theta_t \, dX_t) = 0$. As X is an Ito process, we can write $dX_t = \mu_t \, dt + \sigma_t \, dB_t$ for appropriate μ and σ. By the Diffusion Invariance Principle (Appendix D), there is a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q such that $dX_t = \sigma_t \, dB_t^Q$. Let $Y = \int_0^T \|\theta_t \sigma_t\|^2 \, dt$.

Because θ is in $\mathcal{H}^2(X)$, Y has finite expectation under P. The product of two random variables of finite variance is of finite expectation, so $\frac{dQ}{dP}\sqrt{Y}$ is also of finite expectation under P. Thus, $E^Q(\sqrt{Y}) < \infty$. Proposition 5B then implies that $\int \theta_t \, dX_t$ is a Q-martingale, so $E^Q(\int_0^T \theta_t \, dX_t) = 0$. The remainder of the proof for this case is covered by the arguments used for bounded θ.

For the case of $\theta \in \underline{\Theta}(X)$, the arguments used in the proof of Proposition 6C imply that the wealth process W, defined by $W_t = \theta_t \cdot X_t$, is a supermartingale under Q, so that $E^Q(\theta_T \cdot X_T) \leq \theta_0 \cdot X_0$, implying that θ cannot be an arbitrage. ∎

In most cases, the theorem is applied along the lines of the following corollary, a consequence of the corollary to the Numeraire Invariance Theorem of Section 6B.

Corollary. *If there is a deflator Y such that the deflated price process X^Y admits an equivalent martingale measure, then there is no arbitrage in $\mathcal{H}^2(X^Y)$ or $\underline{\Theta}(X^Y)$.*

If there is a short rate process r, it is typical in applications to take the deflator Y defined by $Y_t = \exp\left(-\int_0^t r_s \, ds\right)$. If r is bounded, then we have $\mathcal{H}^2(X^Y) = \mathcal{H}^2(X)$ and $\underline{\Theta}(X^Y) = \underline{\Theta}(X)$, so the previous result can be stated in a more natural form.

6F State Prices and Martingale Measures

We now investigate the relationship between equivalent martingale measures and state-price deflators. They turn out to be effectively the same concept. We take as given the setup of Section 6A, including a price process X for N securities.

For a probability measure Q equivalent to P, the *density process* ξ for Q is the martingale defined by

$$\xi_t = E_t\left(\frac{dQ}{dP}\right), \qquad t \in [0, T], \tag{9}$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P. As stated in Appendix C, for any times t and $s > t$, and any $\mathcal{F}_s$-measurable random variable W such that $E^Q(|W|) < \infty$,

$$E_t^Q(W) = \frac{E_t(\xi_s W)}{\xi_t}, \qquad t \in [0, T]. \tag{10}$$

Proposition. *Suppose there is a short-rate process r and let Y be defined by $Y_t = \exp\left(-\int_0^t r_s\, ds\right)$. Suppose, after deflation by Y, that there is an equivalent martingale measure with density process ξ. Then a state-price deflator π is defined by $\pi_t = \xi_t Y_t$, provided $\mathrm{var}(\pi_t) < \infty$ for all t. Conversely, suppose π is a state-price deflator and let ξ be defined by*

$$\xi_t = \exp\left(\int_0^t r_s\, ds\right)\frac{\pi_t}{\pi_0}, \qquad t \in [0, T]. \tag{11}$$

Then, provided $\mathrm{var}(\xi_T)$ is finite, ξ is the density process for an equivalent martingale measure.

Proof: Suppose, after deflation by Y, that there is an equivalent martingale measure Q with density process ξ. Let $\pi = \xi Y$. Then, for any times t and $s > t$, using (10),

$$E_t(\pi_s X_s) = E_t(\xi_s X_s^Y) = \xi_t E_t^Q(X_s^Y) = \xi_t X_t^Y = \pi_t X_t. \tag{12}$$

(These expectations exist because both X_s and π_s have finite variances.) This shows that X^π is a martingale, so π is indeed a state-price deflator.

Conversely, suppose π is a state-price deflator, and let ξ be defined by (9). By applying the definition of a state-price deflator to the price process $\beta = 1/Y$, we see that ξ is a strictly positive martingale with

$$E(\xi_T) = E(\beta_T \pi_T) = \beta_0 = 1,$$

so ξ is indeed the density of some equivalent probability measure Q. It is provided in the statement of the proposition that $\xi_T = \frac{dQ}{dP}$ has finite variance. We need only show that X^Y is a Q-martingale, but this follows by applying (12). ∎

The general equivalence between state-price deflators and equivalent martingale measures was shown in the simpler setting of Chapter 2 without technical qualification. An exercise further pursues the equivalence in this setting.

6G Girsanov and Market Prices of Risk

We now look for convenient conditions on X supporting the existence of an equivalent martingale measure. We will also see how to calculate such a measure, and conditions for the uniqueness of such a measure.

Suppose Q is any given probability measure equivalent to P, with density process ξ. By the martingale representation theorem (Appendix D), we can express ξ as a stochastic integral of the form

$$d\xi_t = \gamma_t \, dB_t,$$

for some adapted process $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(d)})$ with $\int_0^T \gamma_t \cdot \gamma_t \, dt < \infty$. *Girsanov's Theorem* (Appendix D) states that a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q is defined by $B_0^Q = 0$ and $dB_t^Q = dB_t - \eta_t \, dt$, where $\eta_t = -\gamma_t/\xi_t$. Because X is an Ito process, we can write $dX_t = \mu_t \, dt + \sigma_t \, dB_t$, and therefore

$$dX_t = (\mu_t - \sigma_t \eta_t) \, dt + \sigma_t \, dB_t^Q.$$

As X is in H^2, it is necessary and sufficient for X to be a Q-martingale that its drift is zero, which means that, almost everywhere,

$$\sigma(\omega, t)\eta(\omega, t) = \mu(\omega, t), \qquad (\omega, t) \in \Omega \times [0, T]. \tag{13}$$

Thus, the existence of a solution η to the system (13) of linear equations (almost everywhere) is necessary for the existence of an equivalent martingale measure for X. Under additional technical conditions, we will find that it is also sufficient.

We can also view a solution η to (13) as providing a proportional relationship between mean rates of change of prices (μ) and the amounts (σ) of "risk" in price changes stemming from the underlying d Brownian motions. For this reason, any such solution η is called a *market-price-of-risk process* for X. The idea is that $\eta_i(t)$ is the "unit price," measured in price drift, of bearing exposure to the increment of $B^{(i)}$ at time t.

A *numeraire deflator* is a deflator that is the reciprocal of the price process of one of the securities. It is usually the case that one first chooses some numeraire deflator Y, and then calculates the market price of risk for the deflated price process X^Y. This is technically convenient because one of the securities, the "numeraire," has a price that is always 1 after such a deflation. If there is a short-rate process r, a typical numeraire deflator is given by Y, where $Y_t = \exp\left(-\int_0^t r_s \, ds\right)$.

If there is no market price of risk, one may guess that something is "wrong," as the following result confirms.

Lemma. *Suppose, for some numeraire deflator Y, that there is no market-price-of-risk process for X^Y. Then there are arbitrages in both $\underline{\Theta}(X^Y)$ and $\mathcal{H}^2(X^Y)$, and there is no equivalent martingale measure for X^Y.*

Proof: Suppose X^Y has drift process μ^Y and diffusion σ^Y, and that there is no solution η to $\sigma^Y \eta = \mu^Y$. Then, as a matter of linear algebra, there exists an adapted process θ taking values that are row vectors in $\mathbb{R}^N$ such that $\theta\sigma^Y \equiv 0$ and $\theta\mu^Y \neq 0$. By replacing $\theta(\omega, t)$ with zero for any (ω, t) such that $\theta(\omega, t)\mu^Y(\omega, t) < 0$, we can arrange to have $\theta\mu^Y > 0$. (This works provided the resulting process θ is not identically zero; in that case the same procedure applied to $-\theta$ works.) These properties for θ are preserved after multiplication by any strictly positive adapted "scaling" process, so we can assume without loss of generality that θ is in $\mathcal{H}^2(X^Y)$. Finally, because the numeraire security associated with the deflator has a price that is identically equal to 1 after deflation, we can also choose the trading strategy for the numeraire so that, in addition to the above properties, θ is self-financing. That is, assuming without loss of generality that the numeraire security is the first security, we can adopt the same idea used in (4) and let

$$\theta_t^{(N)} = -\left[\sum_{i=1}^{N-1} \theta_t^{(i)} X_t^{Y,(i)} + \int_0^t \theta_s^{(i)}\, dX_s^{Y,(i)}\right].$$

It follows that θ is a self-financing trading strategy in $\mathcal{H}^2(X^Y)$, with $\theta_0 \cdot X_0^Y = 0$, whose wealth process W, defined by $W_t = \theta_t \cdot X_t^Y$, is increasing and not constant. In particular, θ is in $\underline{\Theta}(X^Y)$. It follows that θ is an arbitrage for X^Y, and therefore (by Numeraire Invariance) for X.

Finally, the reasoning leading to (13) implies that if there is no market-price-of-risk process, then there can be no equivalent martingale measure for X^Y. ∎

If $\sigma(\omega, t)$ is of rank less than d, there can be multiple solutions $\eta(\omega, t)$ to (13). There may, therefore, be more than one market-price-of-risk process. If there is at least one market price of risk, we can always single out for convenience the unique "minimum-norm" market price of risk, denoted η^X. For concreteness, we could construct η^X defined as follows. Let $\hat{\sigma}_t$ and $\hat{\mu}_t$ be obtained, respectively, by eliminating (ω by ω) as many linearly dependent rows from σ_t as possible and by eliminating the corresponding elements of

μ_t. Regardless of how this is done, if X has a market price of risk, then a particular market price of risk η^X is uniquely defined by

$$\eta_t^X = \hat{\sigma}_t^\top \left(\hat{\sigma}_t \hat{\sigma}_t^\top\right)^{-1} \hat{\mu}_t.$$

For any $\mathbb{R}^d$-valued adapted process η, we let

$$\nu(\eta)_t = \frac{1}{2} \int_0^t \eta_s \cdot \eta_s \, ds,$$

and, if $\nu(\eta)_T$ is finite almost surely, we let

$$\mathcal{E}(-\eta)_t = \exp\left[-\int_0^t \eta_s \, dB_t - \nu(\eta)_t\right], \quad 0 \leq t \leq T.$$

The process $\mathcal{E}(-\eta)$ is known as the *stochastic exponential* of $-\eta$. Lett ξ be defined by $\xi_t = \mathcal{E}(-\eta)_t$, Ito's Formula implies that

$$d\xi_t = -\xi_t \eta_t \, dB_t,$$

so ξ is a local martingale. Novikov's Condition (Appendix D), which is a sufficient technical condition for ξ to be a martingale, is that $\exp[\nu(\eta)_T]$ has a finite expectation. If Novikov's Condition holds and $\mathcal{E}(-\eta)_T$ has moreover finite variance, then we say that η is L^2*-reducible*. For example, it suffices that η is bounded. It can be shown as an exercise that if η is L^2-reducible, then so is η^X, so it suffices to check this property for η^X.

All of this sets up L^2-reducibility as a convenient condition for the existence of an equivalent martingale measure.

Theorem. *If X has market price of risk process that is L^2-reducible (it suffices that there is a bounded market price of risk for X), then there is an equivalent martingale measure for X, and there is no arbitrage in $\mathcal{H}^2(X)$ or $\underline{\Theta}(X)$.*

Proof: If X has an L^2-reducible market price of risk, we let $\xi = \mathcal{E}(-\eta^X)$. By Novikov's Condition, ξ is a positive martingale. We have $\xi_0 = \mathcal{E}(-\eta)_0 = e^0 = 1$, so ξ is indeed the density process of an equivalent probability measure Q defined by $\frac{dQ}{dP} = \xi_T$. By assumption, the variance of $\frac{dQ}{dP}$ is finite. It remains to show that X is Q-martingale.

By Girsanov's Theorem, a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q is defined by $dB_t^Q = dB_t - \eta_t\, dt$. Thus $dX_t = \sigma_t\, dB_t^Q$. As $\frac{dQ}{dP}$ has finite variance and each security price process $X^{(i)}$ is in H^2, we know that

$$E^Q\left[\left(\int_0^T \sigma^{(i)}(t)\cdot\sigma^{(i)}(t)\ dt\right)^{1/2}\right] < \infty,$$

by the same argument used in the proof of Theorem 6E. Thus, $X^{(i)}$ is a Q-martingale by Proposition 5B, and Q is therefore an equivalent martingale measure. The lack of arbitrage in $\mathcal{H}^2(X)$ or $\underline{\Theta}(X)$ follows from Theorem 6E. ▮

Putting this result together with the previous lemma, we see that the existence of a market-price-of-risk process is necessary and, coupled with a technical integrability condition, sufficient for the absence of "well-behaved" arbitrages and the existence of an equivalent martingale measure.

For uniqueness of equivalent martingale measures, we can use the fact that for any such measure Q, Girsanov's Theorem implies that we must have $\frac{dQ}{dP} = \mathcal{E}(-\eta)_T$, for some market price of risk η. If $\sigma(\omega, t)$ is of maximal rank d, however, there can be at most one solution $\eta(\omega, t)$ to (13). This maximal rank condition is equivalent to the condition that the span of the rows of $\sigma(\omega, t)$ is all of $\mathbb{R}^d$, which is reminiscent of the uniqueness condition for equivalent martingale measures found in Chapter 2.

Proposition. *If rank(σ) $= d$ almost everywhere, then there is at most one market price of risk and at most one equivalent martingale measure. If there is a unique market-price-of-risk process, then rank(σ) $= d$ almost everywhere.*

6H Black-Scholes Again

Suppose the given security-price process is $X = (S^{(1)}, \ldots, S^{(N-1)}, \beta)$, where, for $S = (S^{(1)}, \ldots, S^{(N-1)})$,

$$dS_t = \mu_t\, dt + \sigma_t\, dB_t$$

and

$$d\beta_t = r_t \beta_t\, dt; \qquad \beta_0 > 0,$$

where μ, σ, and r are adapted processes (valued in $\mathbb{R}^{N-1}$, $\mathbb{R}^{(N-1)\times d}$, and $\mathbb{R}$ respectively). We also suppose for technical convenience that the short-rate process r is bounded. Then $Y = \beta^{-1}$ is a convenient numeraire deflator, and we let $Z = SY$. By Ito's Formula,

$$dZ_t = \left(-r_t Z_t + \frac{\mu_t}{\beta_t}\right) dt + \frac{\sigma_t}{\beta_t}\, dB_t.$$

In order to apply Theorem 6G to the deflated price process $\hat{X} = (Z, 1)$, it would be enough to know that Z has an L^2-reducible market price of risk. Given this, there would be an equivalent martingale measure Q and no arbitrage in $\mathcal{H}^2(X)$ or $\underline{\Theta}(X)$. Suppose, for the moment, that this is the case. By the Diffusion Invariance result of Appendix D, there is a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q such that

$$dZ_t = \frac{\sigma_t}{\beta_t}\, dB_t^Q.$$

Because $S = \beta Z$, another application of Ito's Formula yields

$$dS_t = r_t\, S_t\, dt + \sigma_t\, dB_t^Q. \tag{14}$$

Equation (14) is an important intermediate result for arbitrage-free asset pricing, giving an explicit expression for security prices under a probability measure Q with the property that the "discounted" price process S/β is a martingale. For example, this leads to an easy recovery of the Black-Scholes formula, as follows.

Suppose that, of the securities with price processes $S^{(1)}, \ldots, S^{(N-1)}$, one is a call option on another. For convenience, we denote the price process of the call option by U and the price process of the underlying security by V, so that $U_T = (V_T - K)^+$, for expiration at time T with some given exercise price K. Because UY is by assumption a martingale under Q, we have

$$U_t = \beta_t E_t^Q \left(\frac{U_T}{\beta_T}\right) = E_t^Q \left[\exp\left(-\int_t^T r_s\, ds\right)(V_T - K)^+\right]. \tag{15}$$

The reader is asked to verify as an exercise that this is the Black-Scholes formula for the case of $d = 1$, $N = 3$, $V_0 > 0$, and with constants $\overline{r}$ and non-zero $\overline{\sigma}$ such that for all t, $r_t = \overline{r}$ and $dV_t = V_t \mu_V(t)\, dt + V_t \overline{\sigma}\, dB_t$, where μ_V is a bounded adapted process. Indeed, in this case, Z has an L^2-reducible market price of risk process, an exercise, so the assumption of an equivalent martingale measure is justified. To be more precise, it is sufficient for the absence of arbitrage that the option-price process is given by (15). Necessity of the Black-Scholes formula for the absence of arbitrages in $\mathcal{H}^2(X)$ or $\underline{\Theta}(X)$ is formally addressed in Section 6J. We can already see, however, that the expectation in (15) defining the Black-Scholes formula does not depend on which equivalent martingale measure Q one chooses, so one should expect that the Black-Scholes formula (15) is also necessary for the absence of arbitrage. If (15) is not satisfied, for instance, there cannot be an equivalent martingale measure for S/β. Unfortunately, and for purely technical reasons, this is not enough to imply directly the necessity of (15) for the absence of well-behaved arbitrage, because we do not have a precise equivalence between the absence of arbitrage and the existence of equivalent martingale measures. Section 6J shows that other methods can be used to show necessity.

In the Black-Scholes setting, we have at most one equivalent martingale measure because $\overline{\sigma}$ is non-zero, implying that σ is of maximal rank $d = 1$ almost everywhere. Thus, from Proposition 6G, there is exactly one equivalent martingale measure.

The detailed calculations of Girsanov's Theorem appear nowhere in the actual solution (14) for the "risk-neutral behavior" of arbitrage-free security prices, which can be given by inspection in terms of σ and r only. The results extend to the case of an infinite horizon under technical conditions given in sources cited in the Notes.

6I Complete Markets

We say that a random variable W can be *replicated* by a self-financing trading strategy θ if it is obtained as the terminal value $W = \theta_T \cdot X_T$. Our basic objective in this section is to give a simple spanning condition on the diffusion σ of the price process X under which, up to technical integrability conditions, any random variable can be replicated (without resporting to "doubling strategies").

Proposition. *Suppose Y is a numerator deflator and Q is an equivalent martingale measure for the deflated price process X^Y. Suppose the diffusion σ^Y of X^Y is of full rank d almost everywhere. Let W be any random variable with $E^Q(|WY|) < \infty$. Then there is a self-financing trading strategy θ that replicates W and whose deflated market-value process $\{\theta_t \cdot X_t^Y : 0 \leq t \leq T\}$ is a Q-martingale.*

Proof: We can suppose that, without loss of generality, the numeraire is the last of the N securities and write $X^Y = (Z, 1)$. Let B^Q be the standard Brownian motion in $\mathbb{R}^d$ under Q obtained by Girsanov's Theorem. Because B^Q has the martingale representation property under Q, there is some φ such that

$$E_t^Q(WY_T) = E^Q(WY_T) + \int_0^t \varphi_s \, dB_s^Q, \quad t \in [0,T]. \tag{16}$$

By the rank assumption on σ^Y, there are adapted processes $\theta^{(1)}, \ldots, \theta^{(N-1)}$ solving

$$(\theta_t^{(1)}, \ldots, \theta_t^{(N-1)})\sigma_t^Y = \varphi_t^\top, \qquad t \in [0,T]. \tag{17}$$

Let $\theta^{(N)}$ be defined by

$$\theta_t^{(N)} = E^Q(WY_T) + \sum_{i=1}^{N-1} \left(\int_0^t \theta_s^{(i)} \, dZ_s^{(i)} - \theta_t^i Z_t^{(i)} \right). \tag{18}$$

Then $\theta = (\theta^{(1)}, \ldots, \theta^{(N)})$ is self-financing and $\theta_T \cdot X_T^Y = WY_T$. By the Numeraire Invariance Theorem, θ is also self-financing with respect to X and $\theta_T \cdot X_T = W$. As $\int \varphi \, dB^Q$ is by construction a Q-martingale, (16)-(18) imply that $\{\theta_t \cdot X_t^Y : 0 \le t \le T\}$ is a Q-martingale. ∎

In order to further explore the dynamic spanning properties of the price process X, we let $\Theta(X)$ denote the space of self-financing trading strategies in $\mathcal{H}^2(X)$. The *marketed space* of X is

$$M(X) = \{\theta_T \cdot X_T : \theta \in \Theta(X)\}.$$

We know that $M(X)$ is a subset of $L^2(P)$, the space of all random variables with finite variance (and therefore finite expectation), because Θ is a subset of $\mathcal{H}^2(X)$. We say that markets are *complete* if the marketed space $M(X)$ is actually equal to the space $L^2(P)$. Our objective now is to extend Proposition 6I with necessary and sufficient conditions for complete markets.

To say that $M(X)$ is *closed* means that if $W_1, W_2, \ldots$ is a sequence in $M(X)$, and if W is some random variable such that $E[(W - W_n)^2] \to 0$, then W is also in $M(X)$. (This would mean that W is also replicated by some trading strategy in Θ.) For technical reasons, this closedness property is useful. The following result is from a source cited in the Notes.

Lemma. *Suppose that Y is a numeraire deflator and that there is a bounded market-price-of-risk process for X^Y. Then $M(X^Y)$ is closed.*

For the remainder of this section, we suppose there is a bounded short-rate process r, and let $X = (S^{(1)}, \ldots, S^{(N-1)}, \beta)$, as in Section 6H. We will work with the usual deflated price process X^Y, where $Y_t = \exp(-\int_0^t r_s \, ds)$. We can now exploit the previous lemma to obtain a simple condition for complete markets.

Theorem. *For $dX_t = \mu_t \, dt + \sigma_t \, dB_t$, suppose there is a bounded market-price-of-risk process for X^Y. Then markets are complete if and only if rank$(\sigma) = d$ almost everywhere.*

Proof: Let η be a bounded market-price-of-risk process for X^Y and Q be the associated equivalent martingale measure.

Suppose $\text{rank}(\sigma) = d$ almost everywhere. Let $Z = S^Y$, so that $X^Y = (Z, 1)$. Let B^Q be the standard Brownian motion in $\mathbb{R}^d$ under Q defined by $dB_t^Q = dB_t - \eta_t \, dt$. Let W be a bounded random variable.

By Ito's Formula, the diffusion σ^Y of $X^Y = XY$ has the same span as σ, (ω, t) by (ω, t), and is therefore of rank d almost everywhere. By Propositon 6I, there is a self-financing trading strategy θ such that $\theta_T \cdot X_T = W$, and whose deflated market-value process V is a Q-martingale. Because W is bounded and $dV_t = \varphi_t \, dB_t^Q$, where φ is given by (16), Proposition 5B implies that $\theta\sigma = \varphi$ is essentially bounded. (That is, there is a constant k such that, letting $\delta_t = 1$ whenever $\|\varphi_t\| \geq k$ and zero otherwise, we have $E(\int_0^T \delta_t \, dt) = 0$. By the definition of a market-price-of-risk process, $\theta\mu = \theta\sigma\eta$. By assumption, η is bounded, so $\theta\mu$ is essentially bounded, and therefore θ is in $\mathcal{H}^2(X)$. This proves that any bounded W can be replicated by some θ in $\Theta(X)$.

Now suppose that W is in $L^2(P)$. For each positive integer n, we approximate W with the bounded random variable W_n defined by $W_n(\omega) = W(\omega)$ whenever $|W(\omega)| \leq n$ and $W_n(\omega) = 0$ otherwise. As W_n is in the marketed space for all n, and because $E[(W - W_n)^2] \to 0$, we have W in $M(X)$ by the previous lemma. Thus $M(X) = L^2(P)$.

Conversely, suppose that it is not true that $\text{rank}(\sigma) = d$ almost everywhere. We will show that markets are not complete. By the rank assumption on σ and the fact that the diffusion σ^Y of X^Y and the diffusion σ of X have the same span for all (ω, t), there is some bounded adapted process φ such that there is no solution $\theta^{(1)}, \ldots, \theta^{(N-1)}$ to (17). Then there is no trading strategy θ in $\mathcal{H}^2(X)$ that is self-financing with respect to $(Z, 1)$ such that $\theta_T \cdot (Z_T, 1) = \int_0^T \varphi_t \, dB_t^Q$. By the Numeraire Invariance Theorem, there is no θ in $\Theta(X)$ with $\theta_T \cdot (S_T, \beta_T) = W$, where $W = \beta_T \int_0^T \varphi_t \, dB_t^Q$. Because φ, β, and η are bounded, W is in $L^2(P)$. ∎

6J Redundant Security Pricing

We return to the Black-Scholes example of Section 6H. We recall that the underlying Brownian motion B is 1-dimensional, and that there are two primitive securties with prices processes V and β, where V is a geometric Brownian motion and $\beta_t = e^{\overline{r}t}$ for a constant interest rate $\overline{r}$. For a market with these two securities alone, there is a bounded market-price-of-risk process. It follows that markets are complete, that there is an equivalent martingale measure Q after deflating by β, and that there is no arbitrage in $\Theta(X)$.

Now, consider an option at strike price K, paying $(V_T - K)^+$ at time T. We would like to conclude that the Black-Scholes Formula applies, meaning that the option has the price process

$$U_t = E_t^Q \left[e^{-\overline{r}(T-t)} (V_T - K)^+ \right].$$

In Section 6H, we showed that this pricing formula is sufficient for the absence of a well-behaved arbitrage with respect to (β, V, U). Now we show that this is the unique arbitrage-free price process for the option with that property. (This was already shown, in effect, in Chapter 5, but the following argument leads to a more general theorem.)

We proceed as follows. As V_T has finite variance, so does the option payoff $(V_T - K)^+$. Suppose, to set up a contradiction, that the actual option price process $\hat{U}$ is not U. For any constant $\epsilon > 0$, let A_ϵ^+ denote the event that $\hat{U}_t - U_t \geq \epsilon$ for some t in $[0, T]$. Let A_ϵ^- denote the event that $U_t - \hat{U}_t \geq \epsilon$ for some t in $[0, T]$. Because U and $\hat{U}$ are assumed to be different processes, there is some $\epsilon > 0$ such that at least one of the events A_ϵ^+ or A_ϵ^- has strictly positive probability. Without loss of generality, suppose that $P(A_\epsilon^+) > 0$, and let $\tau = \inf\{t : \hat{U}_t - U_t \geq \epsilon\}$, a stopping time that is valued in $[0, T]$ with strictly positive probability.

By Proposition 6I, there is a self-financing trading strategy $\theta = (\theta^{(0)}, \theta^{(1)})$ in $\mathcal{H}^2(\beta, V)$ that replicates $(V_T - K)^+$. From the fact that θ is self-financing, numeraire invariance, and the fact that Q is an equivalent martingale measure for $(1, V/\beta)$, we have

$$\theta_t^{(0)}\beta_t + \theta_t^{(1)}V_t = E_t^Q\left(e^{-\overline{r}(T-t)}(V_T - K)^+\right) = U_t.$$

Let φ be the trading strategy defined by $\varphi_t = 0$, $t < \tau$, and

$$\varphi_t = (\theta_t^{(0)} + e^{\overline{r}(t-\tau)}\epsilon, \theta_t^{(1)}, -1), \qquad t \geq \tau,$$

where $(\theta^{(0)}, \theta^{(1)})$ is the option-replicating strategy described above. It can be checked that φ is self-financing and that $\varphi_T \cdot (\beta_T, V_T, \hat{U}_T) > 0$, implying that φ is an arbitrage that is in $\mathcal{H}^2(\beta, V, U)$.

More broadly, given some general price process X for the N "primitive" securities, we say that a security with price process U is *redundant* if its final value U_T can be replicated by a trading strategy θ in $\Theta(X)$. Complete markets implies that any security (with finite-variance price process) is redundant.

Theorem. *Suppose X admits an equivalent martingale measure Q. Given X, consider a redundant security with price process U in H^2. Then $(X, U) \equiv (X^{(1)}, \ldots, X^{(N)}, U)$ admits no arbitrage in $\mathcal{H}^2(X, U)$ if and only if U is a Q-martingale.*

Proof: If U is a Q-martingale, then Q is an equivalent martingale measure for (X, U), implying no arbitrage in $\mathcal{H}^2(X, U)$ by Theorem 6E. Conversely, suppose U is not a Q-martingale. The arguments used for the preceding Black-Scholes case extend directly to this setting so as to imply the existence of an arbitrage in $\mathcal{H}^2(X, U)$. ∎

One would typically apply this result after deflation. In the definition of a redundant security, one could have as easily substituted the credit-constrained class $\underline{\Theta}(X, U)$ of trading strategies for $\mathcal{H}^2(X, U)$, allowing a like substitution in the statement of the theorem.

6K Martingale Measures From No Arbitrage

So far, we have exploited the existence of an equivalent martingale measure as a sufficient condition for the absence of well-behaved arbitrage. Now we turn to the converse issue: Does the absence of well-behaved arbitrages imply the existence of an equivalent martingale measure? In the finite-dimensional setting of Chapter 2, we know that the answer is always: "After a change of numeraire, yes." Only technicalities stand between this finite-dimensional equivalence and the infinite-dimensional case we face here. Because of these technicalities, this section can be skipped on a first reading.

Given a price process X for the N securities, suppose there is no arbitrage in $\Theta(X)$. Then, for each W in $M(X)$ (that is, each $W = \theta_T \cdot X_T$ for some θ in $\Theta(X)$), let $\psi(W) = \theta_0 \cdot X_0$ denote the unique initial investment required to obtain the payoff Z. We know that this function $\psi : M(X) \to \mathbb{R}$ is uniquely well defined because, if there are two trading strategies θ and φ in $\Theta(X)$ with $\theta_T \cdot X_T = \varphi_T \cdot X_T$ and $\theta_0 \cdot X_0 > \varphi_0 \cdot X_0$, then $\varphi - \theta$ is an arbitrage. The function ψ is linear because stochastic integration is linear. Finally, again from the absence of arbitrage, ψ is strictly increasing, meaning that $\psi(W) > \psi(W')$ whenever $W > W'$. The marketed space $M(X)$ is a *linear subspace* of $L^2(P)$ because, whenever $Z = \theta_T \cdot X_T$ and $W = \varphi_T \cdot X_T$ are in $M(X)$, then $aZ + bW$ is also in M for any constants a and b. (This follows from the fact that $a\theta + b\varphi$ is a self-financing strategy, using the linearity of stochastic integration.)

Although the existence of an equivalent martingale measure does not follow from the absence of arbitrage in $\mathcal{H}^2(X)$, we can resort to the notion of an *approximate arbitrage*: a sequence $\{Z_n\}$ in $M(X)$ with $\psi(Z_n) \leq 0$ for all n, such that there exists some sequence $\{Z'_n\}$ in $L^2(P)$ with $Z'_n \leq Z_n$ for all n, and with $E[(Z'_n - Z')^2] \to 0$ for some $Z' > 0$. The idea is that no Z_n has positive market value, yet Z_n is larger than Z'_n, which in turns converges to a positive, nonzero, random value. For example, suppose θ is an arbitrage in $\Theta(X)$ with $\theta_T \cdot X_T > 0$. Then the (trivial) sequence $\{Z_n\}$ defined by $Z_n = \theta_T \cdot X_T$ for all n is an approximate arbitrage. (Just take $Z'_n = \theta_T \cdot X_T$ for all n.) Provided there is a bounded short rate process, or under other weak assumptions, the absence of approximate arbitrage is indeed a stronger assumption than the absence of arbitrage in $\Theta(X)$, and the difference is only important (for technical reasons) in this infinite-dimensional setting. If we strengthen the assumption of no arbitrage in $\mathcal{H}^2(X)$ to the assumption of no approximate arbitrage, we can recover the existence of an equivalent martingale measure.

Proposition. *Suppose $X^{(1)} \equiv 1$. Then there is no approximate arbitrage for X if and only if there is an equivalent martingale measure for X.*

Proof: Suppose there is no approximate arbitrage. Then there is no arbitrage, and the pricing functional $\psi : M(X) \to \mathbb{R}$ for X is well defined, linear, and strictly increasing. By a technical result cited in the Notes, ψ can be extended to a strictly increasing linear functional $\Psi : L^2(P) \to \mathbb{R}$. By "extension," we mean that for any W in $M(X)$, $\Psi(W) = \psi(W)$. Because Ψ is increasing and linear, the Riesz Representation Theorem for $L^2(P)$ (Exercise 6.8) implies that there is a unique π in $L^2(P)$ such that

$$\Psi(W) = E(\pi W), \qquad W \in L^2(P).$$

Because $X^{(1)} \equiv 1$, we have $E(\pi X_T^{(1)}) = X_0^{(1)} = 1$, so $E(\pi) = 1$. Let Q be the probability measure defined by $\frac{dQ}{dP} = \pi$. Because Ψ is strictly increasing, $\pi \gg 0$, so Q is equivalent to P.

Obviously $X^{(1)}$ is a martingale. To show that $X^{(i)}$ is a Q-martingale for each $i > 1$, let τ be an arbitrary stopping time valued in $[0, T]$, and let θ be the trading strategy defined by

(a) $\theta^j = 0$ for $j \neq i$ and $j \neq 1$.

(b) $\theta_t^i = 1$ for $t \leq \tau$, and $\theta_t^i = 0$ for $t > \tau$.

(c) $\theta_t^1 = 0$ for $t \leq \tau$, and $\theta_t^1 = X_\tau^i$ for $t > \tau$.

This means that θ is the strategy of buying the i-th security at time 0 and selling it at time τ, holding the proceeds of the sale in the numeraire security. It is easily seen that θ is in $\Theta(X)$ and that $\theta_T \cdot X_T = X_\tau^{(i)}$, with initial investment $X_0^{(i)} = \psi(X_\tau^{(i)}) = E(\pi X_\tau^{(i)}) = E^Q(X_\tau^{(i)})$. This characterizes $X^{(i)}$ as a Q-martingale, by Doob's Optional Sampling Theorem (Appendix C). Thus, because $\frac{dQ}{dP}$ is of finite variance, Q is an equivalent martingale measure for X.

Conversely, if there is an equivalent martingale measure Q, then there is no arbitrage in $\mathcal{H}^2(X)$ by Theorem 6E, and the linear functional $\Psi : L^2(P) \to \mathbb{R}$ defined by $\Psi(W) = E^Q(W)$ is an extension of the pricing functional ψ. Suppose, for purposes of contradiction, that $\{W_n\}$ is an approximate arbitrage. Then there is some sequence $\{W_n'\}$ in $L^2(P)$ such that $E^Q(W_n') \leq E^Q(W_n) \leq 0$ and $E^Q(W_n')$ converges to a strictly positive number. This is impossible, so there is no approximate arbitrage. ∎

Corollary. *Suppose there is a bounded short-rate process r. Then there is no approximate arbitrage for X if and only if there is an equivalent martingale measure for the deflated price process X^Y, where $Y_t = \exp(-\int_0^t r_s\, ds)$.*

Proof: After deflation by Y, one of the securities has a price identically equal to 1. Because r is bounded, a trading strategy θ is an approximate arbitrage for X if and only if it is an approximate arbitrage for X^Y. The Theorem then applies. ▮

6L Arbitrage Pricing with Dividends

This section and the next extend the basic arbitrage-pricing approach to securities with dividends paid during $[0, T]$. Consider an Ito process D for the *cumulative dividend* of a security. This means that the cumulative total amount of dividends paid by the security until time t is D_t. For example, if $D_t = \int_0^t \delta_s \, ds$, then δ represents the *dividend-rate process*, as treated in Exercises 5.7, 5.8, and 5.9. Given a cumulative-dividend process D and the associated security-price process X, the *gain process* $G = X + D$ measures the total (capital plus dividend) gain generated by holding the security. A trading strategy is now defined to be a process θ in $\mathcal{L}(G)$, allowing one to define the stochastic integral $\int \theta \, dG$ representing the total gain generated by θ. By the linearity of stochastic integrals, if $\int \theta \, dX$ and $\int \theta \, dD$ are well defined, then $\int \theta dG = \int \theta \, dX + \int \theta \, dD$, once again the sum of capital gains and dividend gains.

Suppose we are given N securities defined by the price process $X = (X^1, \dots, X^N)$ and cumulative-dividend process $D = (D^1, \dots, D^N)$, with the associated gain process $G = X + D$. For now, we assume, for each j, that X^j and D^j are Ito processes in H^2. A trading strategy θ is *self-financing*, extending our earlier definition, if

$$\theta_t \cdot X_t = \theta_0 \cdot X_0 + \int_0^t \theta_s \, dG_s, \qquad t \in [0, T].$$

As before, an *arbitrage* is a self-financing trading strategy θ with $\theta_0 \cdot X_0 \leq 0$ and $\theta_T \cdot X_T > 0$, or with $\theta_0 \cdot X_0 < 0$ and $\theta_T \cdot X_T \geq 0$.

We can extend our earlier results characterizing security prices in the absence of arbitrage. An equivalent martingale measure for the dividend-price pair (D, X) is defined as an equivalent probability measure Q under which $G = X + D$ is a martingale, and such that $\frac{dQ}{dP}$ has finite variance. The existence of an equivalent martingale measure implies, by the same arguments used in the proof of Theorem 6E, that there is no arbitrage in $\mathcal{H}^2(G)$ or in $\underline{\Theta}(X)$.

Given a trading strategy θ, if there is an Ito process D^θ such that

$$D_t^\theta = \theta_0 \cdot X_0 + \int_0^t \theta_s \, dG_s - \theta_t \cdot X_t, \qquad t \in [0,T],$$

then we say that D^θ is the cumulative dividend process *generated* by θ. Suppose there exists an equivalent martingale measure Q for (D, X), and consider an additional security defined by the cumulative dividend process H and price process V. Both H and V are assumed to be in H^2. Suppose that the additional security is *redundant*, in that there exists some trading strategy θ in $\mathcal{H}^2(G)$ such that $D^\theta = H$ and $\theta_T \cdot X_T = V_T$. The absence of arbitrage involving all $N+1$ securities implies that, for all t, we have $V_t = \theta_t \cdot X_t$ almost surely. From this, the gain process $V + H$ of the redundant security is also a martingale under Q. The proof is a simple extension of that of Theorem 6J.

Under an equivalent martingale measure Q for (D, X), we have, for any time $t \in [0,T]$,

$$X_t + D_t = G_t = E_t^Q(G_T) = E_t^Q(X_T + D_T),$$

which implies that $X_t = E_t^Q(X_T + D_T - D_t)$. For example, if D is defined by $D_t = \int_0^t \delta_s \, ds$, then

$$X_t = E_t^Q \left(X_T + \int_t^T \delta_s \, ds \right). \tag{19}$$

Given the dividend-price pair (D, X), there should be no economic effect, in principle, from a change of numeraire given by a deflator Y. We can write $dY_t = \mu_Y(t)\, dt + \sigma_Y(t)\, dB_t$ for appropriate μ_Y and σ_Y, and $dD_t = \mu_D(t)\, dt + \sigma_D(t)\, dB_t$ for appropriate μ_D and σ_D. The *deflated cumulative dividend process* D^Y is defined by $dD_t^Y = Y_t \, dD_t + \sigma_D(t) \cdot \sigma_Y(t)\, dt$. The *deflated gain process* G^Y is defined by $G_t^Y = D_t^Y + X_t Y_t$. This leads to a slightly more general version of numeraire invariance, whose proof is left as an exercise.

Lemma (Numeraire Invariance). *Suppose θ is a trading strategy with respect to (D, X) that generatess an Ito dividend process D^θ. If Y is a deflator, then the deflated dividend process $(D^\theta)^Y$ is the dividend process generated by θ with respect to (D^Y, X^Y).*

The term "$\sigma_D(t) \cdot \sigma_Y(t)\, dt$" in the definition of dD_t^Y might seem puzzling at first. This term is in fact dictated by numeraire invariance. In all applications that appear in this book, however, we have either $\sigma_D = 0$ or $\sigma_Y = 0$, implying the more "obvious" definition $dD_t^Y = Y_t \, dD_t$, which can be intuitively treated as the dividend "increment" dD_t deflated by Y_t.

Suppose that $X = (S, \beta)$, with $S = (S^{(1)}, \ldots, S^{(N-1)})$ and

$$\beta_t = \beta_0 \exp\left(\int_0^t r_s \, ds\right); \qquad \beta_0 > 0,$$

where r is a bounded short-rate process. Consider the deflator Y defined by $Y_t = \beta_t^{-1}$. If, after deflation by Y, there is an equivalent martingale measure Q, then (19) implies the convenient pricing formula

$$S_t = E_t^Q \left[\exp\left(\int_t^T -r_u \, du\right) S_T + \int_t^T \exp\left(\int_t^s -r_u \, du\right) dD_s\right]. \tag{20}$$

Proposition 6I and Theorem 6J extend in the obvious way to this setting.

6M Lumpy Dividends and Term Structures

By means beyond the scope of this book, one can extend (20) to the case of finite-variance cumulative-dividend process of the form $D = Z + V - W$, for an Ito process Z and increasing adapted processes V and W that are right continuous. By *increasing*, we mean that $V_s \geq V_t$ whenever $s \geq t$. By *right continuous*, we mean that for any t, $\lim_{s \downarrow t} V_s = V_t$. The *jump* ΔV_t of V at time t, as depicted in Figure 6.1, is defined by $\Delta V_t = V_t - V_{t-}$, where $V_{t-} \equiv \lim_{s \uparrow t} V_s$ denotes the left limit. By convention, $V_{0-} = V_0 = 0$. The jump $\Delta D_t \equiv D_t - D_{t-}$ of the total dividend process D represents the lump-sum dividend paid at time t.

Each of the above implications of the absence of arbitrage for security prices has a natural extension to this case of "lumpy" dividends. In particular, (20) applies as stated, with $\int \theta \, dD$ defined by $\int \theta \, dZ + \int \theta \, dV - \int \theta \, dW$ whenever all three integrals are well defined, the first as a stochastic integral and latter two as *Stieltjes integrals*. A reader unfamiliar with the Stieltjes integral may consult sources given in the Notes. Happily, the stochastic integral and the Stieltjes integral coincide whenever both are well defined. In this book, we only consider applications that involve the following two trivial examples of the Stieltjes integral $\int \theta \, dV$.

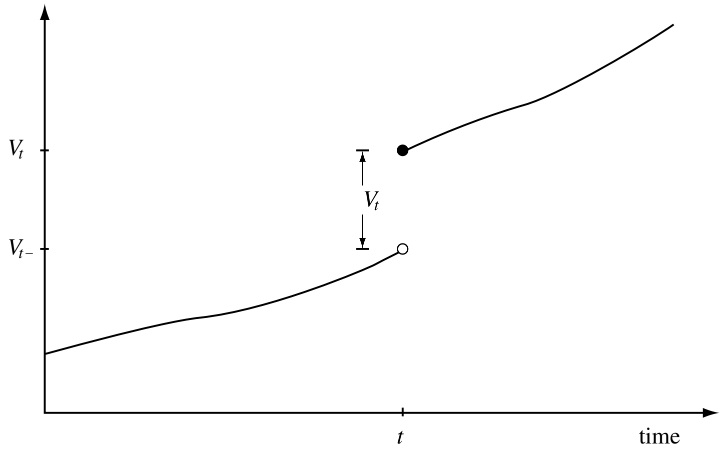

Figure6.1: A Right-Continuous Increasing Sample Path

(a) For the first example of a Stieltjes integral, we let $V = \int \delta_t \, dt$ for some δ in $\mathcal{L}^1$, in which case $\int_0^t \theta_s \, dV_s = \int_0^t \theta_s \delta_s \, ds$.

(b) In the second case, for some stopping time τ, we have $V_t = 0$, $t < \tau$, and $V_t = v$, $t \geq \tau$, where $v = \Delta V_\tau$ is the jump of V at time τ. For this second case, we have $\int_0^t \theta_s \, dV_s = 0$, $t < \tau$, and $\int_0^t \theta_s \, dV_s = \theta_\tau \Delta V_\tau$, $t \geq \tau$, which is natural for our purposes.

We continue to take (D, X) to be a dividend-price pair if $X + D$ is an Ito process. Because of the possibility of jumps in dividends, it is now necessary to take an explicit stance, however, on whether security prices will be measured *ex dividend* or *cum dividend*. We opt for the former convention, which means that for a dividend-price pair (D, X), a trading strategy θ is self-financing if

$$\theta_t \cdot (X_t + \Delta D_t) = \theta_0 \cdot X_0 + \int_0^t \theta_s \, dG_s, \qquad t \in [0, T],$$

where $G = X + D$. With this, an *arbitrage* is defined as self-financing trading strategy θ with $\theta_0 \cdot X_0 \leq 0$ and $\theta_T \cdot (X_T + \Delta D_T) > 0$, or with $\theta_0 \cdot X_0 < 0$ and $\theta_T \cdot (X_T + \Delta D_T) \geq 0$.

Extending our earlier definition to allow for lumpy dividends, a trading strategy θ finances a dividend process D^θ if D^θ is a right-continuous process satisfying

$$\theta_t \cdot (X_t + \Delta D_t) = \theta_0 \cdot X_0 + \int_0^t \theta_s \, dG_s - D^\theta_{t-}, \qquad t \in [0, T],$$

with $\Delta D^\theta_T = \theta_T \cdot (X_T + \Delta D_T)$.

With these new definitions in place, the *term structure* can be characterized from (20) as follows. Given a bounded short-rate process r, suppose that Q is an equivalent martingale measure after deflation by Y, where $Y_t = \exp(-\int_0^t r_s \, ds)$. A unit zero-coupon riskless bond maturing at time τ is defined by the cumulative-dividend process H with $H_s = 0$, $s < \tau$, and $H_s = 1$, $s \geq \tau$. Because $dH_s = 0$ for $s \neq \tau$, and because $\Delta H_\tau = 1$, we know from case (b) above of the Stieltjes integral that

$$\int_t^T \exp\left(\int_t^s -r_u \, du\right) dH_s = \exp\left(\int_t^\tau -r_u \, du\right).$$

Then (20) implies that the price at time t of a unit zero-coupon riskless bond maturing at time $\tau > t$ is given by

$$\Lambda_{t,\tau} = E^Q_t \left[\exp\left(\int_t^\tau -r_u \, du\right)\right]. \tag{21}$$

The solution for the term structure given by (21) is based on the implicit assumption that the price of a bond after its maturity date is zero. This is also consistent with our earlier analysis of option prices, where we have implicitly equated the terminal cum-dividend price of an option with its terminal dividend payment. For example, with an option expiring at T on a price process S with exercise price K, we set the terminal option price at its expiration value $(S_T - K)^+$. This seems innocuous. Had we actually allowed for the possibility that the terminal cum-dividend option price might be something other than $(S_T - K)^+$, however, we would have needed a more complicated model and further analysis to conclude from the absence of arbitrage that the $(S_T - K)^+$ is indeed the cum-dividend expiration value. This issue of terminal security prices is further pursued in a source cited in the Notes.

Exercises

Exercise 6.1 Provide the details left out of the proof provided for Lemma 6G.

Exercise 6.2 Verify relation (14).

Exercise 6.3 We let $(\Omega, \mathcal{F}, P)$ be a probability space on which is defined a standard Brownian motion B. We let $\{\mathcal{F}_t : 0 \le t \le T\}$ denote the standard filtration of B. Suppose the Ito price process S of an underlying asset is S, where

$$dS_t = S_t \mu_t \, dt + S_t \sigma_t \, dB_t,$$

where μ and σ are bounded, and σ is bounded away from zero. The short rate process r is bounded. The price process β, defined by $\beta_t = \exp\left(\int_0^t r_s \, ds\right)$, is that associated with investment at the short rate.

(A) Show that there is an equivalent martingale measure Q for the two securities defined by the normalized price process S/β of the underlying and the normalized value $1 = \beta/\beta$ for investment at the short rate. Compute $\frac{dQ}{dP}$, and provide the drift of S under Q.

(B) Consider a European call option on the underlying asset, with expiration at T, and strike K. If we rule out arbitrages that have a market value bounded below, provide an expression for the price process Y of the option.

(C) For the price process U that you found in part (B), show that

$$U_0 = c_1 Q_1(A) - c_2 K Q_2(A),$$

where A is the event that the option expires in the money, Q_1 and Q_2 are probability measures equivalent to P, and c_1 and c_2 are constants. All of c_1, c_2, Q_1, and Q_2 do not depend on the strike price K. Provide $\frac{dQ_i}{dQ}$ and the constants c_1 and c_2. Explain why c_1 and c_2 can often be obtained without calculation.

(D) Show that the solution U_0 for the option price given in part (C) corresponds to the Black-Scholes option pricing formula in the case of constant r and σ. Hint: Use Girsanov's formula for the distribution of $\log S_T$ under Q_i for each i.

(E) Provide an explicit solution U_0 for the option price given in part (C), for the case in which r and σ are deterministic (but not necessarily constant), expressing the solution in terms of the Black-Scholes formula with an adjusted interest rate parameter and volatility parameter.

Exercise 6.4 Prove Theorem 6J.

Exercise 6.5 Suppose that the return process R^θ for a self-financing trading strategy θ is well defined as an Ito process, as at the end of Section 6D. Show, as claimed there, that R^θ satisfies the state-price restriction (6).

Exercise 6.6 Extend the arguments of Sections 6I and Section 6J to the case of intermediate dividends, as follows. First, consider a particular security with a dividend-rate process δ in $\mathcal{H}^2$. The cumulative-dividend process H is thus defined by $H = \int_0^t \delta_s \, ds$, $t \in [0,T]$. Suppose that the security's price process V satisfies $V_T = 0$. Suppose that Q is an equivalent martingale measure with density ξ. Let π be defined by $\pi_0 = 1$ and (11). The fact that $H^Y + V^Y$ is a Q-martingale is equivalent to

$$V_t = \frac{1}{Y_t} E_t^Q \left(\int_t^T Y_s \delta_s \, ds \right), \qquad t \in [0,T].$$

(A) From the definition of ξ, Fubini's Theorem, the law of iterated expectations, and the fact that ξ is a martingale, show each of the equalities

$$\begin{aligned}
V_t = \quad & \tfrac{1}{\xi_t Y_t} E_t \left(\xi_T \int_t^T Y_s \delta_s \, ds \right) \\
= \quad & \tfrac{1}{\xi_t Y_t} E_t \left(\int_t^T \xi_T Y_s \delta_s \, ds \right) \\
= \quad & \tfrac{1}{\xi_t Y_t} \int_t^T E_t(\xi_T Y_s \delta_s) \, ds \\
= \quad & \tfrac{1}{\xi_t Y_t} \int_t^T E_t[E_s(\xi_T Y_s \delta_s)] \, ds \\
= \quad & \tfrac{1}{\xi_t Y_t} \int_t^T E_t(\xi_s Y_s \delta_s) \, ds \\
= \quad & \tfrac{1}{\xi_t Y_t} E_t \left(\int_t^T \xi_s Y_s \delta_s \, ds \right) \\
= \quad & \tfrac{1}{\pi_t} E_t \left(\int_t^T \pi_s \delta_s \, ds \right).
\end{aligned}$$

This calculation shows that $H^\pi + V^\pi$ is a martingale, consistent with the definition of π as a state-price deflator. Reversing the calculations shows that if π is a state-price deflator and $\text{var}(\pi_T) < \infty$, then $H^Y + V^Y$ is a Q-martingale, where Q is the probability measure defined by its density process ξ from (11).

(B) Extend to the case of V_T not necessarily zero. That is, suppose Q is an equivalent probability measure whose density process ξ is of finite variance. Show that $V^Y + H^Y$ is a Q-martingale if and only if $V^\pi + H^\pi$ is a P-martingale.

(C) Extend to the case of a cumulative-dividend process H that is a bounded Ito process. (Although beyond the scope of this book, an extension of Ito's Formula applying to general dividend processes that are not necessarily Ito processes shows that one need not assume that H is an Ito process.)

Exercise 6.7 Extend Exercise 6.6 to allow for cumulative-dividend processes, as follows. Recall that the cumulative-dividend process D^θ generated by a trading strategy θ is defined by $\Delta D_T^\theta = W_T^\theta$ and $W_t^\theta = W_0^\theta + \int_0^t \theta_s \, dG_s - D_{t-}^\theta$, where $W_t^\theta = \theta_t \cdot (X_t + \Delta D_t)$ and G is the gain process of the given securities. Let G^θ denote the gain process generated by θ, defined by $G_t^\theta = W_t^\theta + D_{t-}^\theta$. Assuming that an Ito return process R^θ for θ is well defined by $dR_t^\theta = (W_t^\theta)^{-1} \, dG_t^\theta$, show that R^θ satisfies the return restriction (6).

Exercise 6.8 Extend the proof of Proposition 6K to allow for general dividend processes. Add technical conditions as necessary.

Notes

The basic approach of this chapter is from Harrison and Kreps [1979] and Harrison and Pliska [1981], who coined most of the terms and developed most of the techniques and basic results. Huang [1985b] and Huang [1985a] generalized the basic theory. The development here differs in some minor ways. On numeraire invariance, see Huang [1985a]. This result extends more generally.

The notion of a doubling strategy, as described here in terms of coin tosses, appears in Harrison and Kreps [1979]. The actual continuous-time "doubling" strategy (3)–(4), and proof that the associated stopping time τ is valued in $(0, T)$, is from Karatzas [1993], as is a version of Lemma 6G. The relevance of the credit-constrained class of trading strategies $\underline{\Theta}(X)$, and results such as Proposition 6C, originates with Dybvig and Huang [1988]. Hindy [1995] explores further the implications of a nonnegative wealth constraint.

Banz and Miller [1978] and Breeden and Litzenberger [1978] explore the ability to deduce state prices from the valuation of derivative securities. Huang and Pagès [1992] give an extension to the case of an infinite-time horizon. The Stieltjes integral, mentioned in Section 6M, can be found in an analysis text such as Royden [1968]. Choulli, Krawczyk, and Stricker [1997] address the role of martingales that are stochastic exponentials (such as a density process) in financial applications.

In order to see a sense in which the absence of arbitrage implies that terminal ex-dividend prices are zero, see Ōhashi [1991]. This issue is especially delicate in non-Brownian information settings, since the event that $X_T \neq 0$, in some informational sense not explored here, can be suddenly revealed at time T, and therefore be impossible to exploit with a simultaneous trade. For further discussion of the terminal arbitrage issue, see Ōhashi [1991].

The main technical result used in Section 6K, on the extension of positive linear functionals, is inspired by Kreps [1981], and can be found specifically in Clark [1993]. Related results leading to technical conditions for the existence of an equivalent martingale measure, based on Harrison and Kreps [1979], and Harrison and Pliska [1981], or related concepts, can be found in Ansel and Stricker [1992], Ansel and Stricker [1994], Back and Pliska [1987], Cassese [1996] Dalang, Morton, and Willinger [1990], Delbaen [1992], Delbaen and Schachermayer [1994b], Delbaen and Schachermayer [1994b], Delbaen and Schachermayer [1994a], Delbaen and Schachermayer [1995a], Delbaen and Schachermayer [1995c], Delbaen and Schachermayer [1995b], Delbaen and Schachermayer [1996a], Delbaen and Schachermayer [1996b], Delbaen and Schachermayer [1998], Duffie and Huang [1986], El Karoui and Quenez [1993], Frittelli and Lakner [1995] Jacod and Shiryaev [1998], Kabanov [1996], Kabanov and Kramkov [1993], Kabanov and Kramkov [1995], Kusuoka [1992a], Lakner [993a], Lakner [1993], Levental and Skorohod [1994], Rogers [1993]. Schachermayer [1992], Schachermayer [1994], Schachermayer [98], Schweizer [1992], and Stricker [1990]. Of these, Delbaen and Schachermayer [1998] is essentially definitive in a continuous-time setting. Further references are found in Duffie [1988b]. Most of the results in this chapter extend to an abstract filtration, not necessarily generated by Brownian motion. Artzner [1995a] treats completeness and uniqueness of equivalent martingale measures. Rydberg [1996] discusses uniqueness in a Markovian setting. On market completeness under a change of measure in the abstract case, see Duffie [1985].

On the relationship between complete markets and equivalent martingale measures, see Artzner [1995a], Artzner and Heath [1990], Jarrow and Madan [1991], Müller [1985], and Stricker [1984].

Amendinger [1999] treats martingale representation for enlarged filtrations. Dritschel and Protter [1998] apply martingale representation in a financial setting involving Azema's martingale.

Babbs and Selby [1996], Buhlmann, Delbaen, Embrechts, and Shyriayev [1996], and Föllmer and Schweizer [1990] suggest some criteria or parameterization for the selection of an equivalent martingale measures in incomplete markets. In particular, Artzner [1995b], Bajeux-Besnainou and Portait [1997a], Dijkstra [1996], Johnson [1994], and Long [1990], address the *numeraire portfolio*, also called *growth-optimal portfolio*, as a device for selecting a state-price deflator.

For various notions of counterexamples to the existence of an equivalent martingale measure in the absence of arbitrage, see Stricker [1990], Back and Pliska [1991], ?], Schachermayer [1993], and Levental and Skorohod(1994). The notion of an approximate arbitrage is a slight variation on the notion of a *free lunch*, introduced by Kreps (1981). Loewenstein and Willard [998a] and Loewenstein and Willard [998b] treat the implications of local-martingale versions of a "density process" for what would, as a martingale, define an equivalent martinglae measure.

Carr and Jarrow [1990] show a connection between *local time* and the Black-Scholes model. See, also, Bick [1993]. Delbaen, Monat, Schachermayer, Schweizer, and Stricker [1994], Monat and Stricker [1993a], Monat and Stricker [1993b], provide conditions for L^2-closedness of the marketed space of contingent claims, a property used in the last section and in the proof of Proposition 6I.

Analogues to some of the results in Chapter 5 or in this chapter for the case of market imperfections such as portfolio constraints or transactions costs are provided by Ahn, Dayal, Grannan, and Swindle [1995], Avellaneda and Parás [1994], Bergman [1995], Boyle and Vorst [1992], Carassus and Jouini [1995], Chen [1994], Clewlow and Hodges [1996] Constantinides [1993], Constantinides and Zariphopoulou [1999], Cvitanić and Karatzas [1993], Davis and Clark [1993], Davis and Panas [1991], Davis, Panas, and Zariphopoulou [1993], Edirisinghe, Naik, and Uppal [1991], Grannan and Swindle [1992], Henrotte [1991], Jouini and Kallal [1991], Jouini and Kallal [993a], Karatzas and Kou [1994], Korn [1992], Kusuoka [1992b], Kusuoka [1993], Leland [1985], Levental and Skorohod [1995] Luttmer [1991], Munk [1997], Soner, Shreve, and Cvitanić [1994], Taleb [1997], and Whalley and Wilmott [1994]. Many of these results are asymptotic, for "small" proportional transactions costs. Additional implications of transactions costs and portfolio constraints for optimal portfolio and consumption choice are cited in the Notes of Chapter 9.

An application to international markets is given by Delbaen and Shirakawa [1994]. General treatments of some of the issues covered in this chapter can be found in Babbs and Selby [1996], Back and Pliska [1991], Christensen [1987], Christensen [1991], Conze and Viswanathan [991a], Dothan [1990], El Karoui and Quenez [1991] El Karoui and Quenez [1993], Geman, El Karoui, and Rochet [1991], Jarrow and Madan [1994], Jouini and Kallal [1991], Karatzas [1993], Müller [1985], Protter [1999], and Rady [1993].

Chapter 7

Term-Structure Models

THIS CHAPTER REVIEWS models of the term structure of interest rates that are used for the pricing and hedging of *fixed-income securities*, those whose future payoffs are contingent on future interest rates. Term-structure modeling is one of the most active and sophisticated areas of application of financial theory to everyday business problems, ranging from managing the risk of a bond portfolio to the design and pricing of collateralized mortgage obligations.

Included in this chapter are such standard examples as the Merton, Ho-Lee, Dothan, Brennan-Schwartz, Vasicek, Black-Derman-Toy, Black-Karasinski, and Cox-Ingersoll-Ross models, and variations of these "single-factor" term-structure models, so named because they treat the entire term structure of interest rates at any time as a function of a single state variable, the short rate of interest. We will also review multifactor models, including multifactor affine models, extending the Cox-Ingersoll-Ross and Vasicek models.

All of the named single-factor and multifactor models can be viewed in terms of marginal forward rates rather than directly in terms of interest rates, within the Heath-Jarrow-Morton (HJM) term-structure framework. The HJM framework allows, under technical conditions, any initial term structure of forward interest rates and any process for the conditional volatilities and correlations of these forward rates.

Numerical tractability is essential for practical applications. The "calibration" of model parameters and the pricing of term-structure derivatives are typically done by such numerical methods as "binomial trees" (Chapter 3), Fourier transform methods (Chapter 8), Monte-Carlo simulation (Chapter 11), and finite-difference solution of PDEs (Chapter 11).

This chapter makes little direct use of the pricing theory developed in Chapter 6 beyond the basic idea of an equivalent martingale measure, which can therefore be treated as a "black box" for those readers not familiar with Chapter 6. One need only remember that with probabilities assigned by an equivalent martingale measure, the expected rate of return on any security is the short rate of interest. Since the existence of an equivalent martingale measure is, under purely technical conditions, equivalent to the absence of arbitrage, we find it safe and convenient to work almost from the outset under an assumed equivalent martingale measure. Sufficient conditions for an equivalent martingale measure are reviewed in Chapter 6. An equilibrium example is given in Chapter 10.

7A The Term Structure

We fix a Standard Brownian Motion $B = (B^1, \ldots, B^d)$ in $\mathbb{R}^d$, for some dimension $d \geq 1$, restricted to some time interval $[0, T]$, on a given probability space $(\Omega, \mathcal{F}, P)$. We also fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ of B, as defined in Section 5I.

We take as given an adapted short-rate process r with $\int_0^T |r_t| \, dt < \infty$. Conceptually, r_t is the continually compounding rate of interest on riskless securities at time t. This is formalized by taking $\exp\left(\int_t^s r_u \, du\right)$ to be the market value at time s of an investment made at time t of 1 unit of account, continually reinvested at the short rate between t and s.

Consider a *zero-coupon bond* maturing at some future time $s > t$. By definition, the bond pays no dividends before time s, and offers a fixed lump-sum payment at time s that we can take without loss of generality to be 1 unit of account. Although it is not always essential to do so, we assume throughout the chapter that such a bond exists for each maturity date s. One of our main objectives is to characterize the price $\Lambda_{t,s}$ at time t of the s-maturity bond, and its behavior over time.

In the absence of arbitrage, purely technical conditions reviewed in Chapter 6 are required for the existence of an equivalent martingale measure. Such a probability measure Q has the property that any security whose dividend is in the form of a lump sum payment of Z at some time s has a price, at any time $t < s$ of

$$E_t^Q \left[\exp\left(\int_t^s -r_u \, du \right) Z \right], \tag{1}$$

where E_t^Q denotes $\mathcal{F}_t$-conditional expectation under Q. Here, Z would be $\mathcal{F}_s$-measurable, and such that the expectation (1) is well defined. A review of Theorem 2G justifies the easy finite-dimensional version of (1). In particular, taking $Z = 1$ in (1), the price at time t of the zero-coupon bond maturing at s is

$$\Lambda_{t,s} \equiv E_t^Q \left[\exp\left(\int_t^s -r_u \, du \right) \right]. \tag{2}$$

The doubly-indexed process Λ is sometimes known as the *discount function*, or more loosely as the *term structure of interest rates*. The term structure is often expressed in terms of the *yield curve*. The *continuously compounding yield* $y_{t,\tau}$ on a zero-coupon bond maturing at time $t + \tau$ is defined by

$$y_{t,\tau} = -\frac{\log(\Lambda_{t,t+\tau})}{\tau}.$$

The term structure can also be represented in terms of forward interest rates, as explained in Section 7J.

In most of this chapter, we review conventional models of the behavior of the short rate r under a fixed equivalent martingale measure Q. In each case, r is modeled in terms of the standard Brownian motion B^Q in $\mathbb{R}^d$ under Q that is obtained from B via Girsanov's Theorem (Appendix D). The Notes cite more general models. We will characterize the term structure and the pricing of term-structure derivatives, securities whose payoffs depend on the term structure.

7B One-Factor Term-Structure Models

We begin with *one-factor term-structure models*, by which we mean models of the short rate r given by an SDE of the form

$$dr_t = \mu(r_t, t)\, dt + \sigma(r_t, t)\, dB_t^Q, \tag{3}$$

where $\mu : \mathbb{R} \times [0, T] \to \mathbb{R}$ and $\sigma : \mathbb{R} \times [0, T] \to \mathbb{R}^d$ satisfy technical conditions guaranteeing the existence of a solution to (3) such that for all t and $s \geq t$, the price $\Lambda_{t,s}$ of the zero-coupon bond maturing at s is finite and well defined by (2). For simplicity, we can take $d = 1$.

The one-factor models are so named because the Markov property (under Q) of the solution r to (3) implies from (2) that the short rate is the only state variable, or "factor," on which the current yield curve depends. That is, for all t and $s \geq t$, we can write $\Lambda_{t,s} = F(t, s, r_t)$, for some fixed F : $[0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$.

Table 7.1 shows most of the parametric examples of one-factor models appearing in the literature, with their conventional names. Each of these models is a special case of the SDE

$$dr_t = [K_0(t) + K_1(t)r_t + K_2(t)r_t \log(r_t)]\ dt + [H_0(t) + H_1(t)r_t]^\nu\ dB_t^Q,$$

for continuous functions K_0, K_1, K_2, H_0, and H_1 on $[0, T]$ into $\mathbb{R}$, and for some exponent $\nu \in [0.5, 1.5]$. Coefficient restrictions, and restrictions on the space of possible short rates, are needed for the existence and uniqueness of solutions. For each model, Table 7.1 shows the associated exponent ν, and uses the symbol "•" to indicate those coefficients that appear in nonzero form. We can view a negative coefficient function K_1 as a *mean-reversion* parameter, in that a higher short rate generates a lower drift, and vice versa. Empirically speaking, mean reversion is widely believed to be a useful attribute to include in single-factor short-rate models.

In most cases, the original versions of these models had constant coefficients, and were only later extended to allow $K_i(t)$ and $H_i(t)$ to depend on t, for practical reasons, such as calibration of the model to a given set of bond and option prices, as described in Section 11M. For example, with time-varying coefficients, the *Merton model* of the term structure is often called the *Ho-Lee model*. A popular special case of the *Black-Karasinski model* is the *Black-Derman-Toy model*, defined in Exercise 7.1. References to the literature are given in the Notes.

Each of these single-factor models has its own desirable properties, some of which will be reviewed below. It tends to depend on the application which of these, if any, is used in practice. The Notes cite some of the empirical evidence regarding these single-factor models, in some cases strongly pointing toward multifactor extensions, which we will turn to later in this chapter.

For essentially any single-factor model, the term structure can be computed (numerically, if not explicitly) by taking advantange of the Feynman-Kac relationship between PDEs and SDEs given in Appendix E. Fixing for convenience the maturity date s, the Feynman-Kac approach implies from (2), under technical conditions on μ and σ, that for all t,

$$\Lambda_{t,s} = f(r_t, t), \tag{4}$$

Table 7.1. Common Single-Factor Model Parameters

Model	K_0	K_1	K_2	H_0	H_1	ν
Cox-Ingersoll-Ross	•	•			•	0.5
Pearson-Sun	•	•		•	•	0.5
Dothan					•	1.0
Brennan-Schwartz	•	•			•	1.0
Merton (Ho-Lee)	•			•		1.0
Vasicek	•	•		•		1.0
Black-Karasinski		•	•		•	1.0
Constantinides-Ingersoll					•	1.5

where $f \in C^{2,1}(\mathbb{R} \times [0, T))$ solves the PDE

$$\mathcal{D}f(x,t) - xf(x,t) = 0, \qquad (x,t) \in \mathbb{R} \times [0,s), \tag{5}$$

with boundary condition

$$f(x,s) = 1, \qquad x \in \mathbb{R}, \tag{6}$$

where

$$\mathcal{D}f(x,t) = f_t(x,t) + f_x(x,t)\mu(x,t) + \frac{1}{2}f_{xx}(x,t)\sigma(x,t)^2.$$

According to the results in Appendix E, in order for (4)–(5)–(6) to be consistent, it is enough that r is nonnegative and that μ and σ satisfy Lipschitz conditions in x and have derivatives μ_x, σ_x, μ_{xx}, and σ_{xx} that are continuous and satisfy growth conditions in x. These conditions are not necessary and can be weakened. We note that the Lipschitz condition is violated for several of the examples considered in Table 7.1, such as the Cox-Ingersoll-Ross model, which must be treated on a case-by-case basis.

The PDE (5)–(6) can be quickly solved using numerical algorithms described in Chapter 11. If μ and σ do not depend on t, then, for any calendar time t and any time $u < s$ remaining to maturity, we can also view the solution f to (5)-(6) as determining the price $f(r_t, s-u) = \Lambda_{t,t+u}$ at time t of the zero-coupon bond maturing at $t+u$, so that a single function f describes the entire term structure at any time.

7C The Gaussian Single-Factor Models

A subset of the models considered in Table 7.1, those with $K_2 = H_1 = 0$, are *Gaussian*, in that the short rates $\{r(t_1), \ldots, r(t_k)\}$ at any finite set $\{t_1, \ldots, t_k\}$ of times have a joint normal distribution under Q. This follows from the properties of linear stochastic differential equations reviewed in Appendix E. Special cases are the Merton (often called "Ho-Lee") and Vasicek models.

For the Gaussian model, we can show that bond-price processes are log-normal (under Q) by defining a new process y satisfying $dy_t = -r_t\, dt$, and noting that (r, y) is the solution of a two-dimensional linear stochastic differential equation, in the sense of Appendix E. Thus, for any t and $s \geq t$, the random variable $y_s - y_t = -\int_t^s r_u\, du$ is normally distributed. Under Q, the mean $m(t,s)$ and variance $v(t,s)$ of $-\int_t^s r_u\, du$, conditional on $\mathcal{F}_t$, are easily computed in terms of r_t, K_0, K_1, and H_0. From the results for linear SDEs in Appendix E, the conditional variance $v(t,s)$ is deterministic and the conditional mean $m(t,s)$ is of the form $a(t,s) + \beta(t,s) r_t$, for coefficients $a(t,s)$ and $\beta(t,s)$ whose calculation is left as an exercise. It follows that

$$\begin{aligned} \Lambda_{t,s} &= E_t^Q \left[\exp\left(-\int_t^s r_u\, du \right) \right] \\ &= \exp\left(m(t,s) + \frac{v(t,s)}{2} \right) \\ &= e^{\alpha(t,s) + \beta(t,s) r(t)}, \end{aligned} \tag{7}$$

where $\alpha(t,s) = a(t,s) + v(t,s)/2$. Because r_t is normally distributed under Q, this means that any zero-coupon bond price is log-normally distributed under Q. Using this property, a further exercise requests explicit computation of bond-option prices in this setting, along the lines of the original Black-Scholes formula. Aside from the simplicity of the Gaussian model, this explicit computation is one of its main advantages in applications.

An undesirable feature of the Gaussian model is that it implies (for H_0 everywhere nonzero) that the short rate and yields on bonds of any maturity are negative with positive probability at any future date. While negative interest rates are sometimes plausible when expressed in "real" (consumption numeraire) terms, it is common in practice to express term structures in nominal terms, relative to the price of money. In nominal terms, negative bond yields imply a kind of arbitrage. In order to describe this arbitrage, we can formally view *money* as a security with no dividends whose price process is identically equal to 1. If a particular zero-coupon bond were to offer a negative yield, consider a short position in the bond and a long position of an equal number of units of money, both held to the maturity of the bond. With a negative bond yield, the initial bond price is larger than 1, implying that this position is an arbitrage. Of course, the proposed alternative of everywhere positive interest rates, along with money, implies that the opposite strategy is an arbitrage if money can be freely shorted. One normally assumes that money is a special kind of security that cannot be shorted. (Indeed, the fact that money has a strictly positive price despite having no dividends means that shorting money is itself a kind of arbitrage.) To address properly the role of money in supporting nonnegative interest rates would therefore require a rather wide detour into monetary theory and the institutional features of money markets. It may suffice for our purposes to point out that money conveys certain special advantages, for example the ability to undertake certain types of transactions immediately, or with reduced transactions costs, which would imply a fee in equilibrium for the shorting of money. Let us merely leave this issue with the sense that allowing negative interest rates is not necessarily "wrong," but is somewhat undesirable. Gaussian short-rate models are nevertheless useful, and frequently used, because they are relatively tractable and in light of the low likelihood that they would assign to negative interest rates within a reasonably short time, with reasonable choices for the coefficient functions.

7D The Cox-Ingersoll-Ross Model

One of the best-known single-factor term-structure models is the *Cox-Ingersoll-Ross* (CIR) model indicated in Table 7.1. For constant coefficient functions K_0, K_1, and H_1, the CIR drift and diffusion functions, μ and σ, may be written in the form

$$\mu(x,t) = \kappa(\overline{x} - x); \qquad \sigma(x,t) = C\sqrt{x}, \qquad x \geq 0, \tag{8}$$

for constants κ, $\overline{x}$, and C. Provided κ and $\overline{x}$ are positive, there is a nonnegative solution to the SDE (3), based on a source cited in the Notes. (Obviously, nonnegativity is important, if only for the fact of the square root in the diffusion). Of course, we assume that $r_0 \geq 0$, and treat (5)–(6) as applying only to a short rate x in $[0, \infty)$. Given r_0, under Q, r_t has a non-central χ^2

distribution with parameters that are known explicitly. In particular, the drift $\mu(x,t)$ indicates reversion toward a stationary risk-neutral mean $\overline{x}$ at a rate κ, in the sense that

$$E^Q(r_t) = \overline{x} + e^{-\kappa t}(r_0 - \overline{x}),$$

which tends to $\overline{x}$ as t goes to $+\infty$. Additional properties of this model are discussed later in this chapter and in Section 10I, where the coefficients κ, $\overline{x}$, and C are calculated in a general equilibrium setting in terms of the utility function and endowment of a representative agent. For the CIR model, it can be verified by direct computation of the derivatives that the solution for the term-structure PDE (5)–(6) is given by

$$f(x,t) = \exp\left[\alpha(t,s) + \beta(t,s)x\right], \tag{9}$$

where

$$\alpha(t,s) = \frac{2\kappa\overline{x}}{C^2}\left[\log\left(2\gamma e^{(\gamma+\kappa)(s-t)/2}\right) - \log\left((\gamma+\kappa)(e^{\gamma(s-t)}-1)+2\gamma\right)\right] \tag{10}$$

$$\beta(t,s) = \frac{2(1-e^{\gamma(s-t)})}{(\gamma+\kappa)(e^{\gamma(s-t)}-1)+2\gamma}, \tag{11}$$

for $\gamma = (\kappa^2 + 2C^2)^{1/2}$. We will later consider multi-factor versions of the CIR model.

7E The Affine Single-Factor Models

The Gaussian and Cox-Ingersoll-Ross models are special cases of single-factor models with the property that the solution f of the term-structure PDE (5)–(6) is given in the exponential-affine form (9) for some coefficients $\alpha(t,s)$ and $\beta(t,s)$ that are continuously differentiable in s. For all t, the yield $-\log[f(x,t)]/(s-t)$ obtained from (9) is affine in x. We therefore call any such model an *affine term-structure model.* (A function $g : \mathbb{R}^k \to \mathbb{R}$, for some k, is *affine* if there are constants a and b in $\mathbb{R}^k$ such that for all x, $g(x) = a + b \cdot x$.)

We can use the PDE (5) to characterize the drift and diffusion functions, μ and σ, underlying any affine model. Specifically, substituting (9) into (5) and simplifying leaves, for each $(x,t) \in \mathbb{R} \times [0,s)$,

$$\beta(t,s)\mu(x,t) = [1 - \beta_t(t,s)]x - \alpha_t(t,s) - \frac{1}{2}\beta^2(t,s)\sigma^2(x,t), \tag{12}$$

where subscripts indicate partial derivatives. We will use (12) to deduce how $\mu(x,t)$ and $\sigma(x,t)$ depend on x. Suppose, for simplicity, that $\mu(x,t)$ and $\sigma(x,t)$ do not depend on t. Applying (12) at two possible maturity dates, say s_1 and s_2, we have the two linear equations in the two unknowns $\mu(x)$ and $\sigma^2(x)$:

$$A(s_1,s_2)\begin{pmatrix}\mu(x)\\ \sigma^2(x)\end{pmatrix}=\begin{pmatrix}-\alpha_t(t,s_1)+[1-\beta_t(t,s_1)]x\\ -\alpha_t(t,s_2)+[1-\beta_t(t,s_2)]x\end{pmatrix}, \tag{13}$$

where

$$A(s_1,s_2)=\begin{pmatrix}\beta(t,s_1) & \beta^2(t,s_1)/2\\ \beta(t,s_2) & \beta^2(t,s_2)/2\end{pmatrix}.$$

Except at maturity dates s_1 and s_2 chosen so that $A(s_1,s_2)$ is singular, we can conclude from (13) that $\mu(x)$ and $\sigma^2(x)$ must themselves be affine in x.

Going the other way, suppose that μ and σ^2 are affine in x, in that

$$\mu(x,t)=K_0(t)+K_1(t)x; \qquad \sigma^2(x,t)=H_0(t)+H_1(t)x.$$

Then we can recover an affine term-structure model by showing that the solution to (5)–(6) is of the affine form (9). Such a solution applies if there exists (α,β) solving (12). The terms proportional to x in (12) must sum to zero, for otherwise we could vary x and contradict (12). This supplies us with an ordinary differential equation (ODE) for β:

$$\beta_t(t,s)=1-K_1(t)\beta(t,s)-\frac{1}{2}H_1(t)\beta^2(t,s); \qquad \beta(s,s)=0, \tag{14}$$

whose boundary condition $\beta(s,s)=0$ is dictated by (6) and (9). The ODE (14) is a form of what is known as a *Ricatti equation*. Solutions are finite given technical conditions on K_1 and K_2.

Likewise, the "intercept" term in (12), the term that is not dependent on x, must also be zero. Having solved for β from (14), this gives us:

$$\alpha_t(t,s)=-K_0(t)\beta(t,s)-\frac{1}{2}H_0(t)\beta^2(t,s).$$

Again, the boundary condition $\alpha(s,s)=0$ is from (6) and (9). Thus, by integrating $\alpha_u(u,s)$ with respect to u, we have

$$\alpha(t,s)=\int_t^s\left[K_0(u)\beta(u,s)+\frac{1}{2}H_0(u)\beta^2(u,s)\right]du. \tag{15}$$

Thus, technicalities aside, μ and σ^2 are affine in x if and only if the term structure is itself affine in x. Numerical solutions of the ODE (14), for example by discretization methods such as Runge-Kutta, are easy and given in a source cited in the Notes. Then (15) can solved by numerical integration. The special cases associated with the Gaussian model and the CIR model have explicit solutions for α and β.

We have shown, basically, that affine term-structure models are easily classified and solved. This idea is further pursued in a multifactor setting later in this chapter and in sources cited in the Notes.

From the above characterization, we know that the "affine class" of term-structure models includes those shown in Table 7.1 with $K_2 = 0$ and $\nu = 0.5$, including

(a) The *Vasicek model*, for which $H_1 = 0$.

(b) The *Cox-Ingersoll-Ross model*, for which $H_0 = 0$.

(c) The *Merton (Ho-Lee)* model, for which $K_1 = H_1 = 0$.

(d) The *Pearson-Sun model*.

For affine models with $H_1 \neq 0$, existence of a solution to the SDE (3) requires coefficients (H, K) with

$$K_0(t) - K_1(t)\frac{H_0(t)}{H_1(t)} \geq 0, \qquad t \in [0, T]. \tag{16}$$

This condition guarantees the existence of a solution r to the SDE (3) with $r(t) \geq -H_0(t)/H_1(t)$ for all t.

7F Term-Structure Derivatives

We return to the general one-factor model (3) and consider one of its most important applications, the pricing of derivative securities. Suppose a derivative has a payoff at some given time s defined by $g(r_s)$. By the definition of an equivalent martingale measure, the price at time t for such a security is

$$F(r_t, t) \equiv E_t^Q\left[\exp\left(-\int_t^s r_u\,du\right) g(r_s)\right].$$

The Feynman-Kac PDE results of Appendix E give technical conditions on μ, σ, and g under which F solves the PDE, for $(x,t) \in \mathbb{R} \times [0,s)$,

$$F_t(x,t) + F_x(x,t)\mu(x,t) + \frac{1}{2}F_{xx}(x,t)\sigma(x,t)^2 - xF(x,t) = 0, \tag{17}$$

with boundary condition

$$F(x,s) = g(x), \qquad x \in \mathbb{R}. \tag{18}$$

Some examples follow, abstracting from many institutional details.

(a) A European option expiring at time s on a zero-coupon bond maturing at some later time u, with strike price p, is a claim to $(\Lambda_{s,u} - p)^+$ at s. The valuation of the option is given, in a one-factor setting, by the solution F to (17)–(18), with $g(x) = [f(x,s) - p]^+$, where $f(x,s)$ is the price at time s of a zero-coupon bond maturing at u.

(b) A *forward-rate agreement* (FRA) calls for a net payment by the fixed-rate payer of $c^* - c(s)$ at time s, where c^* is a fixed payment and $c(s)$ is a floating-rate payment for a time-to-maturity δ, in *arrears*, meaning that $c(s)$ is the simple interest rate $\Lambda_{s-\delta,s}^{-1} - 1$ applying at time $s - \delta$ for loans maturing at time s. In practice, we usually have a time to maturity δ or one quarter or one half year. When originally sold, the fixed-rate payment c^* is usually set so that the FRA is *at market*, meaning of zero market value.

(c) An *interest-rate swap* is a portfolio of FRAs maturing at a given increasing sequence $t(1), t(2), \ldots, t(n)$ of coupon dates. The inter-coupon interval $t(i) - t(i-1)$ is usually 3 months or 6 months. The associated FRA for date $t(i)$ calls for a net payment by the fixed-rate payer of $c^* - c(t(i))$, where the floating-rate payment received is $c(t(i)) = \Lambda_{t(i-1),t(i)}^{-1} - 1$, and the fixed-rate payment c^* is the same for all coupon dates. At initiation, the swap is usually *at market*, meaning that the fixed rate c^* is chosen so that the swap is of zero market value. Ignoring default risk, this would imply, as can be shown as an exercise, that the fixed-rate coupon c^* is the par coupon rate. That is, the at-market swap rate c^* is set at the origination date t of the swap so that

$$1 = c^* \left(\Lambda_{t,t(1)} + \cdots + \Lambda_{t,t(n)}\right) + \Lambda_{t,t(n)},$$

meaning that c^* is the coupon rate on a *par* bond, one whose face value and initial market value are the same.

(d) A *cap* can be viewed as portfolio of "caplet" payments of the form $(c(t(i)) - c^*)^+$, for a sequence of payment dates $t(1), t(2), \ldots, t(n)$ and floating rates $c(t(i))$ that are defined as for a swap. The fixed rate c^* is set with the terms of the cap contract.

(e) A *floor* is defined symmetrically with a cap, replacing $(c(t(i)) - c^*)^+$ with $(c^* - c(t(i)))^+$.

Path-dependent derivative securities, such as mortgage-backed securities, sometimes call for additional state variables. Some interest-rate derivative securities are based on the yields of bonds that are subject to some risk of default, in which case the approach must be modified by accounting for default risk, as in Chapter 9.

There are relatively few cases of practical interest for which the PDE (17)–(18) can be solved explicitly. Chapters 8 and 11 review some numerical solution techniques.

7G The Fundamental Solution

Based on the results of Appendix E, under technical conditions we can also express the solution F of the PDE (17)–(18) for the value of a derivative term-structure security in the form

$$F(x,t) = \int_{-\infty}^{+\infty} G(x,t,y,s)g(y)\,dy, \tag{19}$$

where G is the *fundamental solution* of the PDE (17). Some have called G the *Green's function* associated with (17), although that terminology is not rigorously justified. From (19), for any time $s > t$ and any interval $[y(1), y(2)]$,

$$\int_{y(1)}^{y(2)} G(r_t, t, y, s)\,dy$$

is the price at time t of a security that pays one unit of account at time s in the event that r_s is in $[y(1), y(2)]$. For example, the current price $\Lambda_{t,s}$ of the zero-coupon bond maturing at s is given by $\int_{-\infty}^{+\infty} G(r_t, t, y, s)\,dy$.

One can compute the fundamental solution G by solving a PDE that is "dual" to (5)–(6), in the following sense. As explained in Appendix E, under technical conditions, for each (x,t) in $\mathbb{R} \times [0,T)$, a function $\psi \in C^{2,1}(\mathbb{R} \times (0,T])$ is defined by $\psi(y,s) = G(x,t,y,s)$, and solves the *forward Kolmogorov equation* (also known as the *Fokker-Planck equation*):

$$\mathcal{D}^*\psi(y,s) - y\psi(y,s) = 0, \tag{20}$$

where

$$\mathcal{D}^*\psi(y,s) = -\psi_t(y,s) - \frac{\partial}{\partial y}\left[\psi(y,s)\mu(y,s)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\psi(y,s)\sigma(y,s)^2\right].$$

The "intuitive" boundary condition for (20) is obtained from the role of G in pricing securities. Imagine that the current short rate at time t is x, and consider an instrument that pays one unit of account immediately, if and only if the current short rate is some number y. Presumably this contingent claim is valued at 1 unit of account if $x = y$, and otherwise has no value. From continuity in s, one can thus think of $\psi(\,\cdot\,, s)$ as the density at time s of a measure on $\mathbb{R}$ that converges as $s \downarrow t$ to a probability measure ν with $\nu(\{x\}) = 1$, sometimes called the *dirac measure* at x. Although this initial boundary condition on ψ can be made more precise, we leave that to sources cited in Appendix E. An implementation of this boundary condition for a numerical solution of (20) is spelled out in Chapter 11. A discrete-time analogue is found in Chapter 3, where we provided an algorithm for computing the fundamental solutions for the Black-Derman-Toy and Ho-Lee models.

Given the fundamental solution G, the derivative asset price function F is more easily computed by numerically integrating (19) than from a direct numerical attack on the PDE (17)–(18). Thus, given a sufficient number of derivative securities whose prices must be computed, it may be worth the effort to compute G. Some numerical methods for calculating F and G are indicated in Chapter 11.

A lengthy argument given by a source cited in the Notes shows that the fundamental solution G of the Cox-Ingersoll-Ross model (8) is given explicitly in terms of the parameters κ, $\overline{x}$, and C by

$$G(x,0,y,t) = \frac{\varphi(t) I_q\left(\varphi(t)\sqrt{xye^{-\gamma t}}\right)}{\exp\left[\varphi(t)(y + xe^{-\gamma t}) - \eta(x + \kappa\overline{x}t - y)\right]}\left(\frac{e^{\gamma t}y}{x}\right)^{q/2},$$

where $\gamma = (\kappa^2 + 2C^2)^{1/2}$, $\eta = (\kappa - \gamma)/C^2$,

$$\varphi(t) = \frac{2\gamma}{C^2(1 - e^{-\gamma t})}, \qquad q = \frac{2\kappa\overline{x}}{C^2} - 1,$$

and $I_q(\,\cdot\,)$ is the modified Bessel function of the first kind of order q. The same source gives explicit solutions for the fundamental solutions of other models. For time-independent μ and σ, as with the CIR model, we have, for all t and $s > t$, $G(x, t, y, s) = G(x, 0, y, s - t)$.

7H Multifactor Models

The one-factor model (3) for the short rate is limiting. Even a casual review of the empirical properties of the term structure, some of which can be found in papers cited in the Notes, shows the significant potential improvements in fit offered by a *multifactor term-structure model.* While terminology varies from place to place, by a "multifactor" model, we mean a model in which the short rate is of the form $r_t = R(X_t, t)$, $t \geq 0$, where X is an Ito process in $\mathbb{R}^k$ solving a stochastic differential equation of the form

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dB_t^Q, \tag{21}$$

where the given functions R, μ, and σ on $\mathbb{R}^k \times [0, \infty)$ into $\mathbb{R}$, $\mathbb{R}^k$, and $\mathbb{R}^{k \times d}$, respectively, satisfy enough technical regularity to guarantee that (21) has a unique solution and that the term structure (2) is well defined. (Sufficient conditions are given in Appendix E.) In empirical applications, one often supposes that the state process X also satisfies a stochastic differential equation under the probability measure P, in order to exploit the time-series behavior of observed prices and price-determining variables in estimating the model. Examples are indicated in the Notes.

An interpretation of the role of the "state variables" is left open for the time being. For example, in an equilibrium model such as later considered in Chapter 10, some elements of the state vector X_t are sometimes *latent*, that is, unobservable to the modeler, except insofar as they can be inferred from prices that depend on the levels of X. This latent-variable approach has been popular in much of the empirical literature on term-structure modeling. Another approach is to take some or all of the state variables to be directly observable variables, such as macro-economic determinants of the business cycle and inflation, that are thought to play a role in determining the term structure. This approach has also been explored in the empirical literature. In many examples, one of the component processes $X^{(1}, \ldots, X^{(k)}$ is singled out as the short-rate process r, whose drift and diffusion are allowed to depend on the levels of the other component processes.

A derivative security is, in this setting, given by some real-valued terminal payment function g on $\mathbb{R}^k$, for some maturity date $s \leq T$. By the definition of an equivalent martingale measure, the associated derivative security price is given from (1) by

$$F(X_t, t) = E_t^Q \left[\exp\left(-\int_t^s R(X_u, u)\, du \right) g(X_s) \right].$$

Extending (17)–(18), under technical conditions given in Appendix E, we have the PDE characterization

$$\mathcal{D}F(x,t) - R(x,t)F(x,t) = 0, \qquad (x,t) \in \mathbb{R}^k \times [0,s), \tag{22}$$

with boundary condition

$$F(x,s) = g(x), \qquad x \in \mathbb{R}^k, \tag{23}$$

where

$$\mathcal{D}F(x,t) = F_t(x,t) + F_x(x,t)\mu(x,t) + \frac{1}{2}\mathrm{tr}\left[\sigma(x,t)\sigma(x,t)^\top F_{xx}(x,t)\right].$$

The case of a zero-coupon bond is $g(x) \equiv 1$. Under technical conditions, we can also express the solution F, as in (19), in terms of the fundamental solution G of the PDE (22), as discussed in Appendix E.

7I Affine Term Structure Models

A rich and tractable sub-class of multi-factor models are the *affine term-structure models*, defined by taking (21) with

$$\mu(x,t) = K_0 + K_1 x \tag{24}$$

for some $K_0 \in \mathbb{R}^k$ and $K_1 \in \mathbb{R}^{k\times k}$, and by taking, for each i and j in $\{1,\ldots,k\}$,

$$(\sigma(x,t)\sigma(x,t)^\top)_{ij} = H_{0ij} + H_{1ij}\cdot x, \tag{25}$$

for $H_{0ij} \in \mathbb{R}$ and $H_{1ij} \in \mathbb{R}^k$. One can also allow the coefficient functions $H = (H_0, H_1)$ and $K = (K_0, K_1)$ to depend on t; we ignore that for notational simplicity.

Given the coefficients (H, K), a natural statespace $D \subset \mathbb{R}^K$ for this affine model is set by the obvious requirement that $\left(\sigma(x,t)\sigma(x,t)^\top\right)_{ii} \geq 0$ for all x in D. Thus, given H, we choose the state space

$$D = \left\{x \in \mathbb{R}^k : \; H_{0ii} + H_{1ii} \cdot x \geq 0, \quad i \in \{1, \ldots, k\}\right\}. \tag{26}$$

Conditions cited in the Notes on the coefficients (H, K) ensure existence of a unique solution X to (21) that is valued in D.

An example is the "multi-factor CIR" model, defined by

$$dX_{it} = A_i(\overline{x}_i - X_{it})\,dt + C_i\sqrt{X_{it}}\,dB_t^{Q,(i)}; \qquad X_{i0} > 0, \tag{27}$$

where A_i, $\overline{x}_i$, and C_i are positive constants playing the same respective roles as A, $\overline{x}$, and C in the one-factor CIR model (8). Given the independence under Q of $B^{Q,(1)}, \ldots, B^{Q,(k)}$, if we let $R(x,t) = x_1 + \cdots + x_k$, the multi-factor CIR model generates the zero-coupon bond price $f(x,t)$ for maturity date s given by

$$f(x,t) = \exp\left[\alpha(t,s) + \beta_1(t,s)x_1 + \cdots + \beta_k(t,s)x_k\right], \tag{28}$$

where $\alpha(t,s) = \alpha_1(t,s) + \cdots + \alpha_k(t,s)$, and where $\alpha_i(t,s)$ and $\beta_i(t,s)$ are the solution coefficients of the univariate CIR model with coefficients $(\kappa_i, \overline{x}_i, C_i)$.

More generally, we suppose that

$$R(x,t) = \rho_0 + \rho_1 \cdot x, \tag{29}$$

for coefficients $\rho_0 \in \mathbb{R}$ and $\rho_1 \in \mathbb{R}^k$. For a fixed maturity date s, we expect a solution $f(X_t, t)$ for the price at time t of a zero-coupon bond maturing at time s to be of the exponential-affine form

$$f(x,t) = e^{\alpha(t) + \beta(t) \cdot x}, \tag{30}$$

for deterministic $\alpha(t)$ and $\beta(t)$. For notational simplicity, we suppress the maturity date s from the notation for α and β, and we let $\beta(t)^\top H_1(t)\beta(t)$ denote the vector in $\mathbb{R}^k$ whose n-th element is $\sum_{i,j} \beta_i(t)^\top H_{ijn}\beta_j(t)$. After substituting the candidate solution (30) into the PDE (22), extending from the single-factor case, we conjecture that β satisfies the k-dimensional ordinary differential equation, analogous to (14), given by

$$\beta'(t) = \rho_1 - K_1\beta(t) - \frac{1}{2}\beta(t)^\top H_1\beta(t), \tag{31}$$

with the boundary condition $\beta(s) = 0$ determined by (30) and the requirement that $f(x, s) = 1$. We have repeatedly used a *separation-of-variables* idea: If $a + b \cdot x = 0$ for all x in some open subset of $\mathbb{R}^k$, then a and b must be zero.

The ODE (31) is, as with the single-factor affine models, a Riccati equation. Solutions are finite given technical conditions on K_1 and H_1. In some cases, an explicit solution is possible. One can alternatively apply a numerical ODE solution method, such as Runge-Kutta.

Likewise, we find that

$$\alpha'(t) = \rho_0 - K_0\beta(t) - \frac{1}{2}\beta(t)^\top H_0\beta(t), \tag{32}$$

with the boundary condition $\alpha(s) = 0$. One integrates (32) to get

$$\alpha(t) = \int_t^s \left[-\rho_0 + K_0\beta(u) + \frac{1}{2}\beta(u)^\top H_0\beta(u) \right] du. \tag{33}$$

Numerical integration is an easy and fast method for treating (33) when explicit solutions are not at hand.

This affine class of term structure models extends to allow for time-dependent coefficients (K, H, ρ) and to cases with jumps in the state process X, as cited in the Notes. As we shall see in Chapter 8, one can also analytically solve for the transition distribution of an affine state-variable process, and for the associated prices of options on zero-coupon bonds, using Fourier-transform methods. The affine model, moreover, is used extensively for the analysis of asset-pricing applications going well beyond a term-structure setting.

7J The HJM Model of Forward Rates

In modeling the term structure, we have so far taken as the primitive a model of the short rate process of the form $r_t = R(X_t, t)$, where (under some equivalent martingale measure) X solves a given stochastic differential equation. (In the one-factor case, one usually takes $r_t = X_t$.) This approach has the advantage of a finite-dimensional state-space. For example, with this state-space approach one can compute certain derivative prices by solving PDEs.

An alternative approach is to directly model the stochastic behavior of the entire term structure of interest rates. This is the essence of the Heath-Jarrow-Morton (HJM) model. The remainder of this section is a summary of the basic elements of the HJM model. The following section, the exercises, and sources cited in the Notes, provide many extensions and details.

The forward price at time t of a zero-coupon bond for delivery at time $\tau \geq t$ with maturity at time $s \geq \tau$ is (in the absence of arbitrage) given by $\Lambda_{t,s}/\Lambda_{t,\tau}$, the ratio of zero-coupon bond prices at maturity and delivery, respectively. Proof of this is left as an exercise. The associated *forward rate* is defined by

$$\Phi_{t,\tau,s} \equiv \frac{\log(\Lambda_{t,\tau}) - \log(\Lambda_{t,s})}{s-\tau}, \tag{34}$$

which can be viewed as the continuously compounding yield of the bond bought forward. The *instantaneous forward rate*, when it exists, is defined for each time t and forward delivery date $\tau \geq t$, by

$$f(t,\tau) = \lim_{s \downarrow \tau} \Phi_{t,\tau,s}. \tag{35}$$

Thus, the instantaneous forward-rate process f exists (and is an adapted process) if and only if, for all t, the discount $\Lambda_{t,s}$ is differentiable with respect to s.

From (34) and (35), we arrive at the ordinary differential equation

$$\frac{d}{ds}\,\Lambda_{t,s} = -\Lambda_{t,s} f(t,s),$$

with the boundary condition $\Lambda(t,t) = 1$. The solution,

$$\Lambda_{t,s} = \exp\left(-\int_t^s f(t,u)\,du\right), \tag{36}$$

is the price at time t of a zero-coupon bond maturing at s. The term structure can thus be recovered from the instantaneous forward rates, and vice versa.

Given a stochastic model f of forward rates, we will assume that the short rate process r is defined by $r_t = f(t,t)$, the limit of bond yields as maturity goes to zero. Justification of this assumption can be given under technical conditions cited in the Notes.

We first fix a maturity date s and model the one-dimensional forward-rate process $f(\,\cdot\,,s) = \{f(t,s) : 0 \leq t \leq s\}$. We suppose that $f(\,\cdot\,,s)$ is an Ito process, meaning that

$$f(t,s) = f(0,s) + \int_0^t \mu(u,s)\,du + \int_0^t \sigma(u,s)\,dB_u^Q, \qquad 0 \le t \le s, \tag{37}$$

where $\mu(\,\cdot\,,s) = \{\mu(t,s) : 0 \le t \le s\}$ and $\sigma(\,\cdot\,,s) = \{\sigma(t,s) : 0 \le t \le s\}$ are adapted processes valued in $\mathbb{R}$ and $\mathbb{R}^d$ respectively such that, almost surely, $\int_0^s |\mu(t,s)|\,dt < \infty$ and $\int_0^s \sigma(t,s) \cdot \sigma(t,s)|\,dt < \infty$.

It turns out that there is an important consistency relationship between μ and σ. Under purely technical conditions, it must be the case that

$$\mu(t,s) = \sigma(t,s) \cdot \int_t^s \sigma(t,u)\,du. \tag{38}$$

Under technical conditions, this risk-neutral drift restriction on forward rates will be shown at the end of this section. For now, let us point out that knowledge of the initial forward rates $\{f(0,s) : 0 \le s \le T\}$ and the forward-rate "volatility" process σ is enough to determine all bond and interest-rate derivative price processes. That is, given (38), we can use the definition $r_t = f(t,t)$ of the short rate to obtain

$$r_t = f(0,t) + \int_0^t \sigma(v,t) \cdot \int_v^t \sigma(v,u)\,du\,dv + \int_0^t \sigma(v,t)\,dB_v^Q, \tag{39}$$

assuming that this process exists and is adapted. We can see that if σ is everywhere zero, the spot and forward rates must coincide, in that $r(t) = f(0,t)$ for all t, as one would expect from the absence of arbitrage in a deterministic bond market! From (39), we can price any term-structure related security using the basic formula (1), calculated numerically if necessary by Monte Carlo simulation. Indeed, aside from the Gaussian special case studied in Exercise 7.6 and certain other restrictive special cases, most valuation work in the HJM setting is done numerically. Special cases aside, there is no finite-dimensional state variable for the HJM model, so PDE-based computational methods cannot be used. Instead, one can build an analogous model in discrete time with a finite number of states, and compute prices from "first principles." For the discrete model, the expectation analogous to (1) is obtained by constructing all sample paths for r from the discretization of (39), and by computing the probability (under Q) of each. Sources given in the Notes provide details. Monte Carlo simulation can also be used, as explained in Chapter 11.

It remains to confirm the key relationship (38) between the drifts and diffusions of forward rates. Consider the Q-martingale M defined by

$$
\begin{aligned}
M_t &= E_t^Q\left[\exp\left(-\int_0^s r_u\,du\right)\right] &(40)\\
nonum &= \exp\left(-\int_0^t r_u\,du\right)\Lambda_{t,s} &(41)\\
&= \exp\left(X_t + Y_t\right), &(42)
\end{aligned}
$$

where, using (36),

$$X_t = -\int_0^t r_u\,du; \qquad Y_t = -\int_t^s f(t,u)\,du. \tag{41}$$

In order to continue, we want to show that Y, as an infinite sum of the Ito processes for forward rates over all maturities ranging from t to s, is itself an Ito process. From Fubini's Theorem for stochastic integrals (Appendix D), this is true under technical conditions on μ and σ. For example, it is certainly sufficient for Fubini's Theorem that, in addition to our previous assumptions, $\mu(t,u,\omega)$ and $\sigma(t,u,\omega)$ are uniformly bounded and, for each ω, continuous in (t,u). Under these or weaker conditions for Fubini's Theorem, moreover, we can calculate that $dY_t = \mu_Y(t)\,dt + \sigma_Y(t)\,dB_t^Q$, where

$$\mu_Y(t) = f(t,t) - \int_t^s \mu(t,u)\,du, \tag{42}$$

and

$$\sigma_Y(t) = -\int_t^s \sigma(t,u)\,du. \tag{43}$$

This is natural, given the linearity of stochastic integration explained in Chapter 5. Provided Fubini's Theorem does apply to give us (42)-(43), we can apply Ito's Formula in the usual way to $M_t = e^{X(t)+Y(t)}$ and obtain the drift under Q of M as

$$\mu_M(t) = M_t\left(\mu_Y(t) + \frac{1}{2}\sigma_Y(t)\cdot\sigma_Y(t) - r_t\right). \tag{44}$$

Because M is a Q-martingale, we must have $\mu_M = 0$, so, substituting (42) into (44), we obtain

$$\int_t^s \mu(t,u)\,du = \frac{1}{2}\left(\int_t^s \sigma(t,u)\,du\right)\cdot\left(\int_t^s \sigma(t,u)\,du\right). \tag{45}$$

Taking the derivative of each side of (42) with respect to s then leaves the risk-neutral drift restriction (38).

7K Markovian Yield-Curves and SPDEs

Specializing the HJM model, we can view the forward-rate curve $f(t) = \{f(t, t+u) : 0 \leq u \leq \infty\}$, in each state ω of Ω, as an element of the state space $C^1([0,\infty))$ of real-valued continuously differentiable functions on $[0,\infty)$. We may then suppose that

$$\sigma(t,s) = \Sigma\left(f(t), t, s-t\right), \tag{46}$$

where $\Sigma : C([0,T]\times[0,\infty)\times[0,\infty) \to \mathbb{R}^d$. For the forward-rate process $\{f(t) : t \geq 0\}$ to be well defined under Q, we would like the "volatility function" Σ to satisfy enough regularity for existence and uniquess of solutions to

$$f(t,s) = f(0,s) + \int_0^t M(f(u), u, s)\, du + \int_0^t \Sigma(f(u), u, s-u)\, dB_u, \tag{47}$$

where

$$M(\overline{f}, t, \tau) = \Sigma(\overline{f}, t, \tau) \cdot \int_t^{t+\tau} \Sigma(\overline{f}, t, u)\, du.$$

This formulation (47) is an example of a *stochastic partial differential equation* (SPDE), a general class of infinite-dimensional stochastic differential equations treated in sources cited in the Notes. While the technicalities are rather onerous and left to those sources, the idea is rather elegant. Provided the SPDE (47) is well defined, the entire forward rate curve $f(t)$ is a sufficient statistic for its future evolution. That is, the function-valued process $\{f(t) : t \geq 0\}$ is Markovian with state space $C^1([0,\infty))$, or some suitable sub-space of smooth functions in $C^1([0,\infty))$.

Applications of the SPDE approach include the *market model*, which specifies the proportional-volatility case

$$\Sigma(f(t), t, u) = \overline{\sigma}(t,u) f(t, t+u) \tag{48}$$

for some deterministic $\overline{\sigma}(t,u)$, at least for certain specified u. This special case allows for zero-coupon bond options (and therefore conventional caps) to be priced explicitly by a version of the Black-Scholes option-pricing formula that is explored in an exercise. This is a matter of some convenience in practice, for the prices of many caps and bond-options are quoted in terms of Black-Scholes implied volatilities.

Exercises

Exercise 7.1 The *Black-Derman-Toy model* is normally expressed in the form

$$r_t = U(t) \exp[\gamma(t) B^Q(t)], \tag{49}$$

for some functions U and γ in $C^1(\mathbb{R}_+)$. Find conditions on K_1, K_2, and H_2 under which the parameterization for the Black-Karasinski model shown in Table 7.1 specializes to the Black-Derman-Toy model (49).

Exercise 7.2 For the Vasicek model, as specified in Table 7.1, show that

$$\Lambda_{t,s} = \exp[\alpha(t,s) + \beta(t,s) r_t],$$

and provide $\alpha(t,s)$ and $\beta(t,s)$ in the case of time-independent K_0, K_1, and H_0.

Exercise 7.3 For the Vasicek model, for time-independent K_0, K_1, and H_0, compute the price at time zero of a zero-coupon bond call option. The underlying bond matures at time s. The option is European, struck at $c \in (0,1)$, and expiring at τ. That is, compute the price of a derivative that pays $(\Lambda_{\tau,s} - c)^+$ at time τ. Hint: Choose as a numeraire the zero-coupon bond maturing at time τ. Relative to this numeraire, the short rate is zero and the payoff of the option is unaffected since $\Lambda_{\tau,\tau} = 1$. Use Ito's Formula and the solution to the previous exercise to write a stochastic differential expression for the normalized bond price $p_t \equiv \Lambda_{t,s}/\Lambda_{t,\tau}$, $t \leq \tau$. As such, p is a "log-normal" process. Now apply the approach taken in Chapter 5 or Chapter 6. Express the solution for the bond option price in the form of the Black-Scholes option-pricing formula, replacing the usual arguments with new expressions based on K_0, K_1, and H_0. Do not forget to renormalize to the original numeraire! This exercise is extended below to the Heath-Jarrow-Morton setting.

Exercise 7.4 Show, as claimed in Section 7J, that in the absence of arbitrage the forward price at time t for delivery at time τ of a zero-coupon bond maturing at time $s > \tau$ is given by $\Lambda_{t,s}/\Lambda_{t,\tau}$. Show that if the short-rate process r is nonnegative then the forward interest rates defined by (34) and instantaneous forward rates defined by (35) are nonnegative. Finally, show (36).

Exercise 7.5 Let $\lambda_{t,\tau,s}$ denote the forward price at time t for delivery at time τ of one zero-coupon bond maturing at time s. Now consider the forward price F_t at time t for delivery at time s of a security with price process S and deterministic dividend rate process δ. Show, assuming integrability as needed, that the absence of arbitrage implies that

$$F_t = \frac{S_t}{\Lambda_{t,s}} - \int_t^s \lambda_{t,\tau,s}^{-1} \delta_\tau \, d\tau.$$

Do not assume the existence of an equivalent martingale measure.

Exercise 7.6 Consider the Gaussian forward-rate model, defined by taking the HJM model of Section 7J with coefficients $\mu(t,s)$ and $\sigma(t,s)$ of (37) that are deterministic and differentiable with respect to s. Let $d = 1$ for simplicity. Calculate the arbitrage-free price at time t of a European call option on a unit zero-coupon bond maturing at time s, with strike price K and expiration date τ, with $t < \tau < s$. To be specific, the option has payoff $(\Lambda_{\tau,s} - K)^+$ at time τ. Hint: Consider the numeraire deflator defined by normalizing prices relative to the price $\Lambda_{t,\tau}$ of the pure discount bond maturing at τ. With this deflation, compute an equivalent martingale measure $P(\tau)$ and the stochastic differential equation under $P(\tau)$ for the deflated bond-price process Z defined by $Z_t = \Lambda_{t,s}/\Lambda_{t,\tau},\ t \le \tau$ and $Z_t = \Lambda_{t,s},\ t > \tau$. Show that Z_τ is log-normally distributed under $P(\tau)$. Using the fact that $\Lambda_{\tau,\tau} = 1$, show that the relevant option price is $\Lambda_{t,\tau} E_t^{P(\tau)}[(\Lambda_{\tau,s} - K)^+]$. An explicit solution is then obtained by exploiting the Black-Scholes option-pricing formula. Under $P(\tau)$, conditioning on $\mathcal{F}_t$, one needs to compute the variance of $\log \Lambda_{\tau,s}$, which is normally distributed.

Exercise 7.7 Verify the claim that the at-market coupon rate c^* on a swp is the par fixed coupon rate.

Exercise 7.8 In the context of the HJM model with forward-rate process f, consider a bond issued at time t that pays a dividend process $\{\delta_s : t \le s \le \tau\}$ until some maturity date τ, at which time it pays 1 unit of account. Suppose that, for all s, we have $\delta_s = f(t,s)$. Show that, barring arbitrage, the price at time t of this bond is 1.

Exercise 7.9 We can derive the equivalent martingale measure Q for the HJM model as follows, at the same time obtaining conditions under which an arbitrage-free instantaneous forward-rate model f is defined in terms of the Brownian motion B under the original measure P, for each fixed maturity s, by

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)\,du + \int_0^t \sigma(u,s)\,dB_u, \qquad t \leq s. \tag{50}$$

Here, $\{\alpha(t,s) : 0 \leq t \leq s\}$ and $\{\sigma(t,s) : 0 \leq t \leq s\}$ are adapted processes valued in $\mathbb{R}$ and $\mathbb{R}^d$ respectively such that (50) is well defined as an Ito process. This proceeds as follows.

(A) For each fixed s, suppose an $\mathbb{R}^d$-valued process a^s and a real-valued process b^s are well defined by

$$a_t^s = -\int_t^s \sigma(t,v)\,dv; \qquad b_t^s = \frac{\|a_t^s\|^2}{2} - \int_t^s \alpha(t,v)\,dv, \qquad t \leq s. \tag{51}$$

Show, under additional technical conditions, that for each fixed s,

$$\Lambda_{t,s} = \Lambda_{0,s} + \int_0^t \Lambda_{u,s}(r_u + b_u^s)\,du + \int_0^t \Lambda_{u,s}\, a_u^s\,dB_u, \qquad 0 \leq t \leq s. \tag{52}$$

Now, taking an arbitrary set $\{s(1), \ldots, s(d)\}$ of d different maturities, consider the deflated bond price processes $Z^1, \ldots, Z^d$, defined by

$$Z_t^i = \exp\left(-\int_0^t r_u\,du\right)\Lambda_{t,s(i)}, \qquad t \leq s(i). \tag{53}$$

For the absence of arbitrage involving these d bonds until time $S \equiv \min\{s(i) : 1 \leq i \leq d\}$, Chapter 6 shows that it suffices, and in a sense is almost necessary, that there exists an equivalent martingale measure Q for $Z = (Z^1, \ldots, Z^d)$. For this, it is sufficient that Z has a market price of risk that is L^2-reducible, in the sense of Section 6G. The question of L^2-reducibility hinges on the drift and diffusion processes of Z. For the remainder of the exercise, we restrict ourselves to the time interval $[0, S]$.

(B) Show that, for all i,

$$dZ_t^i = Z_t^i b_t^{s(i)}\,dt + Z_t^i a_t^{s(i)}\,dB_t, \qquad t \in [0, S]. \tag{54}$$

For each $t \leq S$, let A_t be the $d \times d$ matrix whose (i, j)-element is the j-th element of the vector $a_t^{s(i)}$, and let λ_t be the vector in $\mathbb{R}^d$ whose i-th element is $b_t^{s(i)}$. We can then consider the system of linear equations

$$A_t \eta_t = \lambda_t, \qquad t \in [0, S], \tag{55}$$

to be solved for an $\mathbb{R}^d$-valued process η in $\mathcal{L}^2$. Assuming such a solution η to (55) exists, and letting $\nu(Z) = \int_0^S \eta_t \cdot \eta_t \, dt/2$ and $\xi(Z) = \exp[\int_0^S -\eta_t \, dB_t - \nu(Z)]$, Proposition 6G implies that α and σ are consistent with the absence of arbitrage, and that there exists an equivalent martingale measure for Z that is denoted $Q(S)$, provided $\exp[\nu(Z)]$ has finite expectation and $\xi(Z)$ has finite variance. In this case, we can let

$$\frac{dQ(S)}{dP} = \xi(Z). \tag{56}$$

Provided A_t is nonsingular almost everywhere, $Q(S)$ is uniquely defined. Of course, S is arbitrary. A sufficient set of technical conditions for each of the above steps is cited in the Notes.

(C) Suppose A_t is everywhere nonsingular. Using Girsanov's Theorem of Appendix D, show that, for $t \leq s$,

$$f(t, s) = f(0, s) + \int_0^t [\alpha(u, s) - \sigma(u, s)\eta_u] \, du + \int_0^t \sigma(u, s) \, dB_u^Q, \tag{57}$$

where B^Q is the standard Brownian in $\mathbb{R}^d$ under $Q(S)$ arising from Girsanov's Theorem. That is, $dB_t^Q = dB_t + \eta_t \, dt$.

(D) Show that (57) and (38) are consistent.

Exercise 7.10 (Foreign Bond Derivatives) Suppose you are to price a foreign bond option. The underlying zero-coupon bond pays one unit of foreign currency at some maturity date T, and has a dollar price process of S. With an expiration date for the option of τ and a strike price of K, the bond option pays $(S_\tau - K)^+$ dollars at time τ. Our job is to obtain the bond option-price process C.

The foreign currency price process, say U_t, is given. The foreign currency is defined as a security having a continuous dividend process of $U_t R_t$, where

R is the foreign short-rate process. The exchange-rate process U is assumed to be a strictly positive Ito process of the form

$$dU_t = \alpha_t U_t \, dt + U_t \beta_t \, dB_t,$$

where α, a real-valued adapted process, and β, an $\mathbb{R}^d$-valued adapted process, are both bounded.

Foreign interest rates are given by a forward-rate process F, as in the HJM setting. That is, the price of the given zero-coupon foreign bond at time t, in units of foreign currency, is $\exp\left(\int_t^T -F(t,u)\,du\right)$, and we have $R_t = F(t,t)$. It follows that the bond-price process in dollars is given by

$$S_t = U_t \exp\left(\int_t^T -F(t,u)\,du\right),$$

and that the price of the bond option, in dollars, is

$$C_t = E_t^Q\left[\exp\left(\int_t^\tau -r_u\,du\right)(S_\tau - K)^+\right], \tag{58}$$

where r is the dollar short-rate process and Q is an equivalent martingale measure. In general we assume that the vector X of security-price processes (in dollars) for all available securities is such that $\exp\left(\int_0^t -r_u\,du\right)X_t$ defines a Q-martingale, consistent with the definition of Q as an equivalent martingale measure. It is assumed that for each t and $s \ge t$,

$$F(t,s) = F(0,s) + \int_0^t a(u,s)\,du + \int_0^t b(u,s)\,dB_u^Q,$$

where B^Q is a standard Brownian motion in $\mathbb{R}^d$ under Q, and where the s-dependent drift process $a(\,\cdot\,,s) : \Omega \times [0,T] \to \mathbb{R}$ and the s-dependent diffusion process $b(\,\cdot\,,s) : \Omega \times [0,T] \to \mathbb{R}^d$ are assumed to satisfy sufficient regularity conditions for Q to indeed be an equivalent martingale measure and for foreign bond price processes to be well defined.

(A) Demonstrate the risk-neutral drift restriction on the foreign forward-rate process F is given by $a(t,s) = b(t,s)\cdot\left[\int_t^s b(t,u)\,du - \beta_t\right]$.

(B) Suppose the domestic forward-rate process f is also of the HJM form. That is, we have $r_t = f(t,t)$, where

$$f(t,s) = f(0,s) + \int_0^t \mu(u,s)\,du + \int_0^t \sigma(u,s)\,dB_u^Q, \tag{59}$$

and where the s-dependent drift process $\mu(\,\cdot\,, s) : \Omega \times [0, T] \to \mathbb{R}$ and the s-dependent diffusion process $\sigma(\,\cdot\,, s) : \Omega \times [0, T] \to \mathbb{R}^d$ are assumed to satisfy regularity conditions analogous to a and b, respectively. Suppose the coefficient processes β, b, and σ are all deterministic. Derive a relatively explicit expression for the foreign bond-option price.

(C) An *international yield spread option* is a derivative security that promises a dollar payoff that depends on the difference $\delta = F - f$ between the foreign and domestic forward-rate curves. For various reasons, it has been proposed to develop a model directly for the *spread curve* δ. We will have (59) and

$$\delta(t, s) = \delta(0, s) + \int_0^t m(u, s)\, du + \int_0^t v(u, s)\, dB_u^Q, \tag{60}$$

where the s-dependent drift process $m(\,\cdot\,, s) : \Omega \times [0, T] \to \mathbb{R}$ and the s-dependent diffusion process $v(\,\cdot\,, s) : \Omega \times [0, T] \to \mathbb{R}^d$ are assumed to satisfy regularity conditions analogous to a and b, respectively. Develop the drift restriction on δ. That is, obtain an expression for m that does not explicitly involve a and b. Do not assume deterministic coefficient processes β, σ, and v.

Notes

The relationship between forwards and futures in Sections 8B, 8C, and 8D was developed by Cox, Ingersoll, and Ross [981b]. The derivation given here for the martingale property (8) of futures prices is original, although the formula itself is due to Cox, Ingersoll, and Ross (1981b), as is the subsequent replication strategy. For additional work in this vein, see Bick [1994], Dezhbakhsh [1994], Duffie and Stanton [1988], and Myneni [992b]. An explicit Gaussian example is given by Jamshidian [993b] and Jamshidian and Fein [1990]. Grinblatt and Jegadeesh [1993] derived the futures prices for bonds in the setting of a Cox-Ingersoll-Ross model of the term structure. Grauer and Litzenberger [1979] give an example of the equilibrium determination of commodity forward prices. Carr [1989] provides option-valuation models for assets with stochastic dividends, in terms of the stochastic model for forward prices on the underlying asset. Carr and Chen [1993] treat the valuation of the *cheapest-to-deliver option* in Treasury Bond futures, sometimes called the *quality option*, and the associated problem of determining the futures price. For the related wildcard option, see Fleming and Whaley [1994]. **?**] treat the case of complex options.

Black [1976] showed how to extend the Black-Scholes option-pricing formula to the case of futures options. See, also, Bick [1988]. Carr [1993] and Hemler [1987] value the option to deliver various grades of the underlying asset against the futures contract. This problem is related to that of valuing compound options, and options on the maximum or minimum of several assets, which was solved (in the Black-Scholes setting) by Geske [1979], Johnson [1987], Margrabe [1978], Selby and Hodges [1987], and Stulz [1982]. On put-call parity and symmetry, see Carr [993b].

McKean [1965], Merton [1973], Harrison and Kreps [1979], and Bensoussan [1984] did important early work on American option pricing. Proposition 8G is from Karatzas [1988], although his technical conditions are slightly different. Karatzas defines the *fair price* of an American security, which turns out to be equal to the arbitrage-free price when both exist, and also extends Merton's analysis of *perpetual options*, those with no expiration. Jaillet, Lamberton, and Lapeyre [1988] and Jaillet, Lamberton, and Lapeyre [1990] review the treatment of the optimal stopping valuation problem as a variational inequality, which can be written in the form (20)–(21). A decomposition of the American option in terms of an *early exercise premium* (19) was proposed in a collection of papers by Jamshidian [989c], Jacka [1991], Kim [1990], and Carr, Jarrow, and Myneni [1992], working from McKean's (1965) formulation of the *free boundary problem*, sometimes called a *Stefan problem*. Moerbeke [1976] was the first to demonstrate, among other results, that the optimal stopping boundary S^* is continuously differentiable. In this regard, see also Ait-Sahlia [1995]. Jorgensen [994a] and Myneni [992a] survey this and other literature on American put option pricing in the Black-Scholes setting. Approximate solutions to the American option price are given by Allegretto, Barone-Adesi, and Elliott [1993], Broadie and Detemple [993a], Carr [1994], Geske and Johnson [1984], Gandhi, Kooros, and Salkin [993a], and Barone-Adesi and Elliott [1991] (who cite related literature). The behavior of the *optimal exercise boundary* near expiration is treated by Ait-Sahlia [1995], Barles, Burdeau, Romano, and Samsoen [1993], Lamberton [1993], and Charretour, Elliott, Myneni, and Viswanathan [1992]. Option pricing in a jump-diffusion setting was originated by Merton [1976], and more recently treated by Amin [993a], **?**], and Zhang [1994]. de Matos [1993] gives a method-of-simulated-moments estimation technique for American options. Yu [1993] provides additional results on American option valuation. Broadie and Detemple [993b] and Broadie and Detemple [1995] provide pricing for American capped call options, and for options on multiple assets.

Term-structure models such as those applied in Chapter 7 have been applied to commodity option valuation by Jamshidian [991b] and Jamshidian (1993b). The sell-at-the-max and buy-at-the-min lookback option valuation is from Goldman, Sosin, and Gatto [1979]. The particular representation of the sell-at-the-max put formula is copied from Conze and Viswanathan [991b]. The distribution of the maximum of a Brownian motion path between two dates, and related results on the distribution of first passage times, can be found in Chuang [1994], Dassios [1994], Harrison [1985], and Ricciardi and Sato [1988]. For other lookback option valuation results, see Conze and Viswanathan (1991b), Duffie and Harrison [1993], and Shepp and Shiryaev [1993]. The *asian option*, based on an arithmetic average of the underlying price process, is analyzed by Geman and Yor [1993], Oliveira [1994], Rogers and Shi [1994], and Yor [1991]. Akahari [1993], Miura [1992], and Yor [1993] treat the related problem of *median-price options.*

The hedging of asian and lookback options is analysed by Kat [993b]. For hedging under leverage constraints, see Naik and Uppal [1992]. For hedging with a "minimax" criterion, see Howe and Rustem [994a] and Howe and Rustem [994b].

Forms of *barrier options*, which are variously known as *knockouts*, *knock-ins*, *down-and-outs*, *up-and-ins*, *limited-risk options*, and *lock-in options* are covered by Carr and Ellis [1994], Conze and Viswanathan (1991b), Merton (1973b), and Yu (1993). On approximation methods for anlaysing path-dependent options, see Kind, Liptser, and Runggaldier [1991].

Beckers (1981) promoted the idea of using implied volatility, as measured by options prices. A generalized version of implied volatility is discussed by Bick and Reisman [1993]. Cherian and Jarrow [1993] explore a related "rationality" issue. Option pricing with stochastic volatility was proposed as an answer to the "smile curve," and analyzed, by Hull and White [1987], Scott [1987], **?**], and Wiggins [1987], and since has been addressed by Amin [993b], Amin and Ng [1993], Ball and Roma [1994], Barles, Romano, and Touzi [1993], Duan [1995], Heston [1993], Hofmann, Platen, and Schweizer [1992], Lu and Yu [1993], Platen and Schweizer [1994], Renault and Touzi [992a], Renault and Touzi [992b], Touzi [1993], and Touzi [1995]. Renault and Touzi (1992b) consider the econometric use of option price data in this setting. Amin and Jarrow [1993] treat the problem of option valuation with stochastic interest rates, in a Heath-Jarrow-Morton setting. Melino and Turnbull [1990] illustrate an application to foreign exchange option pricing. Heynen and Kat [1993] and Heynen, Kemna, and Vorst [1994] provide formulas for prediction of volatility in a Markovian setting. Nelson [1990], Nelson [1991], and Nelson [1992] treat the convergence of ARCH, GARCH, and EGARCH models to stochastic volatility models of the style considered in Section 8E, as well as related issues. Bollerslev, Chou, and Kroner [1992] and Taylor [1994] survey applications in finance for ARCH and ARCH-related models, originated by Engle [1982]. **?**], Harvey and Shephard [1993], and **?**] present related econometric techniques and results. Hobson and Rogers [1993] describe a model for endogenous stochastic volatility. Proposition 8E can be deduced from results in Karatzas and Shreve [1988].

The literature on stochastic volatility and option pricing is often linked with the extensive body of available work on econometric models of *autoregressive conditional heteroskedasticity* (ARCH), and its extensions and variants, GARCH and EGARCH defined in sources cited in the Notes. It has been shown, for example, that typical discrete-time models of heteroskedasticity, including certain ARCH and EGARCH models, converge in a natural way with time periods of shrinking length to the continuous-time stochastic volatility model in which $v_t = \log V_t$ is well defined and satisfies the *Ornstein-Uhlenbeck* stochastic differential equation

$$dv_t = (a + bv_t)\,dt + c\,dv_t, \tag{43}$$

where a, b, and c are coefficients that can be estimated from historical observations of the underlying asset-price process. As (43) is a linear stochastic differential equation, we know from Appendix E that its solution is a Gaussian process (under Q). One must bear in mind, especially for econometric applications, that our analysis has been under an equivalent martingale measure. In order to draw econometric implications, one may also wish to characterize the behavior of stochastic volatility under the original probability measure P. For example, one can adopt parametric assumptions regarding the market price of risk.

Attempts have also been made to extend the econometric model to include observations on option prices in the data set used to estimate the parameters of the stochastic volatility process. In principal, use of options data should improve the econometric efficiency of the estimation, given the one-to-one relationship between v_t and a given option price at time t that follows from the proposition above.

Derman and Kani [1994], Dupire [1992], Dupire [1994], and Rubinstein [1995] also construct implied-tree models of option pricing.

Nielsen and Saá-Requejo [1992] provide an example of a foreign exchange option-valuation model.

The results of Section 8F are based on Duffie, Pan, and Singleton [1997], which builds on the seminal work on transform-based option pricing by ?] and Heston [1993], as well as subsequent work by Bakshi, Cao, and Chen [1997], ?], and ?].

The hedging coefficients, "delta," "gamma," and so on, associated with derivative securities are studied by Carr [1991]. On option pricing with transactions costs and constraints, see references cited in the Notes of Chapter 6. Johnson and Shanno [1987] and Rich [1993] deal with the impact of default risk on the Black-Scholes approach. Brennan and Schwartz [980b] present a model for the valuation of convertible bonds.

General reviews of options, futures, or other derivative markets include those of Cox and Rubinstein [1985], Daigler [1993], Duffie [1989], Hull [1993], Jarrow and Rudd [1983], Rubinstein [1992], Siegel and Siegel [1990], and Stoll and Whaley [1993]. For computational issues, see Chapter 11, or Wilmott, Dewynne, and Howison [1993]. Dixit and Pindyck [1993] is a thorough treatment, with references, of the modeling of *real options*, which arise in the theory of production planning and capital budgeting under uncertainty.

The problem of valuing futures options, as considered in Exercise 8.7, was addressed and solved by Black (1976). The forward and futures prices for bonds in the Cox-Ingersoll-Ross model, addressed in Exercise 8.8, are found in Grinblatt [1994]. A related problem, examined by Carr [1989], is the valuation of options when carrying costs are unknown. The definition and pricing result for the market-timing option is from ?]. Gerber and Shiu [1994] describe a computational approach to option pricing based on the Escher transform.

Bossaerts and Hillion [1994]115 Kat [993a]116 Ait-Sahlia and Lai [1998]117 Ait-Sahalia and Lai [997a]118 Ait-Sahalia and Lai [997b]119 Amin and Morton [1994b]120 Amin and Morton [1994a]121 Andersen [1995]122 Andersen, Andreasen, and Brotherton-Ratcliffe [1997]123 Andersen and Brotherton-Ratcliffe [1995]124 Bakshi and Madan [1997b]125 Bakshi and Madan [1997a]126 Ball and Torous [1985]127 Barles, Burdeau, Romano, and Samsoen [1995]128 Barles and Soner [1996]129 Basak and Cuoco [995b]130 Bates [1996]131 Bates [1994]132 Bates [995a]133 Beckers [1981]134 Beibel and Lerche [1997]135 Beibel and Lerche [1995]136 Bensoussan, Crouhy, and Galai [995a]137 Bensoussan, Crouhy, and Galai [995b]138 Bergman [985a]139 Bergman, Grundy, and Wiener [1996]140 Bick [1997]141 Bottazzi and Hueffmann [1999]142

Buckdahn and Hu [995b]143 Carr [1995]144 Carr and Faguet [1996]145 Chesney, Jeanblanc, and Yor [1994]146 Corradi [1997]147 Davydov and Linetsky [1998]148 Décamps and Koehl [1994]149 Eberlein, Keller, and Prause [1998]150 Karoui, Jeanblanc, and Shreve [1995]151 Flesaker [1991]152 Frey [1996]153 Frey and Sin [1997]154 Ghysels, Gourieroux, and Jasiak [1995]155 Ghysels, Harvey, and Renault [1995]156 Goncalves and Souza [1996]157 Grabbe [1983]158 Gukhal [995a]159 Gukhal [995b]160 Harvey, Ruiz, and Shephard [1994]161 Haug [1999]162 He and Takahashi [1995]163 Heston [1997]164 Heston and Nandi [1997]165 Hobson [1995]166 Hobson and Rogers [1995]167 Huang, Subrahmanyam, and Yu [1995]168 Jackwerth [996a]169 Jackwerth and Rubinstein [1996b]170 Jackwerth and Rubinstein [1996a]171 Jackwerth and Rubinstein [1996c]172 Jakubenas [1999]173 Jamshidian [1997]174 Johnson and Stulz [1987]175 Jorion [1988]176 Ju [997a]177 Ju [997b]178 Kusuoka [996a]179 Lagnado and Osher [1996]180 Lai [1995]181 Lamoureux and Lastrapes [1993]182 **?**] Linetsky [1996]183 Linetsky [1997]184 Lyons [1995]185 Ma and Cvitanić [1996]186 Ma and Yong [1997]187 Madan and Chang [1995]188 Madan and Chang [1996]189 **?**] Maghsoodi [1998]190 Nelson [1993]191 Ordentlich and Cover [1996]192 Pappalardo [1996]193 Pham [1995]194 Pikovsky and Shreve [996a]195 Rady [1995]196 Rady [1996]197 Redekop [1995]198 Redekop and Fisher [1995]199 Romano and Touzi [1997]200 Sbuelz [1998]201 Schroder [1999]202 Schröder [1999]203 Schröder [1999]204 Schwartz [1997]205 Scott [1992]206 Scott [1997]207 Stutzer [1996]208 Subramanian [1997]209 Toft and Brown [1996]210 Topper [1998]211 Zhou [1997]212 Zou and Derman [1996]213

Chapter 8

Derivative Pricing

THIS CHAPTER APPLIES arbitrage-free pricing techniques from Chapter 6 to derivative securities that are not always easily treated by the direct PDE approach of Chapter 5. A derivative security is one whose cash flows are contingent on the prices of other securities, or on closely related indices. After summarizing the essential results from Chapter 6 for this purpose, we study the valuation of forwards, futures, European and American options, and certain exotic options. Option pricing with stochastic volatility is addressed with Fourier-transform methods.

8A Martingale Measures in a Black Box

Skipping over the foundational theory developed in Chapter 6, this section reviews the properties of an equivalent martingale measure, a convenient "black-box" approach to derivative asset pricing in the absence of arbitrage. Once again, we fix a Standard Brownian motion $B = (B^1, \ldots, B^d)$ in $\mathbb{R}^d$ restricted to some time interval $[0, T]$, on a given probability space $(\Omega, \mathcal{F}, P)$. The standard filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ of B is as defined in Section 5I.

We take as given an adapted short-rate process r, with $\int_0^T |r_t| \, dt < \infty$ almost surely, and an Ito security price process S in $\mathbb{R}^N$ with

$$dS_t = \mu_t \, dt + \sigma_t \, dB_t,$$

for appropriate μ and σ. It was shown in Chapter 6 that aside from technical conditions, the absence of arbitrage is equivalent to the existence of a probability measure Q with special properties, called an equivalent martingale measure. For this chapter, we will use a narrow definition of equivalent martingale measures under which all expected rates of return are equivalent to the riskless rate r; a broader definition is given in Chapter 6. This means that, under Q, there is a Standard Brownian Motion B^Q in $\mathbb{R}^d$ such that if the given securities pay no dividends before T, then

$$dS_t = r_t S_t \, dt + \sigma_t \, dB_t^Q, \tag{1}$$

which repeats (6.6). After substituting this "risk-neutral" measure Q for P, one can thus treat every security as though its "instantaneous expected rate of return" is the short rate r.

More generally, suppose the securities with price process S are claims to a cumulative dividend process D. (That is, D_t is the vector of cumulative dividends paid by the N securities up through time t.) In this case, we have

$$S_t = E_t^Q \left[\exp\left(\int_t^T -r_s \, ds \right) S_T + \int_t^T \exp\left(\int_t^s -r_u \, du \right) dD_s \right], \tag{2}$$

which repeats (6.18). For example, suppose that $D_t = \int_0^t \delta_s \, ds$ for some dividend-rate process δ. Then (2) implies that

$$dS_t = (r_t S_t - \delta_t) \, dt + \sigma_t \, dB_t^Q, \tag{3}$$

generalizing (1). For another example, consider a unit discount riskless bond maturing at some time s. The cumulative-dividend process, say H, of this security is characterized by $H_u = 0$ for $u < s$ and $H_u = 1$ for $u \geq s$. The price of this bond at any time $t < s$ is therefore determined by (2) as

$$\Lambda_{t,s} \equiv E_t^Q \left[\exp\left(- \int_t^s r_u \, du \right) \right].$$

This doubly indexed process Λ is sometimes known as the *discount function*, or more loosely as a *term structure model*. Details are given in Chapter 7.

By the definition of an equivalent martingale measure given in Chapter 6, any random variable Z that has finite variance with respect to P has finite expectation with respect to Q, and

$$E_t^Q(Z) = \frac{1}{\xi_t} E_t(\xi_T Z), \tag{4}$$

where

$$\xi_t = \exp\left(-\int_0^t \eta_s \, dB_s - \frac{1}{2}\int_0^t \eta_s \cdot \eta_s \, ds\right),$$

and where η is a market-price-of-risk process, that is, an adapted process in $\mathbb{R}^d$ solving the family of linear equations

$$\sigma_t \eta_t = \mu_t - r_t S_t, \qquad t \in [0, T].$$

The remainder of this chapter applies these concepts to the calculation of derivative asset prices, going beyond the simple cases treated in Chapter 5.

8B Forward Prices

Sections 8B through 8D address the pricing of forward and futures contracts, an important class of derivatives. A discrete-time primer on this topic is given in Exercise 2.17. The forward contract is the simpler of these two closely related securities. Let W be an $\mathcal{F}_T$-measurable finite-variance random variable underlying the claim payable to a holder of the forward contract at its delivery date T. For example, with a forward contract for delivery of a foreign currency at time T, the random variable W is the market value at time T of the foreign currency. The forward-price process F is an Ito process defined by the fact that one forward contract at time t is a commitment to pay the net amount $F_t - W$ at time T, with no other cash flows at any time. In particular, the true price of a forward contract, at the contract date, is zero.

We fix a bounded short-rate process r and an equivalent martingale measure Q. The dividend process H defined by the forward contract made at time t is given by $H_s = 0$, $s < T$, and $H_T = W - F_t$. Because the true price of the forward contract at t is zero, (2) implies that

$$0 = E_t^Q\left[\exp\left(-\int_t^T r_s \, ds\right)(W - F_t)\right].$$

Solving for the forward price,

$$F_t = \frac{E_t^Q\left[\exp\left(-\int_t^T r_s \, ds\right) W\right]}{E_t^Q\left[\exp\left(-\int_t^T r_s \, ds\right)\right]}.$$

If we assume that there exists at time t a zero-coupon riskless bond maturing at time T, then

$$F_t = \frac{1}{\Lambda_{t,T}} E_t^Q \left[\exp\left(- \int_t^T r_s \, ds \right) W \right]. \tag{5}$$

From this, ew see that the forward-price process F is indeed an Ito process.

If r and W are statistically independent with respect to Q, we have the simplified expression $F_t = E_t^Q(W)$, implying that the forward price is a Q-martingale. This would be true, for instance, if the short-rate process r is deterministic.

As an example, suppose that the forward contract is for delivery at time T of one unit of a particular security with price process S and dividend process D. In particular, $W = S_T$. We can obtain a more concrete representation of the forward price than (5), as follows. From (5) and (2),

$$F_t = \frac{1}{\Lambda_{t,T}} \left(S_t - E_t^Q \left[\int_t^T \exp\left(- \int_t^s r_u \, du \right) \, dD_s \right] \right). \tag{6}$$

If the short-rate process r is deterministic, we can simplify further to

$$F_t = \frac{S_t}{\Lambda_{t,T}} - E_t^Q \left[\int_t^T \exp\left(\int_s^T r_u \, du \right) \, dD_s \right]. \tag{7}$$

which is known as the *cost-of-carry formula* for forward prices.

For deterministic r and D, the cost-of-carry formula (7) can be recovered from a direct and simple arbitrage argument. As an alternative to buying a forward contract at time t, one could instead buy the underlying security at t and borrow the required cost S_t by selling riskless zero-coupon bonds maturing at T. If one lends out the dividends as they are received by buying riskless bonds maturing at T, the net payoff to this strategy at time T is the value S_T of the underlying security, less the maturity value $S_t/\Lambda_{t,T}$ of the bonds sold at t, plus the total maturity value $\int_t^T \Lambda_{s,T}^{-1} \, dD_s$ of all of the bonds purchased with the dividends received between t and T. The total is $S_T - S_t/\Lambda_{t,T} + \int_t^T \Lambda_{s,T}^{-1} \, dD_s$. The payoff of the forward contract is $S_T - F_t$. Since these two strategies have no payoffs except at T, and since both F_t and $S_t/\Lambda_{t,T} - \int_t^T \Lambda_{s,T}^{-1} \, dD_s$ are known at time t, there would be an arbitrage unless F_t and $S_t/\Lambda_{t,T} - \int_t^T \Lambda_{s,T}^{-1} \, dD_s$ are equal, consistent with (7).

We have put aside the issue of calculating the equivalent martingale measure Q. The simplest case is that in which the forward contract is redundant, for in this case, the equivalent martingale measure does not depend on the forward price. The forward contract is automatically redundant if the underlying asset is a security with deterministic dividends between the contract date t and the delivery date T, provided there is a zero-coupon bond maturing at T. In that case, the forward contract can be replicated by a strategy similar to that used to verify the cost-of-carry formula directly. Construction of the strategy is assigned as an exercise.

8C Futures and Continuous Resettlement

As with a forward contract, a futures contract with delivery date T is keyed to some delivery value W, which we take to be an $\mathcal{F}_T$-measurable random variable with finite variance. The contract is completely defined by a *futures-price process* Φ with the property that $\Phi_T = W$. As we shall see, the contract is literally a security whose price process is zero and whose cumulative dividend process is Φ. In other words, changes in the futures price are credited to the holder of the contract as they occur. See Exercise 2.17 for an explanation in discrete time.

This definition is an abstraction of the traditional notion of a futures contract, which calls for the holder of one contract at the delivery time T to accept delivery of some asset (whose spot market value is represented here by W) in return for simultaneous payment of the current futures price Φ_T. Likewise, the holder of -1 contract, also known as a *short position* of 1 contract, is traditionally obliged to make delivery of the same underlying assset in exchange for the current futures price Φ_T. This informally justifies the property $\Phi_T = W$ of the futures-price process Φ given in the definition above. Roughly speaking, if Φ_T is not equal to W (and if we continue to neglect transactions costs and other details), there is a *delivery arbitrage*. We won't explicitly define a delivery arbitrage since it only complicates the analysis of futures prices that follows. Informally, however, in the event that $W > \Phi_T$, one could buy at time T the deliverable asset for W, simultaneously sell one futures contract, and make immediate delivery for a profit of $W - \Phi_T$. Thus the potential of delivery arbitrage will naturally equate Φ_T with the delivery value W. This is sometimes known as the principle of *convergence*.

Many modern futures contracts have streamlined procedures that avoid the delivery process. For these, the only link that exists with the notion of delivery is that the terminal futures price Φ_T is contractually equated to some such variable W, which could be the price of some commodity or security, or even some abstract variable of general economic interest such as a price deflator. This procedure, finessing the actual delivery of some asset, is known as *cash settlement*. In any case, whether based on cash settlement or the absence of delivery arbitrage, we shall always take it by definition that the delivery futures price Φ_T is equal to the given delivery value W.

The institutional feature of futures markets that is central to our analysis of futures prices is *resettlement*, the process that generates daily or even more frequent payments to and from the holders of futures contracts based on changes in the futures price. As with the expression "forward price," the term "futures price" can be misleading in that the futures price Φ_t at time t is not the price of the contract at all. Instead, at each resettlement time t, an investor who has held $\overline{\theta}$ futures contracts since the last resettlement time, say s, receives the resettlement payment $\overline{\theta}(\Phi_t - \Phi_s)$, following the simplest resettlement recipe. More complicated resettlement arrangements often apply in practice. The continuous-time abstraction is to take the futures-price process Φ to be an Ito process and a *futures position process* to be some θ in $\mathcal{H}^2(\Phi)$ generating the resettlement gain $\int \theta \, d\Phi$ as a cumulative-dividend process. In particular, as we have already stated in its definition, the futures-price process Φ is itself, formally speaking, the cumulative dividend process associated with the contract. The true price process is zero, since (again ignoring some of the detailed institutional procedures), there is no payment against the contract due at the time a contract is bought or sold.

8D Arbitrage-Free Futures Prices

The futures-price process Φ can now be characterized as follows. We suppose that the short-rate process r is bounded. For all t, let $Y_t = \exp\left(-\int_0^t r_s \, ds\right)$. Because Φ is strictly speaking the cumulative-dividend process associated with the futures contract, and since the true-price process of the contract is zero, from (2) we see that

$$0 = E_t^Q\left(\int_t^T Y_s \, d\Phi_s\right), \qquad t \leq T,$$

from which it follows that the stochastic integral $\int Y \, d\Phi$ is a Q-martingale. Because r is bounded, there are constants $k_1 > 0$ and k_2 such that $k_1 \leq$

$Y_t \leq k_2$ for all t. The process $\int Y \, d\Phi$ is therefore a Q-martingale if and only if Φ is also a Q-martingale. (This seems obvious; proof is assigned as an exercise.) Since $\Phi_T = W$, we have deduced a convenient representation for the futures-price process:

$$\Phi_t = E_t^Q(W), \qquad t \in [0, T]. \tag{8}$$

If r and W are statistically independent under Q, the futures-price process Φ given by (8) and the forward-price process F given by (5) are thus identical. In particular, if r is deterministic, the cost-of-carry formula (7) applies as well to futures prices.

As for how to calculate an equivalent martingale measure Q, it is most convenient if the futures contract is redundant, for then a suitable Q can be calculated directly from the other available securities. We shall work on this approach, originating with an article cited in the Notes, and fix for the remainder of the section an equivalent martingale measure Q. Aside from the case of complete markets, it is not obvious how to establish the redundancy of a futures contract since the futures-price process Φ is itself the cumulative-dividend process of the contract, so any argument might seem circular. Suppose, however, that there is a self-financing strategy (in securities other than the futures contract) whose value at the delivery date T is

$$Z_T = W \exp\left(\int_t^T r_s \, ds\right).$$

We will give an example of such a strategy shortly. From the definition of Q, the market value of this strategy at time t is $Z_t = E_t^Q(W)$. We claim that if Φ_t is not equal to Z_t, then there is an arbitrage. In order to show this, we will construct a trading strategy, involving only the futures contract and borrowing or lending at the short rate, such that the strategy pays off exactly Z_T at time T and requires the investment of Φ_t at time t. It will be clear from this that the absence of arbitrage equates Φ_t and Z_t. The strategy is constructed as follows. Let θ be the (bounded) futures position process defined by $\theta_s = 0, s < t$, and $\theta_s = \exp\left(\int_t^s r_u \, du\right), s \geq t$. Let V_t be the amount invested at the short rate at time t, determined as follows. Let $V_s = 0$, $s < t$, and $V_t = \Phi_t$. After t, let all dividends generated by the futures position be invested at the short rate and "rolled over." That is, let

$$dV_s = r_s V_s \, ds + \theta_s \, d\Phi_s, \qquad s \in [t, T].$$

The total market value at any time $s \geq t$ of this self-financing strategy in futures and investment at the short rate is the amount V_s invested at the short rate, since the true price of the futures contract is zero. We can calculate by Ito's Formula that

$$V_T = \Phi_T \exp\left(\int_t^T r_s\, ds\right) = W \exp\left(\int_t^T r_s\, ds\right) = Z_T, \tag{9}$$

which verifies the claim that the futures contract is redundant.

Summarizing, the futures-price process is uniquely defined by (8) provided there is a self-financing strategy with value $Z_T = W \exp\left(\int_t^T r_s\, ds\right)$ at the delivery date T. It remains to look for examples in which Z_T is indeed the value at time T of some self-financing strategy. That is the case, for instance, if the futures contract delivers a security that pays no dividends before T and if the short-rate process is deterministic. With this, the purchase of $\exp\left(\int_t^T r_s\, ds\right)$ units of the underlying security at time 0 would suffice. More general examples can easily be constructed.

There is one loose end to tidy up. The assumption that the futures-price process Φ is an Ito process played a role in our analysis, yet we have not confirmed that the solution (8) for Φ is actually an Ito process. This can be shown as an application of Girsanov's Theorem (Appendix D).

8E Stochastic Volatility

The Black-Scholes option-pricing formula, as we recall from Chapter 5, is of the form $C(x, p, \overline{r}, t, \overline{\sigma})$, for $C : \mathbb{R}_+^5 \to \mathbb{R}_+$, where x is the current underlying asset price, p is the exercise price, $\overline{r}$ is the short interest rate, t is the time to expiration, and $\overline{\sigma}$ is the volatility coefficient for the underlying asset. For each fixed $(x, p, \overline{r}, t)$, the map from $\overline{\sigma}$ to $C(x, p, \overline{r}, t, \overline{\sigma})$ is strictly increasing, and its range is unbounded. We may therefore invert and obtain the volatility from the option price. That is, we can define an *implied volatility* function $I : \mathbb{R}_+^5 \to \mathbb{R}_+$ by

$$c = C(x, p, \overline{r}, t, I(x, p, \overline{r}, t, c)), \tag{10}$$

for all sufficiently large $c \in \mathbb{R}_+$.

If c_1 is the Black-Scholes price of an option on a given asset at strike p_1 and expiration t_1, and c_2 is the Black-Scholes price of an option on the same asset at strike p_2 and expiration t_2, then the associated implied volatilities

Figure8.1: The Smile Curve

$I(x, p_1, \overline{r}, t_1, c_1)$ and $I(x, p_2, \overline{r}, t_2, c_2)$ must be identical if indeed the assumptions underlying the Black-Scholes formula apply literally, and in particular if the underlying asset-price process has the constant volatility of a geometric Brownian motion. It has been widely noted, however, that actual market prices for European options on the same underlying asset have associated Black-Scholes implied volatilities that vary with both exercise price and expiration date. For example, in certain markets at certain times implied volatilities depend on strike prices in the convex manner illustrated in Figure 8.1, which is often termed a *smile curve.* Other forms of systematic deviation away from constant implied volatilities have been noted, both over time and across various derivatives at a point in time.

Three major lines of modeling address these systematic deviations from the assumptions underlying the Black-Scholes model. In both of these, the underlying log-normal price process is generalized by replacing the constant volatility parameter $\overline{\sigma}$ of the Black-Scholes model with a *volatility process*, an adapted non-negative process V with $\int_0^T V_t \, dt < \infty$ such that the underlying asset price process S satisfies

$$dS_t = r_t S_t \, dt + S_t \sqrt{V_t} \, d\epsilon_t^S, \tag{11}$$

where $\epsilon^S = c_S \cdot B^Q$ is a standard Brownian motion under Q obtained from any c_S in $\mathbb{R}^d$ with unit norm.

In the first class of models, $V_t = v(S_t, t)$, for some function $v : \mathbb{R} \times [0, T] \to \mathbb{R}$ satisfying technical regularity conditions. In practical applications, the function v, or its discrete-time discrete-state analogue, is "calibrated" to the available option prices. This approach, sometimes referred to as the *implied-tree* model, is explored in literature cited in the Notes of this chapter and of Chapter 3.

A second approach, called *generalized autoregressive conditional heteroscedastic*, or GARCH, the volatility depends on the path of squared returns. The model was originally formulated in a discrete time setting by constructing the volatility V_t at time t of the return $R_{t+1} = \log S_{t+1} - \log S_t$ according to the recursive formula

$$V_t = a + bV_{t-1} + cR_t^2,$$

for fixed coefficients a, b, and c satisfying regularity conditions. By takinga time period of length h, normalizing, and taking limits, a natural continous-time limiting behavior for volatility is simply a deterministic mean-reverting process V satisfying the ordinary differential equation

$$\frac{dV(t)}{dt} = \kappa(\overline{v} - V(t)).$$

There is, however, an alternative proposal for the continuous-time limit, discussed in the Notes.

In a third approach, the increments of the volatility process V depend on Brownian motions that are not perfectly correlated with ϵ^S. For example, in a simple "one-factor" setting the volatility process V satisfies a stochastic differential equation of the form

$$dV_t = \mu_V(V_t)\, dt + \sigma_V(V_t)\, d\epsilon_t^V, \tag{12}$$

where $\epsilon^V = c_V \cdot B^Q$ is a standard Brownian motion under Q. As we shall see, the correlation parameter $c_{SV} = c_S \cdot c_V$ has an important influence on option prices.

The Feynman-Kac approach illustrated in Chapter 5 leads, under technical conditions, to a partial differential equation to be solved for a function $f : \mathbb{R}_+ \times \mathbb{R} \times [0, t] \to \mathbb{R}$ that determines the price at time s of a European option at exercise price p and expiration at time t as

$$f(S_s, V_s, s) = E_s^Q \left[e^{-\overline{r}(t-s)}(S_t - p)^+ \right].$$

Methods for solving such a PDE by discretization are cited in Chapter 11.

A special case of the stochastic-volatility model that has sometimes been applied takes the correlation parameter c_{SV} to be zero. This implies that the volatility process V is independently distributed (under Q) with the return-driving Brownian motion ϵ^S. One can then more easily calculate the value an option (or another derivative) on the underlying asset by noting that, conditional on the volatility process V, the underlying asset price process is log-normal under Q. That is, the distribution under Q of $\log S_t$ conditional on the entire volatility process $\{V_s : s \in [0, t]\}$ is normal with standard deviation $\sigma(V)\sqrt{t}$, where

$$\sigma(V) = \frac{1}{\sqrt{t}} \left(\int_0^t V_s \, ds \right)^{1/2},$$

and with mean $\overline{r}t - \sigma(V)^2 t/2$. By the law of iterated expectations, the European call option price, with expiration date t and strike p, is given by

$$\begin{aligned} \mathcal{C}(S_0, V_0, p, \overline{r}, t) &= E^Q \left[e^{-\overline{r}t}(S_t - p)^+ \right] \\ &= E^Q \left(E^Q \left[e^{-\overline{r}t}(S_t - p)^+ \mid \{V_s : s \in [0, t]\} \right] \right) \\ &= E^Q \left[C\left(S_0, p, \overline{r}, t, \sigma(V)\right) \right], \end{aligned} \tag{13}$$

where $C(\,\cdot\,)$ as usual denotes the Black-Scholes formula. Given a particular stochastic model for V, one could evaluate the option price (13) by several numerical methods mentioned in the Notes. One finds that that the implied smile curve is indeed "smile-shaped," although it is difficult to reconcile this special case with the empirical behavior of many types of options. In particular, in many settings, a pronounced *skew* to the smile indicates an important potential role for correlation between the increments of the return-driving and volatility-driving Brownian motions, ϵ^S and ϵ^V. This role is borne out directly by the correlation apparent from time-series data on implied volatilities and returns for certain important asset classes, as indicated in sources cited in the Notes.

A tractable model that allows for the skew effects of correlation is the *Heston model*, for which

$$dV_t = \kappa(\overline{v} - V_t)\, dt + \sigma_v \sqrt{V_t}\, d\epsilon_t^V, \tag{14}$$

for positive coefficients κ, $\overline{v}$, and σ_v that play the same respective roles for V as for a Cox-Ingersoll-Ross interest rate model. (Indeed, (14) is sometimes called a "CIR model" for volatility.) In the original Heston model, the short rate was assumed to be a constant, say $\overline{r}$, and option prices can be computed analytically, using transform methods explained in the next section, in terms of the parameters $(\overline{r}, c_{SV}, \kappa, \overline{v}, \sigma_v)$ of the Heston model, as well as the initial volatility V_0, the initial underlying price S_0, the strike price, and the expriation time. This transform approach also accomodates stochastic interest rates and more general volatility models.

8F Option Valuation by Transform Analysis

This section is devoted to the calculation of option prices with stochastic volatility, in a setting with affine state dynamics of the type introduced for term-structure modeling in Chapter 7, using transform analysis. This will allow for relatively rich and tractable specifications of stochastic interest rates and volatility, and, eventually, for jumps. Repeating from Chapter 7, a state process X in state space $D \subset \mathbb{R}^k$ is affine (under Q) if

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t^Q, \tag{15}$$

where $\mu(x) = K_0 + K_1 x$ for some $K_0 \in \mathbb{R}^k$ and $K_1 \in \mathbb{R}^{k \times k}$ and, for each i and j in $\{1, \ldots, k\}$,

$$\left(\sigma(x)\sigma(x)^\top\right)_{ij} = H_{0ij} + H_{1ij} \cdot x, \tag{16}$$

for some $H_{0ij} \in \mathbb{R}$ and $H_{1ij} \in \mathbb{R}^k$, for the state space

$$D = \{x : H_{0ii} + H_{1ii} \cdot x \geq 0,\ 1 \leq i \leq n\}.$$

The Notes cite technical conditions on the coefficients (H, K) ensuring the existence of a unique solution X to (15). For time-series empirical studies, it is often convenient to suppose that the X is also affine under the data-generating probability measure P, albeit with a different set K^P of drift-related coefficients in place of K. Conditions for this, and extensions to time-dependent coefficients, are explored in exercises.

In this setting, the short-rate process r is assumed to be of the affine form $r_t = \rho_0 + \rho_1 \cdot X_t$, for coefficients ρ_0 in $\mathbb{R}$ and ρ_1 in $\mathbb{R}^k$. Finally, we suppose that the price process S underling the options in question is of the exponential-affine form $S_t = \exp(a_t + b_t \cdot X_t)$, for potentially time-dependent coefficients a_t in $\mathbb{R}$ and b_t in $\mathbb{R}^k$. An example would be the price of an equity, a foreign currency, or, as shown in Chapter 7, the price of a zero-coupon bond.

The Heston model (14) is a special case with constant short rate $\overline{r} = \rho_0$, with $k = d = 2$, with $X_t^{(1)} = Y_t \equiv \log(S_t)$, and $X_t^{(2)} = V_t$. From Ito's Formula,

$$dY_t = \left(\overline{r} - \frac{1}{2}V_t\right) dt + \sqrt{Y_t}\, d\epsilon_t^S, \tag{17}$$

which indeed makes the state vector $X_t = (Y_t, V_t)^\top$ an affine process, whose state space is $D = \mathbb{R} \times [0, \infty)$, and whose coefficients (H, K) can be chosen in terms of the parameters $(\overline{r}, c_{SV}, \kappa, \overline{v}, \sigma_v)$ of the Heston model. The underlying asset price is of the desired exponential-affine form because $S_t = e^{Y(t)}$. We will return to the Heston model shortly with some explicit results on option valuation.

For the general affine case, suppose we are interested in valuing a European call option on the underlying security, with strike price p and exercise date t. We have the initial option price

$$U_0 = E^Q\left[\exp\left(-\int_0^t r_u\, du\right)(S_u - p)^+\right].$$

Letting A denote the event $\{\omega : S(\omega, t) \geq p\}$ that the option is in the money at expiration, we have the option price

$$U_0 = E^Q\left[\exp\left(-\int_0^t r_s\, ds\right)(S_t 1_A - p 1_A)\right].$$

Because $S(t) = e^{a(t)+b(t)\cdot X(t)}$, we have

$$U_0 = e^{a(t)} G(-\log p; t, b(t), -b(t)) - pG(-\log p, t, 0, -b(t)), \tag{18}$$

where, for any $y \in \mathbb{R}$ and for any coefficient vectors d and δ in $\mathbb{R}^k$,

$$G(y; t, d, \delta) = E^Q\left[\exp\left(-\int_0^t r_s\, ds\right) e^{d\cdot X(t)} 1_{\delta\cdot X(t)\leq y}\right].$$

So, with the ability to compute the function G, we can obtain the prices of options of any strike and exercise date. Likewise, the prices of European puts, interest rate caps, chooser options, and many other derivatives can be derived in terms of G, as shown in exercises and sources cited in the Notes.

We note, for fixed (t, d, δ), assuming $E\left(e^{d \cdot X(t)}\right) < \infty$, that $G(\,\cdot\,; t, d, \delta)$ is a bounded increasing function. For any such function $g : \mathbb{R} \to [0, \infty)$, an associated *transform* $\hat{g} : \mathbb{R} \to \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, is defined by

$$\hat{g}(z) = \int_{-\infty}^{+\infty} e^{izy}\, dg(y), \tag{19}$$

where i is the usual imaginary number, often denoted $\sqrt{-1}$. (Appendix H summarizes a few minimal elements of complex arithmetic.) Depending on one's conventions, one may refer to $\hat{g}$ as the *fourier transform* of g. Under the technical condition that $\int_{-\infty}^{+\infty} |\hat{g}(z)|\, dz < \infty$, we have the *Levy Inversion Formula*

$$g(y) = \frac{\hat{g}(0)}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{1}{z} \mathrm{Im}[e^{izy} \hat{g}(iz)]\, dz, \tag{20}$$

where $\mathrm{Im}(c)$ denotes the imaginary part of a complex number c.

For the case $g(y) = G(y; t, d, \delta)$, knowing the transform $\hat{G}(\,\cdot\,; t, d, \delta)$, then we can compute $G(y; t, d, \delta)$ from (20), typically by computing the integral in (20) numerically, and thereby obtain option prices from (18). Our final objective is therefore to compute the transform $\hat{G}$. Fixing z, and applying Fubini's Theorem to (19) implies that $\hat{G}(z; t, d, \delta) = f(X_0, 0)$ where $f : D \times [0, t] \to \mathbb{C}$ is defined by

$$f(X_s, s) = E^Q \left[\exp\left(-\int_s^t r_u\, du \right) e^{d \cdot X(t)} e^{iz\delta \cdot X(t)} \;\Big|\; X_t \right]. \tag{21}$$

From (21), the same separation-of-variables arguments used in Chapter 7 imply, under technical regularity conditions, that

$$f(x, s) = e^{\alpha(s) + \beta(s) \cdot x}, \tag{22}$$

where β solves the Ricatti ordinary differential equation (ODE)

$$\beta'(s) = \rho_1 - K_1 \beta(s) - \frac{1}{2} \beta(s)^\top H_1 \beta(s), \tag{23}$$

with the boundary condition

$$\beta(t) = d + iz\delta, \tag{24}$$

and where

$$\alpha(s) = \int_s^t \left[-\rho_0 + K_0 \cdot \beta(u) + \frac{1}{2} \beta(u)^\top H_0 \beta(u) \right] du. \tag{25}$$

The ODE (23) is identical to that arising in the affine term-structure calculations of Chapter 7, but the solutions for $\alpha(t)$ and $\beta(t)$ are complex numbers, in light of the complex boundary condition (24) for $\beta(t)$. One must keep track of both the real and imaginary parts of $\alpha(s)$ and $\beta(s)$, following the usual rules of complex arithmetic outlined in Appendix H.

Thus, under technical conditions, we have our transform $\hat{G}(z; t, d, \delta)$, evaluated at a particular z. We then have the option-pricing formula (18), where $G(y; t, d, \delta)$ is obtained from the inversion formula (20) applied to the transforms $\hat{G}(\,\cdot\,; t, b(t), -b(t))$ and $\hat{G}(\,\cdot\,; t, 0, -b(t))$, obtained by solving the Riccati equation (23) with the respective boundary conditions $b(t) - izb(t)$ and $-izb(t)$. For cases in which the ODE (23) cannot be solved explicitly, its numerical computation, followed by numerical integration to obtain (20), is somewhat burdensome. Direct PDE or Monte Carlo numerical methods would typically, however, be even more computationally intensive.

For option pricing with the Heston model, we require only the transform $\psi(u) = e^{-\overline{r}t} E^Q[e^{uY(t)}]$, for some particular choices for $u \in \mathbb{C}$, and solving (23) for this case we have

$$\psi(u) = e^{\overline{\alpha}(t,u) + uY(0) + \overline{\beta}(t,u)V(0)},$$

where, letting $b = u\sigma_v c_{SV} - \kappa$, $a = u(1-u)$, and $\gamma = \sqrt{b^2 + a\sigma_v^2}$, we find that

$$\begin{aligned}
\overline{\beta}(t,u) &= -\frac{a\,(1-e^{-\gamma t})}{2\gamma - (\gamma + b)\,(1 - e^{-\gamma t})}, \\
\overline{\alpha}(t,u) &= \overline{r}t(u-1) - \kappa\overline{v}\left(\frac{\gamma + b}{\sigma_v^2}\,t + \frac{2}{\sigma_v^2}\log\left[1 - \frac{\gamma + b}{2\gamma}\left(1 - e^{-\gamma t}\right)\right]\right).
\end{aligned}$$

Other special cases for which one can compute explicit solutions are cited in the Notes, or treated in exercises.

8G American Security Valuation

This section addresses the valuation of American securities, those whose cash flows are determined by the stopping time at which the owner of the American security decides to exercise. As our setup for primitive securities, we take a bounded short-rate process r and suppose that the price process S of the other securities satisfies (1), where B^Q is a standard Brownian motion under a probability measure Q equivalent to P. We also suppose for this section that rank $(\sigma) = d$ almost everywhere, so that any random payoff with finite risk-neutral expectation can be replicated without resorting to "doubling strategies," as shown by Proposition 6I. As indicated in Chapter 2, some sort of dynamic-spanning property of this type is important for the valuation of American securities.

An *American security*, defined by an adapted process U and an expiration time $\overline{\tau}$, is a claim to the payoff U_τ at a stopping time $\tau \leq \overline{\tau}$ chosen by the holder of the security. Such a stopping time is an *exercise policy*. As with the discrete-time treatment in Chapter 2, our objective is to calculate the price process V of the American security and to characterize rational exercise policies. The classic example is the case of a put option on a stock in the Black-Scholes setting of constant-volatility stock prices and constant short rates. In that case, we have $U_t = (K - S_t)^+$, where K is the exercise price and S is the underlying asset price process. More generally, we will rely on the following technical condition.

American Regularity Condition. *U is a adapted continuous process, bounded below, with $E^Q(U_*) < \infty$, where $U_* = \sup_{t \in [0,T]} U_t$.*

This regularity is certainly satisfied for an American put option in standard settings for which the underlying price process is an Ito process.

Given some particular exercise policy τ, Proposition 6I implies that the claim to $U_\tau \exp\left(\int_\tau^T r_s \, ds\right)$ at T can be replicated by a self-financing trading strategy θ whose market-value process V^τ is given by

$$V_t^\tau = E_t^Q \left(\varphi_{t,\tau} U_\tau\right),$$

where $\varphi_{t,\tau} \equiv \exp\left(\int_t^\tau -r_s \, ds\right)$. This implies that the payoff of U_τ at time τ is replicated by the trading strategy θ^τ that is θ until time τ, and zero afterward, generating a lump-sum payment of U_τ at τ.

Following the approach taken in Section 2I, we therefore define a *rational exercise policy* as a solution to the *optimal-stopping problem*

$$V_0^* = \sup_{\tau \in \mathcal{T}(0)} V_0^\tau, \tag{26}$$

where $\mathcal{T}(t)$ denotes the set of stopping times valued in $[t, \overline{\tau}]$. This is the problem of maximizing the initial cost of replication. We will show that there is in fact a stopping time τ^* solving (26), and that the absence of "non-pathological" arbitrages implies that the American security must sell initially for V_0^*.

If $V_0 < V_0^*$, then purchase of the American security for V_0, adoption of a rational exercise policy τ^*, and replication of the payoff $-U(\tau^*)$ at τ^* at an initial payoff of V_0^*, together generate a net initial profit of $V_0^* - V_0 > 0$ and no further cash flow. This is an arbitrage.

In order to rule out the other possibility, that $V_0 > V_0^*$, we will exploit the notion of a *super-replicating trading strategy*, a self-financing trading strategy whose market-value process Y dominates the exercise-value process U, in that $Y_t \geq U_t$ for all t in $[0, \overline{\tau}]$. We will show the existence of a super-replicating trading strategy with initial market value $Y_0 = V_0^*$. If $V_0 > V_0^*$, then sale of the American security and adoption of a super-replicating strategy implies an initial profit of $V_0 - V_0^* > 0$ and the ability to cover the payment U_τ demanded by the holder of the American security at exercise with the market value Y_τ of the super-replicating strategy, regardless of the exercise policy τ used by the holder of the American security. This constitutes an arbitrage. Indeed, then, the unique arbitrage-free American security price would be given by (26). (We have implicitly extended the definition of an arbitrage slightly in order to handle American securities.)

Let $\widehat{U}$ be the deflated exercise-value process, defined by

$$\widehat{U}_t = \exp\left(\int_0^t -r_s \, ds\right) U_t.$$

Let W be the *Snell envelope* of $\widehat{U}$ under Q, meaning that

$$W_t = \operatorname*{ess\ sup}_{\tau \in \mathcal{T}(t)} E_t^Q(\widehat{U}_\tau), \qquad t \leq \overline{\tau}, \tag{27}$$

where, ess sup denotes *essential supremum.* (In other words, for all τ in $\mathcal{T}(t)$, $P(W_t \geq E_t^Q(\widehat{U}_\tau)) = 1$, and if $\hat{W}_t$ is any other $\mathcal{F}_t$-measurable random variable satisfying $P(\hat{W}_t \geq E_t^Q(\widehat{U}_\tau)) = 1$ for all τ in $\mathcal{T}(t)$, then $P(W_t \leq \hat{W}_t) = 1$.)

We recall from Chapter 6 that a trading strategy whose market-value process is bounded below cannot take advantage of certain "pathological" varieties of arbitrage, such as doubling strategies.

Proposition. *There is a super-replicating trading strategy θ^* whose market-value process Y is bounded below, with inital market value $Y_0 = V_0^*$. A rational exercise policy is given by $\tau^0 = \inf\{t : Y_t = U_t\}$.*

Proof: Under the American Regularity Conditions, a source cited in the Notes shows that the Snell envelope W of $\hat{U}$ is a continuous supermartingale under Q, and can therefore be decomposed in the form $W = Z - A$, where Z is a Q-martingale and A is an increasing adapted process with $A_0 = 0$. (This was proved in a discrete-time setting in Chapter 2.) By Proposition 6I and Numerarire Invariance, there is a self-financing trading strategy whose market-value process Y has the final market value $Y_T = Z_{\overline{\tau}} \exp\left(\int_{\overline{\tau}}^T r_t\, dt\right)$ and satisfies

$$Y_t = E_t^Q \left[\exp\left(\int_t^T -r_t\, dt \right) Y_T \right].$$

A Q-martingale $\hat{Y}$ is thus defined by $\hat{Y}_t = Y_t \exp\left(\int_0^t -r_s\, ds\right)$. Because Z is also a Q-martingale, for $t \leq \overline{\tau}$ we have

$$\begin{aligned} Y_t &= \exp\left(\int_0^t r_s\, ds\right) E_t^Q \left[\exp\left(-\int_0^T r_s\, ds \right) Y_T \right] \\ &= \exp\left(\int_0^t r_s\, ds\right) E_t^Q [Z(\overline{\tau})] \\ &= \exp\left(\int_0^t r_s\, ds\right) Z_t \\ &= \exp\left(\int_0^t r_s\, ds\right) (W_t + A_t). \end{aligned} \tag{28}$$

Taking $t = 0$ in (28), we have $Y_0 = W_0 = V_0^*$, as asserted. From (28),

$$Y_t \geq \exp\left(\int_0^t r_s\, ds\right) W_t \geq U_t, \tag{29}$$

using the facts that A_t is nonnegative, the definition of $\widehat{U}_t$, and the fact that $W_t \geq \widehat{U}_t$. Thus the underlying trading strategy is super-replicating. Because U is bounded below, (29) implies that the replicating market-value process Y is bounded below. Moreover, τ^0 is a rational exercise policy because

$$V_0^* = Y_0 = E^Q \left[\exp\left(\int_0^{\tau^0} -r_t\, dt \right) Y(\tau^0) \right] = V_0^{\tau^0},$$

from noting that $\hat{Y}$ is a Q-martingale and that $Y(\tau^0) = U(\tau^0)$. ▮

Putting the various pieces of the story together, the "arbitrage pricing" result is summarized as follows. If the initial price V_0 of the American security $(U, \overline{\tau})$ is strictly larger than

$$V_0^* = \sup_{\tau \in \mathcal{T}(0)} E^Q \left[\exp \left(\int_0^\tau -r(s)\, ds \right) U_\tau \right], \tag{30}$$

then an arbitrage consists of sale of the option and adoption of the super-replicating trading strategy θ^* until whatever exercise time τ chosen by the option holder. Conversely, if $V_0 > V_0^*$, then an arbitrage is made by purchase of the option at time 0, exercise of the option at the rational time τ^0, and adoption of the trading strategy $-\theta^*$ until liquidation at τ^0. Neither of these arbitrage strategies would be "pathological" because the market-value process Y of the super-replicating stragegy is consistent with the given equivalent martingale measure Q, and the position held in the American security is constant until exercise. Thus, the American regularity conditions and complete markets are, in effect, sufficient for (30) to be the unique price for the American security that keeps the markets for all securities free of (pathological) arbitrage.

All of our assumptions are satisfied in the case of an American put in the "Black-Scholes" setting, with a constant short rate $\overline{r}$, and an underlying price process S solving

$$dS_t = \overline{r} S_t\, dt + \overline{\sigma} S_t\, d\epsilon_t^S; \qquad S_0 = x, \tag{31}$$

where ϵ^S is a standard Brownian motion under Q. Thus, an American put with exercise price K and expiration at time $\overline{\tau}$ has the initial arbitrage-free price

$$V_0^* = \max_{\tau \in \mathcal{T}(0)} E^Q \left[e^{-\overline{r}\tau} (K - S_\tau)^+ \right]. \tag{32}$$

In Chapter 11 we review some numerical recipes for approximating this value. There need not in fact be complete markets for our results to apply in this simple setting, for even if the underlying Brownian motion is of dimension $d > 1$, the super-replicating strategy of the Proposition 8G can be constructed in terms of the underlying security with price process S and with funds invested at the short rate $\overline{r}$, and has a market-value process Y that dominates the exercise value $(K - S_\tau)^+$ at any stopping time τ, *even a stopping time τ that is determined by information not generated by the Brownian motion ϵ^S of the underlying price process S.*

By extending our arguments, we can handle an American security that promises a cumulative-dividend process H until exercised at a stopping time $\tau \leq \overline{\tau}$ for a final payoff of U_τ. The same arguments applied previously lead to an initial price of the American security $(H, U, \overline{\tau})$ given, under similar technical regularity, by

$$V_0^* = \sup_{\tau \in \mathcal{T}(0)} E^Q \left(\int_0^\tau e^{\int_0^s -r(u)\, du}\, dH_s + e^{\int_0^\tau -r(u)\, du} U_\tau \right).$$

8H American Exercise Boundaries

We take the case of an American security $(U, \overline{\tau})$ with $U_t = g(X_t, t)$, where $g : \mathbb{R}^k \times [0, T] \to \mathbb{R}$ is continuous and X is state process in $\mathbb{R}^k$ satisfying the SDE (under the equivalent martingale measure Q)

$$dX_t = a(X_t)\, dt + b(X_t)\, dB_t^Q, \tag{33}$$

for continuous functions a and b satisfying Lipschitz conditions. For simplicity, we take the interest-rate process r to be zero, and later show that, aside from technicalities, this is without loss of generality. We adopt the American regularity conditions and again assume redundancy of the American security for any exercise policy. Starting at time t with initial condition $X_t = x$ for (33), the arbitrage-free value is given by

$$h(x, t) \equiv \sup_{\tau \in \mathcal{T}(t)} E_t^Q \left[g(X_\tau, \tau) \right]. \tag{34}$$

By inspection, $h \geq g$. From Proposition 8G, an optimal exercise policy is given by

$$\tau^0 = \inf\{t \in [0, T] : h(X_t, t) = g(X_t, t)\}. \tag{35}$$

By (35), $h(X_t, t) > g(X_t, t)$ for all $t < \tau^0$. Letting

$$\mathcal{E} = \{(x, t) \in \mathbb{R}^k \times [0, T] : h(x, t) = g(x, t)\}, \tag{36}$$

we can write $\tau^0 = \inf\{t : (X_t, t) \in \mathcal{E}\}$, and safely call $\mathcal{E}$ the *exercise region*, and its complement

$$\mathcal{C} = \{(x, t) \in \mathbb{R}^k \times [0, T) : h(x, t) > g(x, t)\}$$

the *continuation region*. In order to solve the optimal exercise problem, it is enough to break $\mathbb{R}^k \times [0,T]$ into these two sets. An optimal policy is then to exercise whenever (X_t, t) is in $\mathcal{E}$, and otherwise to wait. Typically, solving for the exercise region $\mathcal{E}$ is a formidable problem.

For a characterization of the solution in terms of the solution of a partial differential equation, suppose that h is sufficiently smooth for an application of Ito's Formula. Then

$$h(X_t, t) = h(X_0, 0) + \int_0^t \mathcal{D}h(X_s, s)\, ds + \int_0^t h_x(X_s, s) b(X_s)\, dB_s^Q,$$

where

$$\mathcal{D}h(x,t) = h_t(x,t) + h_x(x,t)a(x) + \frac{1}{2}\mathrm{tr}\left[h_{xx}(x,t)b(x)b(x)^\top\right].$$

For any initial conditions (x, t) and any stopping time $\tau \geq t$, we know from the definition of h that $E^Q[h(X_\tau, \tau)] \leq h(x,t)$. From this, it is natural to conjecture that $\mathcal{D}h(x,t) \leq 0$ for all (x,t). Moreover, from the fact that M is a Q-martingale and $M_t = h(X_t, t)$, $t < \tau^0$, it is easy to see that $\mathcal{D}h(x,t) = 0$ for all (x,t) in $\mathcal{C}$. We summarize these conjectured necessary conditions on h, supressing the arguments (x,t) everywhere for brevity. On $\mathbb{R}^k \times [0,T)$

$$h \geq g, \qquad \mathcal{D}h \leq 0, \qquad (h-g)(\mathcal{D}h) = 0. \tag{37}$$

The last of these three conditions means that $\mathcal{D}h = 0$ wherever $h > g$, and conversely that $h = g$ wherever $\mathcal{D}h < 0$. Intuitively, this is a Bellman condition prescribing a policy of not exercising so long as the expected rate of change of the value function is not strictly negative. We also have the boundary condition

$$h(x,T) = g(x,T), \qquad x \in \mathbb{R}^k. \tag{38}$$

Under strong technical assumptions that can be found in sources cited in the Notes, it turns out that these necessary conditions (37)–(38) are also sufficient for h to be the value function. This characterization (37)–(38) of the value function lends itself to a finite-difference algorithm for numerical solution of the value function h.

In order to incorporate nonzero interest rates, suppose that the short-rate process r can be written in the form $r_t = R(X_t)$ for some bounded $R(\,\cdot\,)$. By

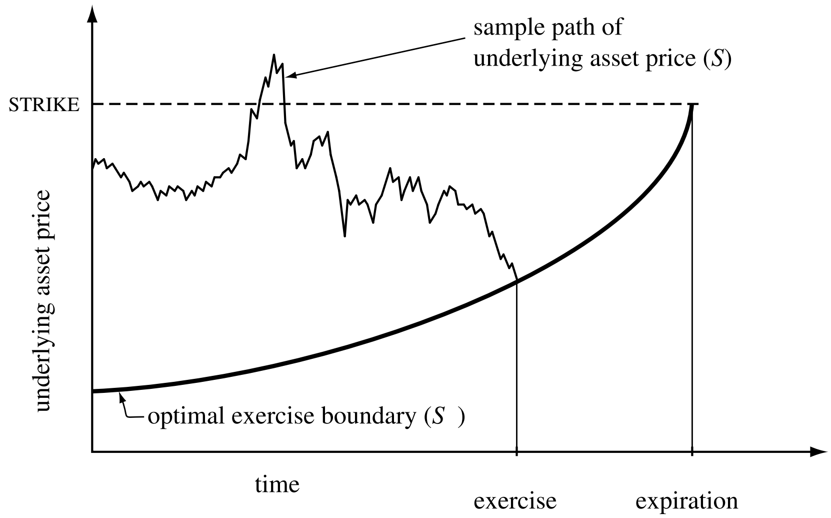

Figure8.2: Optimal Exercise Boundary for American Put

similar arguments, the variational inequality (37)–(38) for the value function h can then be written exactly as before, with the exception that $\mathcal{D}h(x,t)$ is replaced everywhere by $\mathcal{D}h(x,t) - R(x)h(x,t)$.

For the special case of the American put with a constant short rate and the constant-volatility underlying process S given by (31), a series of advances cited in the Notes has led to the following characterization of the solution, taking the state process X to be the underlying proce process S. Because S is nonnegative, the continuation region $\mathcal{C}$ can be treated as a subset of $\mathbb{R}_+ \times [0,T)$. It turns out that there is an increasing continuously differentiable function $\overline{S} : [0,T) \to \mathbb{R}$, called the *optimal exercise boundary*, such that $\mathcal{C} = \{(x,t) : x > \overline{S}_t\}$. Letting, $\overline{S}_T = K$, the optimal exercise policy τ^* is then to exercise when (and if) the stock price S "hits the optimal exercise boundary," as illustrated in Figure 8.2. That is, $\tau^0 = \inf\{t : S_t = \overline{S}_t\}$.

Unfortunately, there is no explicit solution available for the optimal stopping boundary $\overline{S}$. There are, however, numerical methods for estimating the value function h and exercise boundary $\overline{S}$. One is the simple algorithm (3.23) for the "binomial" model, which as we see in Chapter 11 can be taken as an approximation of the Black-Scholes model. The other is a direct finite-difference numerical solution of the associated *partial differential inequality*

(37)–(38), which in this case can be written, with the change of variables from stock price x to its logarithm $y = \log(x)$: On $\mathbb{R} \times [0, T)$,

$$\begin{aligned} h(y,t) - (K - e^y)^+ &\geq 0 \\ \mathcal{D}h(y,t) - \overline{r}h(y,t) &\leq 0 \\ \left[h(y,t) - (K - e^y)^+\right]\left[\mathcal{D}h(y,t) - \overline{r}h(y,t)\right] &= 0, \end{aligned} \tag{39}$$

with the boundary condition

$$h(y,T) = (K - e^y)^+, \qquad y \in \mathbb{R}, \tag{40}$$

where

$$\mathcal{D}h(y,t) = h_t(y,t) + \left(\overline{r} - \frac{\overline{\sigma}^2}{2}\right) h_y(y,t) + \frac{\overline{\sigma}^2}{2} h_{yy}(y,t). \tag{41}$$

Not all of the indicated derivatives of h may exist, but it turns out, from sources cited in the Notes, that there is no essential difficulty in treating (39)–(41) as written for computational purposes.

8I Lookback Options

Certain classes of derivative securities are said to be *path dependent*, in that the payoff of the derivative security at a given time depends on the path taken by the underlying asset price up until that time. An important and illustrative class of examples is given by *lookback options*.

Consider the case of a *sell-at-the-max option*, with exercise date T, a lookback option defined as follows. Let S be an Ito process defining the price of a security, and let r be a bounded adapted short-rate process. For any times t and $s \geq t$, let $S^*_{t,s} = \max\{S_u : t \leq u \leq s\}$ denote the maximum level achieved by the security-price process between times t and s. This particular option offers the right to sell the stock at time T for $S^*_{0,T}$. Of course, $S^*_{0,T} \geq S_T$, so the option is always exercised for a payoff of $S^*_{0,T} - S_T$. Given an equivalent martingale measure Q, the market value of the option at time t is then by definition

$$V_t = E^Q_t \left[\exp\left(- \int_t^T r_s \, ds \right) \left(S^*_{0,T} - S_T\right) \right]. \tag{42}$$

For the case of a log-normal security-price process satisfying (31) and constant short interest rate $\overline{r}$, a closed-form solution can be obtained for the lookback-option price process V. We know that $S_t = S_0 \exp\left(\nu t + \overline{\sigma}\epsilon_t^S\right)$, where ϵ^S is a standard Brownian motion under Q and $\nu = \overline{r} - \overline{\sigma}^2/2$. Thus

$$\begin{aligned} S^*_{0,T} &= \max(S^*_{0,t}, S^*_{t,T}) \\ &= \max\left[S^*_{0,t}, S_t \exp(Z_{t,T})\right], \end{aligned} \tag{43}$$

where

$$Z_{t,T} = \max_{u\in[t,T]} \nu(u-t) + \overline{\sigma}(\epsilon_u^S - \epsilon_t^S). \tag{44}$$

The cumulative distribution function F_{T-t} of $Z_{t,T}$ is known, based on sources cited in the Notes, to be given by

$$F_\tau(z) = 1 - \Phi\left(\frac{z-\nu\tau}{\overline{\sigma}\sqrt{\tau}}\right) + e^{2\nu z/\overline{\sigma}^2}\Phi\left(\frac{-z-\nu\tau}{\overline{\sigma}\sqrt{\tau}}\right), \qquad z > 0, \tag{45}$$

where Φ is the standard normal cumulative distribution function. From (43),

$$E_t^Q\left(S^*_{0,T}\right) = F_{T-t}(z^*)S^*_{0,t} + [1 - F_{T-t}(z^*)]S_t \int_{z^*}^{\infty} e^z F'_{T-t}(z)\,dz, \tag{46}$$

where $z^* = \log(S^*_{0,t}) - \log(S_t)$.

Because there are no dividends, the payment of S_T at T has a market value at time t of S_t. From (42), we thus have

$$V_t = E_t^Q\left(e^{-\overline{r}(T-t)}S^*_{0,T}\right) - S_t.$$

After computing the integral in (46), we have the lookback sell-at-the-max put option price $V_t = p(S_t, S^*_{0,t}; T-t)$, where

$$\begin{aligned} p(x,y;\tau) &= -x\Phi(-D) + e^{-\overline{r}\tau}y\,\Phi(-D + \overline{\sigma}\sqrt{\tau}) \\ &\quad + \frac{\overline{\sigma}^2}{2\overline{r}}e^{-\overline{r}\tau}x\left[-\left(\frac{x}{y}\right)^{-2\overline{r}/\overline{\sigma}^2}\Phi\left(D - \frac{2\overline{r}}{\overline{\sigma}}\sqrt{\tau}\right) + e^{\overline{r}\tau}\Phi(D)\right], \end{aligned}$$

and where

$$D = \frac{1}{\overline{\sigma}\sqrt{\tau}}\left(\log\left(\frac{x}{y}\right) + \overline{r}\tau + \frac{1}{2}\overline{\sigma}^2\tau\right).$$

The analogous arbitrage-free *buy-at-the-min* lookback price is also known in closed form. This is the derivative paying $S_T - S_{*,0,T}$ at time T, where

$$S_{*,t,s} = \min\{S_u : t \le u \le s\}. \tag{47}$$

Other varieties of path-dependent derivatives include

- *knock-outs*, whose payoff is zero if the underlying asset price touches a given boundary before expiration. (See Exercise 2.1)

- *knock-ins*, whose payoff is zero unless the underlying asset price touches a given boundary before expiration.

- *barrier* derivatives, which payoff at the point in time at which a given process, usually the underlying asset price, touches some boundary.

- *asians*, whose payoff depends on the average over time of the sample path of the underlying asset price before expiration.

Path-dependent options are sometimes called *exotic options*, despite the fact that some are commonplace in the market. Exotic options include, however, a much wider variety of options that need not be path dependent. Some relevant sources are indicated in the Notes.

Exercises

Exercise 8.1 Show by use of Girsanov's Theorem that the futures-price process Φ defined by (8) is an Ito process (under the original measure P).

Exercise 8.2 Verify (9) with Ito's Formula.

Exercise 8.3 Show, as claimed in Section 8D, that if Φ is an Ito process and Y is bounded and bounded away from zero, then $\int Y\, d\Phi$ is a martingale if and only if Φ is a martingale. Hint: Use the unique decomposition property of Ito processes.

Exercise 8.4 Suppose there are three traded securities. Security 1 has the price process X given by $X_t = x \exp(\alpha t + \beta \cdot B_t)$, where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^2$, and B is a standard Brownian motion in $\mathbb{R}^2$. Security 2 has the price process Y given by $Y_t = y \exp(at + b \cdot B_t)$, where $a \in \mathbb{R}$, $b \in \mathbb{R}^2$. Security 3 has the price process e^{rt}, where r is a constant. None of the securities pay dividends during $[0, T]$. Consider a contract paying, at some time $T > 0$, either k units of security 1 or one unit of security 2, whichever is preferred by the owner of the contract at that time. Calculate the arbitrage-free price of that contract.

Exercise 8.5 Consider the *rolling spot futures contract*, defined as follows. The contract promises to pay, continually, changes in the spot price S_t of the underlying asset as well as futures resettlement payments. That is, the rolling spot futures-price process associated with the contract, say U, contractually satisfies $U_T = S_T$, and the total gain for the contract between any two times t and $s \geq t$ is $G_s - G_t$, where $G_t = U_t + S_t$. Show that, given the existence of an equivalent martingale measure (which is essentially equivalent to no arbitrage), the rolling spot futures-price process is given by $U_t = 2F_t - S_t$, where F_t is the conventional futures price.

Exercise 8.6 Compute the value of the buy-at-the-min lookback option defined in Section 8I under the same assumptions used to derive the price (46) of the sell-at-the-max option.

Exercise 8.7 Consider an underlying asset price process S satisfying (31), the log-normal model, and a futures option on the asset. That is, we let f be the futures price for delivery of the underlying asset at time T, and consider a security that pays $(f_\tau - K)^+$ at some given time $\tau < T$. Compute the initial price of this futures option.

Exercise 8.8 Suppose the short-rate process r is given by the Cox-Ingersoll-Ross model, defined in Section 7D. Compute the futures price and the forward price that would apply at time zero for delivery at time $\tau < T$ of a zero-coupon bond maturing at time T. Hint: For the futures price, conjecture the form $f(r_t, t) = e^{\alpha(t)+\beta(t)r(t)}$, for $t < \tau$, for time-dependent coefficients α and β. You will determine an ordinary differential equation for β, with a boundary condition at τ. Consider the change of variables given by $b(t) = \beta(t)^{-1}$. Solve for b, then β, then α.

Exercise 8.9 (A Market-Timing Option Valuation Formula) Consider an economy with a constant short rate of r, and a firm whose equity price process is a geometric Brownian motion S. We assume, as usual, that for given real parameters α and σ,

$$S_t = S_0 \exp(\alpha t + \sigma B_t),$$

where B is a standard Brownian motion. A constant-proportional investment strategy is defined by the fraction $b \in \mathbb{R}$ of the value of a portfolio of the riskless and risky securities that is invested in the risky asset, with rebalancing to maintain this fraction constantly over time. (We know, for example, that under HARA utility and technical regularity, such a policy is optimal.)

The wealth process $W(b)$ associated with such a policy coefficient b, with initial wealth $W(b)_0 = 1$, is unambiguously defined for each $b \in \mathbb{R}$.

If one had advance knowledge at time zero of the path of returns, $\{\alpha t + \sigma B_t : t \in [0, T]\}$ taken by the underlying share price, one could take advantage of this knowledge in choosing from among different available constant proportional policies. Such an advanced-knowledge policy is defined by a constant risky-investment fraction f that is an $\mathcal{F}_T$-measurable random variable. Of course, we assume that such advance knowledge is not available. Nevertheless, a broker-dealer decides to offer for a sale a derivative security that pays off the value of the constant proportional investment strategy that would have been chosen with advance knowledge. The broker-dealer calls this a *market-timing option*. This market-timing option pays off $W(b^*)_T$ at a fixed time T, where, for each ω, the fraction $b^*(\omega)$ solves, the problem

$$\max_{b \in [0,1]} W(b, \omega)_T.$$

The requirement that b be chosen in $[0, 1]$ means no short sales of either asset. In other words, the broker-dealer will give the investor the payoff associated with a constant proportional investment policy whose risky fraction is chosen at time T *after seeing what happened*!

(A) Compute the market value of the market-timing option, assuming the absence of arbitrage. In this setting, the type of arbitrage to rule out would be in the form of a self-financing trading strategy $\theta \in \mathcal{H}^2(S, e^{-rt})$ for stock and bank account, combined with a fixed position in the market timing option. Show that there is no arbitrage at the price that you propose, and that there is an arbitrage at any other price.

(B) Suppose that the broker-dealer offers to sell a market-timing option with the one change being that the risky-asset proportion is not constrained to $[0, 1]$, but is completely unconstrained. If there is a finite arbitrage-free price for the market-timing option, compute it and show why, as above. If not, show why not.

Exercise 8.10 (Gold Delivery Contract Pricing) The spot price process S of gold is assumed to be a geometric Brownian motion with constant volatility parameter v. The interest rate is assumed to be a constant $r \geq 0$. We assume for simplicity that gold can be stored costlessly, and pays no dividends. That is, the usual assumptions underlying the Black-Scholes model apply.

A gold-mining firm offers the following delivery contract to special customers. For a price of p, set and paid on date 0, the customer receives the right, but not the obligation, to buy n ounces of gold from the mining firm on dates chosen by the customer between dates $T > 0$ and $U > T$, inclusive, at the spot price S_T set on date T. For each date t between T and U, the decision of whether or not to purchase gold on date t at the price S_T can be made based on all information available on date t. At most k ounces can be purchased per day. (Purchase decisions can be made at a fixed time, say noon, each day, when the spot market for gold is open.) For notational simplicity, you may assume that n/k is an integer and that $U - T \geq n/k$.

(A) Provide, and justify, an explicit formula for the unique arbitrage-free price p of the gold delivery contract. Suggestion: Reduce the difficulty of this problem, in stages, to that of a problem that you know how to solve. Don't apply brute force. Provide your reasoning.

(B) For an extra fee of f, set and paid on date 0, the customer can elect on a given date T_0, between 0 and T, to double the size of the contract to $2n$ ounces by making an additional payment of p on date T_0. State explicitly the arbitrage-free level of f. You may assume for notational simplicity that $U - T \geq 2n/k$.

Exercise 8.11 (Valuation of Options with Bankruptcy) For this exercise, please refer to the appendix on counting processes. Suppose, in a given economy, that there is a constant short rate r. Consider a firm whose equity price process is a geometric Brownian motion X, until bankruptcy. We assume that there is an equivalent martingale measure Q (after normalization of prices, as usual, by e^{-rt}) such that, for given real parameters α and σ, we have

$$X_t = X_0 \exp(\alpha t + \sigma B_t^Q),$$

where B^Q is a standard Brownian motion under Q. Bankruptcy occurs with "risk-neutralized" arrival intensity λ. That is, bankruptcy occurs at the stopping time $\tau = \inf\{t : N_t = 1\}$, where, under Q, N is a Poisson process with intensity λ, independent of B^Q. When bankruptcy occurs, the price of the equity jumps to zero.

(A) Compute the price of a European call option on the equity for expiration at time T and for strike price $K > 0$. *Hint:* The point of the exercise is that the firm may go bankrupt before expiration. The option pays off if the firm does not go bankrupt, and if the price of the equity is above the strike.

(B) Compute the price of a European put option on the equity for expiration at time T and for strike price $K > 0$, without using put-call parity. Now show that put-call parity holds.

Exercise 8.12 For the Heston model with parameter vector $(\overline{r}, c_{SV}, \kappa, \overline{v}, \sigma_v)$, this exercise explores a variation on the transform solution method. For $h = 0.5$ or $h = -0.5$, let $Y^{(h)}$ solve the SDE

$$dY_t^{(h)} = (\overline{r} + hV_t)\ dt + \sqrt{V_t}\ d\epsilon_t^S; \quad Y_0^{(h)} = \log S_0.$$

For each real number y and time t, let

$$g_{h,t}(y) = P(Y_t^{(h)} \geq y). \tag{48}$$

(A) Show that the transform $\hat{g}_{h,t}$ of $g_{h,t}(\,\cdot\,)$ is given by

$$\hat{g}_{h,t}(z) = \exp\left[C(t,z) + D(t,z)V_0 + izY_0^{(h)}\right], \tag{49}$$

where

$$\begin{aligned} C(t,z) &= \overline{r}izt + \frac{\kappa\overline{v}}{\sigma_v^2}\left[(\kappa - c_{SV}\,\sigma_v zi + \varphi)t - 2\log\left(\frac{1-\zeta e^{t\varphi}}{1-\zeta}\right)\right] \\ D(t,z) &= \frac{\kappa - c_{SV}\,\sigma_v i + \varphi}{\sigma_v^2} \cdot \left(\frac{1-e^{t\varphi}}{1-\zeta e^{t\varphi}}\right), \end{aligned}$$

for

$$\zeta = \frac{\kappa - c_{SV}\,\sigma_v zi + \varphi}{\kappa - c_{SV}\,\sigma_v zi - \varphi},$$

with

$$\varphi = \sqrt{(c_{SV}\,\sigma_v zi - \kappa)^2 - \sigma_v^2(2hzi - z^2)}.$$

(B) Provide the price of a call option on the underlying, expiring at time t and struck at p, in terms of $g_{0.5,t}(\,\cdot\,)$ and $g_{-0.5,t}(\,\cdot\,)$. Hint: In order to compute

$G(y; t, b(t), -b(t))$ of the option-pricing formula (18), consider treating the underlying asset price process S as a numeraire.

(C) For the Heston model with parameters

$$(\overline{r}, c_{SV}, \kappa, \overline{v}, \sigma_v) = (0.05, -0.5, 0.5, 0.09, 0.09),$$

let $S_0 = 100$ and $V_0 = 0.09$. Plot or tabulate (at least 10 points) the smile curve (Black-Scholes implied volatilities) for European call options with expiration at time $t = 0.25$. Now re-plot for the correlation parameter $c_{SV} = 0$. You may compute (20) with any convenient numerical-integration software.

Exercise 8.13 (Volatility Hedging) As we know from Chapter 5, the implied-tree model has the property that an option can be replicated, and therefore hedged, by a self-financing trading strategy involving the underlying asset and funds invested at the short rate r. The stochastic-volatility model does not generally have this property. Under regularity conditions, however, one can replicate a given option with a self-financing trading strategy involving the underlying asset, *a different option*, and funds invested at the riskless rate r.

(A) To be specific, consider the model (11) for the underlying price process S and (12) for the volatility process V, with correlation coefficient $c_{SV} \in (0, 1)$. Let $\Phi_t = f(S_t, V_t, t)$ and $\Gamma_t = g(S_t, V_t, t)$ denote the prices at time t of options for expiration at a future time τ, and different strike prices p_1 and p_2, respectively. Suppose that f and g solve the PDEs suggested by the Feynman-Kac approach. Suppose the short rate is a constant $\overline{r}$. Let $\theta = (a, b, c)$ denote a self-financing trading strategy in the price process $(S_t, \Phi_t, e^{\overline{r}t})$ that replicates the payoff of the option with strike p_2 and price process Γ. Solve for θ_t explicitly in terms of f and g and their partial derivatives, making minimal assumptions.

(B) For the Heston model, with parameters

$$(\overline{r}, c_{SV}, \kappa, \overline{v}, \sigma_v) = (0.05, -0.5, 0.5, 0.09, 0.09),$$

let $S_0 = 100$, $V_0 = 0.09$, and $p_1 = 100$. Tabulate (or plot) estimates of the initial replicating portfolio (a_0, b_0, c_0) as the exercise price p_2 of the option to be replicated ranges from 60 to 150, showing at least 10 points. In order to estimate partial derivatives, you may use either analytic methods (show your work), or take the usual discretization approximation, under which, for small Δ, we have $F'(x) \simeq [F(x + \Delta) - F(x - \Delta)]/2\Delta$.

Exercise 8.14 (Option Valuation in an Affine Term-Structure Model)

This problem shows how to calculate the prices of options on bonds in a setting with affine state dynamics. We start slightly differently than in Section 7I by beginning with the behavior of the state process X, not under an equivalent martingale measure Q, but rather under the originally given measure P, assuming that X solves a stochastic differential equation of the form

$$dX_t = [K_0^P + K_1^P X_t]\, dt + \sigma(X_t)\, dB_t, \quad X_0 \in D \subset \mathbb{R}^k, \tag{50}$$

where B is a standard Brownian motion in $\mathbb{R}^d$ under P, and σ satisfies (25). We assume additional conditions on the coefficients $K^P = (K_0^P, K_1^P)$ and $H = (H_{0ij}, H_{1ij})$ that ensure existence of a unique solution X to (50) in the state space D defined by (26). You may assume for simplicity that D contains an open set.

(A) The state-price deflator π for the given economy is assumed to be the process π defined by $\pi_t = \exp(X_t^{(1)})$, the exponential of the first co-ordinate of the state process $X = (X^{(1)}, \ldots, X^{(k)})^\top$. This implies that the short-rate process r is defined by $r_t = \rho_0 + \rho_1 \cdot X_t$ for some $\rho_0 \in \mathbb{R}$ and $\rho_1 \in \mathbb{R}^k$. Compute ρ_0 and ρ_1.

(B) The price at time t of a zero-coupon bond maturing at a given time $s \geq t$ is by definition of π given by $f(X_t, t, s) = E_t(\pi_s)/\pi_t$. Extending the ideas in Section 7I, conjecture the ordinary differential equation (ODE) solved by some $a : [0, s] \to \mathbb{R}$ and $b : [0, s] \to \mathbb{R}^k$ such that $f(x, t, s) = \exp[a(s-t) + b(s-t) \cdot x]$. Do not forget to provide boundary conditions. This sort of ordinary differential equation (which we shall see again below) is easy to solve numerically, provided a solution exists (which we assume). Perform a verification of your candidate solution for f under technical integrability conditions provided by you. Hint: If an Ito process is a martingale, its drift process must be zero.

(C) Suppose there are $n > 0$ such zero-coupon bonds available for trade, with maturity dates $T_1, T_2, \ldots, T_n$. Provide a spanning condition under which, in principle, an additional security with payoff at some time T given by a bounded $\mathcal{F}_T$-measurable random variable Z is redundant, given the opportunity to trade the n bonds and to borrow or lend continuously at the short rate r? Develop notation as you need it.

(D) We are now interested in pricing a zero-coupon bond option that provides the opportunity, but not the obligation, to *sell* at time τ for a given price $p > 0$ the zero-coupon bond maturing at a given time $s > \tau$ (paying one unit of account at maturity). The option is in the money at expiration if and only if $\exp[a(s-\tau) + b(s-\tau) \cdot X_\tau] \leq p$, which is the event A that $b(s-\tau) \cdot X_\tau \leq \log p - a(s-\tau)$. It can be seen under integrability conditions that the price of the option at time 0 is therefore of the form $c_1 Q_1(A) - c_2 Q_2(A)$, for some coefficients c_1 and c_2 and some probability measures Q_1 and Q_2 equivalent to P. Provide the integrability conditions, the Radon-Nikodym derivatives of Q_1 and Q_2 with respect to P, and the definitions of c_1 and c_2. These coefficients should be easily obtained from knowledge of the initial term structure of interest rates. Why?

(E) For each $i \in \{1, 2\}$, the density process ξ_i of Q_i can be shown (under technical integrability conditions) to be of the form $\xi_i(t) = \exp[\alpha_i(t) + \beta_i(t) \cdot X_t]$, for α_i and β_i solving ordinary differential equations. Provide these ordinary differential equations, their boundary equations, and technical integrability conditions justifying this solution. How would you find the coefficient c_1?

(F) Under Q_i, for each $i \in \{1, 2\}$, provide a new stochastic differential equation for the state-vector process X driven by a standard Brownian motion $B^{Q(i)}$ in $\mathbb{R}^d$ under Q_i. Define $B^{Q(i)}$.

(G) By virtue of the previous steps, you have shown that the bond option price can be easily computed if one can compute any probability of the form $Q_i(\gamma \cdot X_T \leq y)$, for any $\gamma \in \mathbb{R}^d$. One can compute $P(Z \leq z)$, for a given random variable Z and number z, if one knows the characteristic function $\psi : \mathbb{R} \to \mathbb{R}$ of Z, defined by $\psi(u) = E[\exp(iuZ)]$, where $i = \sqrt{-1}$ is the usual imaginary number. Thus, option pricing in this setting can be reduced to the calculation of the characteristic function (sometimes called the Fourier transform — the Fourier transform and the characteristic function are identical up to a scalar multiple) of $Z = \gamma \cdot X_T$ under Q_1 and Q_2. Based on the results of Section 8F, we conjecture that, fixing u and d, and defining $\psi(X_t, t) = E_t[\exp(iud \cdot X_T)]$, we have

$$\psi(x, t) = \exp\left[\alpha(t) + \beta(t) \cdot x\right],$$

where α and β are complex-valued coefficient functions that solve ordinary differential equations (ODEs). Taking as given the time-dependent coefficients $(K(t), H(t))$ for X appropriate to the probability measure at hand, your task is now to provide these ODEs for α and β, *with their boundary* *conditions*, and to confirm the conjecture under integrability conditions *provided by you* that will arise as you verify your solution with Ito's Formula. Hint: Remember that $\psi(x,t)$ is a complex number, and apply Ito's Formula to get the drift (both the real and imaginary parts). You will see a restriction on the drift that will give you the desired ODEs. Express the ODEs as tidily as possible. Note: In practice, at this point, you would compute the solutions of the differential equation for each u, separately (there are tricks that can speed this up), and from the resulting characteristic function, numerically compute the needed numbers c_1, c_2, $Q_1(A)$ and $Q_2(A)$.

Notes

The relationship between forwards and futures in Sections 8B, 8C, and 8D was developed by Cox, Ingersoll, and Ross [981b]. The derivation given here for the martingale property (8) of futures prices is original, although the formula itself is due to Cox, Ingersoll, and Ross (1981b), as is the subsequent replication strategy. For additional work in this vein, see Bick [1994], Dezhbakhsh [1994], Duffie and Stanton [1988], and Myneni [992b]. An explicit Gaussian example is given by Jamshidian [993b] and Jamshidian and Fein [1990]. Grinblatt and Jegadeesh [1993] derived the futures prices for bonds in the setting of a Cox-Ingersoll-Ross model of the term structure. Grauer and Litzenberger [1979] give an example of the equilibrium determination of commodity forward prices. Carr [1989] provides option-valuation models for assets with stochastic dividends, in terms of the stochastic model for forward prices on the underlying asset. Carr and Chen [1993] treat the valuation of the *cheapest-to-deliver option* in Treasury Bond futures, sometimes called the *quality option*, and the associated problem of determining the futures price. For the related wildcard option, see Fleming and Whaley [1994]. **?**] treat the case of complex options.

Black [1976] showed how to extend the Black-Scholes option-pricing formula to the case of futures options. See, also, Bick [1988]. Carr [1993] and Hemler [1987] value the option to deliver various grades of the underlying asset against the futures contract. This problem is related to that of valuing compound options, and options on the maximum or minimum of several assets, which was solved (in the Black-Scholes setting) by Geske [1979], Johnson [1987], Margrabe [1978], Selby and Hodges [1987], and Stulz [1982]. On put-call parity and symmetry, see Carr [993b].

McKean [1965], Merton [1973], Harrison and Kreps [1979], and Bensoussan [1984] did important early work on American option pricing. Proposition

8G is from Karatzas [1988], although his technical conditions are slightly different. Karatzas defines the *fair price* of an American security, which turns out to be equal to the arbitrage-free price when both exist, and also extends Merton's analysis of *perpetual options*, those with no expiration. Jaillet, Lamberton, and Lapeyre [1988] and Jaillet, Lamberton, and Lapeyre [1990] review the treatment of the optimal stopping valuation problem as a variational inequality, which can be written in the form (20)–(21). A decomposition of the American option in terms of an *early exercise premium* (19) was proposed in a collection of papers by Jamshidian [989c], Jacka [1991], Kim [1990], and Carr, Jarrow, and Myneni [1992], working from McKean's (1965) formulation of the *free boundary problem*, sometimes called a *Stefan problem*. Moerbeke [1976] was the first to demonstrate, among other results, that the optimal stopping boundary S^* is continuously differentiable. In this regard, see also Ait-Sahlia [1995]. Jorgensen [994a] and Myneni [992a] survey this and other literature on American put option pricing in the Black-Scholes setting. Approximate solutions to the American option price are given by Allegretto, Barone-Adesi, and Elliott [1993], Broadie and Detemple [993a], Carr [1994], Geske and Johnson [1984], Gandhi, Kooros, and Salkin [993a], and Barone-Adesi and Elliott [1991] (who cite related literature). The behavior of the *optimal exercise boundary* near expiration is treated by Ait-Sahlia [1995], Barles, Burdeau, Romano, and Samsoen [1993], Lamberton [1993], and Charretour, Elliott, Myneni, and Viswanathan [1992]. Option pricing in a jump-diffusion setting was originated by Merton [1976], and more recently treated by Amin [993a], **?**], and Zhang [1994]. de Matos [1993] gives a method-of-simulated-moments estimation technique for American options. Yu [1993] provides additional results on American option valuation. Broadie and Detemple [993b] and Broadie and Detemple [1995] provide pricing for American capped call options, and for options on multiple assets.

Term-structure models such as those applied in Chapter 7 have been applied to commodity option valuation by Jamshidian [991b] and Jamshidian (1993b). The sell-at-the-max and buy-at-the-min lookback option valuation is from Goldman, Sosin, and Gatto [1979]. The particular representation of the sell-at-the-max put formula is copied from Conze and Viswanathan [991b]. The distribution of the maximum of a Brownian motion path between two dates, and related results on the distribution of first passage times, can be found in Chuang [1994], Dassios [1994], Harrison [1985], and Ricciardi and Sato [1988]. For other lookback option valuation results, see Conze and Viswanathan (1991b), Duffie and Harrison [1993], and Shepp and Shiryaev

[1993]. The *asian option*, based on an arithmetic average of the underlying price process, is analyzed by Geman and Yor [1993], Oliveira [1994], Rogers and Shi [1994], and Yor [1991]. Akahari [1993], Miura [1992], and Yor [1993] treat the related problem of *median-price options*.

The hedging of asian and lookback options is analysed by Kat [993b]. For hedging under leverage constraints, see Naik and Uppal [1992]. For hedging with a "minimax" criterion, see Howe and Rustem [994a] and Howe and Rustem [994b].

Forms of *barrier options*, which are variously known as *knockouts*, *knock-ins*, *down-and-outs*, *up-and-ins*, *limited-risk options*, and *lock-in options* are covered by Carr and Ellis [1994], Conze and Viswanathan (1991b), Merton (1973b), and Yu (1993). On approximation methods for anlaysing path-dependent options, see Kind, Liptser, and Runggaldier [1991].

Beckers (1981) promoted the idea of using implied volatility, as measured by options prices. A generalized version of implied volatility is discussed by Bick and Reisman [1993]. Cherian and Jarrow [1993] explore a related "rationality" issue. Option pricing with stochastic volatility was proposed as an answer to the "smile curve," and analyzed, by Hull and White [1987], Scott [1987], **?**], and Wiggins [1987], and since has been addressed by Amin [993b], Amin and Ng [1993], Ball and Roma [1994], Barles, Romano, and Touzi [1993], Duan [1995], Heston [1993], Hofmann, Platen, and Schweizer [1992], Lu and Yu [1993], Platen and Schweizer [1994], Renault and Touzi [992a], Renault and Touzi [992b], Touzi [1993], and Touzi [1995]. Renault and Touzi (1992b) consider the econometric use of option price data in this setting. Amin and Jarrow [1993] treat the problem of option valuation with stochastic interest rates, in a Heath-Jarrow-Morton setting. Melino and Turnbull [1990] illustrate an application to foreign exchange option pricing. Heynen and Kat [1993] and Heynen, Kemna, and Vorst [1994] provide formulas for prediction of volatility in a Markovian setting. Nelson [1990], Nelson [1991], and Nelson [1992] treat the convergence of ARCH, GARCH, and EGARCH models to stochastic volatility models of the style considered in Section 8E, as well as related issues. Bollerslev, Chou, and Kroner [1992] and Taylor [1994] survey applications in finance for ARCH and ARCH-related models, originated by Engle [1982]. **?**], Harvey and Shephard [1993], and **?**] present related econometric techniques and results. Hobson and Rogers [1993] describe a model for endogenous stochastic volatility. Proposition 8E can be deduced from results in Karatzas and Shreve [1988].

The literature on stochastic volatility and option pricing is often linked with the extensive body of available work on econometric models of *auto-regressive conditional heteroskedasticity* (ARCH), and its extensions and variants, GARCH and EGARCH defined in sources cited in the Notes. It has been shown, for example, that typical discrete-time models of heteroskedasticity, including certain ARCH and EGARCH models, converge in a natural way with time periods of shrinking length to the continuous-time stochastic volatility model in which $v_t = \log V_t$ is well defined and satisfies the *Ornstein-Uhlenbeck* stochastic differential equation

$$dv_t = (a + bv_t)\,dt + c\,d\zeta_t, \tag{43}$$

where a, b, and c are coefficients that can be estimated from historical observations of the underlying asset-price process. As (43) is a linear stochastic differential equation, we know from Appendix E that its solution is a Gaussian process (under Q). One must bear in mind, especially for econometric applications, that our analysis has been under an equivalent martingale measure. In order to draw econometric implications, one may also wish to characterize the behavior of stochastic volatility under the original probability measure P. For example, one can adopt parametric assumptions regarding the market price of risk.

Attempts have also been made to extend the econometric model to include observations on option prices in the data set used to estimate the parameters of the stochastic volatility process. In principal, use of options data should improve the econometric efficiency of the estimation, given the one-to-one relationship between v_t and a given option price at time t that follows from the proposition above.

Derman and Kani [1994], Dupire [1992], Dupire [1994], and Rubinstein [1995] also construct implied-tree models of option pricing.

Nielsen and Saá-Requejo [1992] provide an example of a foreign exchange option-valuation model.

The results of Section 8F are based on Duffie, Pan, and Singleton [1997], which builds on the seminal work on transform-based option pricing by **?**] and Heston [1993], as well as subsequent work by Bakshi, Cao, and Chen [1997], **?**], and **?**].

The hedging coefficients, "delta," "gamma," and so on, associated with derivative securities are studied by Carr [1991]. On option pricing with transactions costs and constraints, see references cited in the Notes of Chapter 6. Johnson and Shanno [1987] and Rich [1993] deal with the impact of default risk on the Black-Scholes approach. Brennan and Schwartz [980b] present a model for the valuation of convertible bonds.

General reviews of options, futures, or other derivative markets include those of Cox and Rubinstein [1985], Daigler [1993], Duffie [1989], Hull [1993], Jarrow and Rudd [1983], Rubinstein [1992], Siegel and Siegel [1990], and Stoll and Whaley [1993]. For computational issues, see Chapter 11, or Wilmott, Dewynne, and Howison [1993]. Dixit and Pindyck [1993] is a thorough treatment, with references, of the modeling of *real options*, which arise in the theory of production planning and capital budgeting under uncertainty.

The problem of valuing futures options, as considered in Exercise 8.7, was addressed and solved by Black (1976). The forward and futures prices for bonds in the Cox-Ingersoll-Ross model, addressed in Exercise 8.8, are found in Grinblatt [1994]. A related problem, examined by Carr [1989], is the valuation of options when carrying costs are unknown. The definition and pricing result for the market-timing option is from **?**]. Gerber and Shiu [1994] describe a computational approach to option pricing based on the Escher transform.

Bossaerts and Hillion [1994]115 Kat [993a]116 Ait-Sahlia and Lai [1998]117 Ait-Sahalia and Lai [997a]118 Ait-Sahalia and Lai [997b]119 Amin and Morton [1994b]120 Amin and Morton [1994a]121 Andersen [1995]122 Andersen, Andreasen, and Brotherton-Ratcliffe [1997]123 Andersen and Brotherton-Ratcliffe [1995]124 Bakshi and Madan [1997b]125 Bakshi and Madan [1997a]126 Ball and Torous [1985]127 Barles, Burdeau, Romano, and Samsoen [1995]128 Barles and Soner [1996]129 Basak and Cuoco [995b]130 Bates [1996]131 Bates [1994]132 Bates [995a]133 Beckers [1981]134 Beibel and Lerche [1997]135 Beibel and Lerche [1995]136 Bensoussan, Crouhy, and Galai [995a]137 Bensoussan, Crouhy, and Galai [995b]138 Bergman [985a]139 Bergman, Grundy, and Wiener [1996]140 Bick [1997]141 Bottazzi and Hueffmann [1999]142 Buckdahn and Hu [995b]143 Carr [1995]144 Carr and Faguet [1996]145 Chesney, Jeanblanc, and Yor [1994]146 Corradi [1997]147 Davydov and Linetsky [1998]148 Décamps and Koehl [1994]149 Eberlein, Keller, and Prause [1998]150 Karoui, Jeanblanc, and Shreve [1995]151 Flesaker [1991]152 Frey [1996]153 Frey and Sin [1997]154 Ghysels, Gourieroux, and Jasiak [1995]155 Ghysels, Harvey, and Renault [1995]156 Goncalves and Souza [1996]157 Grabbe [1983]158 Gukhal [995a]159 Gukhal [995b]160 Harvey, Ruiz, and Shephard [1994]161 Haug [1999]162 He and Takahashi [1995]163 Heston [1997]164 Heston and Nandi [1997]165 Hobson [1995]166 Hobson and Rogers [1995]167 Huang, Subrahmanyam, and Yu [1995]168 Jackwerth [996a]169 Jackwerth and Rubinstein [1996b]170 Jackwerth and Rubinstein [1996a]171 Jackwerth and Rubinstein [1996c]172 Jakubenas [1999]173 Jamshidian [1997]174 Johnson and Stulz [1987]175 Jorion [1988]176 Ju [997a]177 Ju [997b]178 Kusuoka [996a]179 Lagnado and Osher [1996]180 Lai [1995]181 Lamoureux and Lastrapes [1993]182 **?**] Linetsky [1996]183 Linetsky [1997]184 Lyons [1995]185 Ma and Cvitanić [1996]186 Ma and Yong [1997]187 Madan and Chang [1995]188 Madan and Chang [1996]189 **?**] Maghsoodi [1998]190 Nelson [1993]191 Ordentlich and Cover [1996]192 Pappalardo [1996]193 Pham [1995]194 Pikovsky and Shreve [996a]195 Rady [1995]196 Rady [1996]197 Redekop [1995]198 Redekop and Fisher [1995]199 Romano and Touzi [1997]200 Sbuelz [1998]201 Schroder [1999]202 Schröder [1999]203 Schröder [1999]204 Schwartz [1997]205 Scott [1992]206 Scott [1997]207 Stutzer [1996]208 Subramanian [1997]209 Toft and Brown [1996]210 Topper [1998]211 Zhou [1997]212 Zou and Derman [1996]213

Chapter 9

Portfolio and Consumption Choice

THIS CHAPTER PRESENTS basic results on optimal portfolio and consumption choice, first using dynamic programming, then using general martingale and utility-gradient methods. We begin with a review of the Hamilton-Jacobi-Bellman equation for stochastic control, and then apply it to Merton's problem of optimal consumption and portfolio choice in finite- and infinite-horizon settings. Then, exploiting the properties of equivalent martingale measures from Chapter 6, Merton's problem is solved once again in a non-Markovian setting. Finally, we turn to the general utility-gradient approach from Chapter 2, and show that it coincides with the approach of equivalent martingale measures.

9A Stochastic Control

Dynamic programming in continuous time is often called *stochastic control* and uses the same basic ideas applied in the discrete-time setting of Chapter 3. The existence of well-behaved solutions in a continuous-time setting is a delicate matter, however, and we shall focus mainly on necessary conditions. This helps us to conjecture a solution that, if correct, can often be easily validated.

Given is a standard Brownian motion $B = (B^1, \ldots, B^d)$ in $\mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, P)$. We fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ of B and begin with the time horizon $[0, T\,]$ for some finite $T > 0$. The primitive objects of a stochastic control problem are

- a set $A \subset \mathbb{R}^m$ of *actions*.
- a set $\mathcal{Y} \subset \mathbb{R}^K$ of *states*.
- a set $\mathcal{C}$ of A-valued adapted processes, called *controls*.
- a *controlled drift function* $g : A \times \mathcal{Y} \to \mathbb{R}^K$.
- a *controlled diffusion function* $h : A \times \mathcal{Y} \to \mathbb{R}^{K \times d}$.
- a *running reward function* $f : A \times \mathcal{Y} \times [0, T\,] \to \mathbb{R}$.
- a *terminal reward function* $F : \mathcal{Y} \to \mathbb{R}$.

The K-dimensional set $\mathcal{Y}$ of states of the problem is not to be confused with the underlying set Ω of "states of the world." A control c in $\mathcal{C}$ is *admissible* given an initial state y in $\mathcal{Y}$ if there is a unique Ito process Y^c valued in $\mathcal{Y}$ with

$$dY_t^c = g(c_t, Y_t^c)\, dt + h(c_t, Y_t^c)\, dB_t; \qquad Y_0^c = y. \tag{1}$$

For this, there are of course technical conditions required of c, g, and h.

Let $\mathcal{C}_a(y)$ denote the set of admissible controls given initial state y. We assume that the primitives $(A, Y, \mathcal{C}, g, h, f, F)$ are such that, given any initial state $y \in \mathcal{Y}$, the utility of any admissible control c is well defined as

$$V^c(y) = E\left[\int_0^T f(c_t, Y_t^c, t)\, dt + F(Y_T^c)\right],$$

which we allow to take the values $-\infty$ or $+\infty$. The *value* of an initial state y in $\mathcal{Y}$ is then

$$V(y) = \sup_{c \in \mathcal{C}_a(y)} V^c(y), \tag{2}$$

with $V(y) = -\infty$ if there is no admissible control given initial state y. If $V^c(y) = V(y)$, then c is an *optimal control* at y. (One may note that this forumulation allows for the possibility that an optimal control achieves infinite utility.)

One usually proceeds by conjecturing that $V(y) = J(y, 0)$ for some J in $C^{2,1}(\mathcal{Y} \times [0, T])$ that solves the Bellman equation:

$$\sup_{a \in A} \{\, \mathcal{D}^a J(y, t) + f(a, y, t)\} = 0, \qquad (y, t) \in (\mathcal{Y}, [0, T)), \tag{3}$$

where

$$\mathcal{D}^a J(y,t) = J_y(y,t)g(a,y) + J_t(y,t) + \frac{1}{2}\operatorname{tr}\left[h(a,y)h(a,y)^\top J_{yy}(y,t)\right],$$

with the boundary condition

$$J(y,T) = F(y), \qquad y \in \mathcal{Y}. \tag{4}$$

An intuitive justification of (3) is obtained from an analogous discrete-time, discrete-state, discrete-action setting, in which the Bellman equation would be something like

$$J(y,t) = \max_{a \in A} \left\{ f(a,y,t) + E\left[\, J(Y^c_{t+1}, t+1) \mid Y^c_t = y,\, c_t = a \right] \right\},$$

where $f(a,y,t)$ is the running reward per unit of time. (The reader is invited to apply imagination liberally here. A complete development and rigorous justification of this analogy goes well beyond the goal of illustrating the idea. Sources that give such a justification are cited in the Notes.) For any given control process c, this discrete-time Bellman equation implies that

$$E_t\left[\, J(Y^c_{t+1}, t+1) - J(Y^c_t, t)\right] + f(c_t, Y^c_t, t) \leq 0,$$

which, for a model with intervals of length Δt, may be rewritten

$$E_t\left[\, J(Y^c_{t+\Delta t}, t+\Delta t) - J(Y^c_t, t)\right] + f(c_t, Y^c_t, t)\Delta t \leq 0.$$

Now, returning to the continuous-time setting, dividing the last equation by Δt, and taking limits as $\Delta t \to 0$ leaves, under technical conditions described in Chapter 5,

$$\frac{d}{ds} E_t\left[\, J(Y^c_s, s)\right]\Big|_{s=t+} + f(c_t, Y^c_t, t) = \mathcal{D}^{c(t)} J(Y^c_t, t) + f(c_t, Y^c_t, t) \leq 0,$$

with equality if c_t attains the supremum in the discrete version of the Bellman equation. This leads, again only by this incomplete heuristic argument, to the Bellman equation (3).

The continuous-time Bellman equation (3) is often called the *Hamilton-Jacobi-Bellman (HJB)* equation. One may think of $J(y,t)$ as the optimal utility remaining at time t in state y. Given a solution J to (3)–(4), suppose that a measurable function $C : \mathcal{Y} \times [0,T] \to A$ is defined so that for each

(y, t), the action $C(y, t)$ solves (3). The intuitive idea is that if the time is t and the state is y, then the optimal action is $C(y, t)$. In order to verify the optimality of choosing actions in this manner for the original problem (2), we turn this feedback form of control policy function C into a control in the sense of problem (2), that is, a process in the set $\mathcal{C}$. For this, suppose that there is a $\mathcal{Y}$-valued solution Y^* to the stochastic differential equation (SDE)

$$dY_t^* = g[C(Y_t^*, t), Y_t^*]\, dt + h[C(Y_t^*, t), Y_t^*]\, dB_t; \qquad Y_0^* = y.$$

Conditions on the primitives of the problem sufficient for the existence of a unique solution Y^* to this SDE are often difficult to formulate because C depends on J, which is not usually an explicit function. Sources indicated in the Notes address this existence issue. Given Y^*, we can conjecture that an optimal control c^* is given by $c_t^* = C(Y_t^*, t)$, and attempt to verify this conjecture as follows. Let $c \in \mathcal{C}_a(y)$ be an arbitrary admissible control. We want to show that $V^{c^*}(y) \geq V^c(y)$. By (3),

$$\mathcal{D}^{c(t)} J(Y_t^c, t) + f(c_t, Y_t^c, t) \leq 0, \qquad t \in [0, T]. \tag{5}$$

By Ito's Formula,

$$J(Y_T^c, T) = J(y, 0) + \int_0^T \mathcal{D}^{c(t)} J(Y_t^c, t)\, dt + \int_0^T \beta_t\, dB_t, \tag{6}$$

where $\beta_t = J_y(Y_t^c, t) h(c_t, Y_t^c)$, $t \in [0, T)$. Supposing that the local martingale $\int \beta\, dB$ is in fact a martingale (which is usually verified on a problem-by-problem basis or circumvented by special tricks), we know that $E(\int_0^T \beta_t\, dB_t) = 0$. We can then take the expectation of each side of (6) and use the boundary condition (4) and inequality (5) to see that

$$V^c(y) = E\left[\int_0^T f(c_t, Y_t^c, t)\, dt + F(Y_T^c)\right] \leq J(y, 0). \tag{7}$$

The same calculation applies with $c = c^*$, except that the inequalities in (5) and (7) may be replaced with equalities, implying that

$$J(y, 0) = V^{c^*}(y). \tag{8}$$

Then (7) and (8) imply that $V(y) = J(y, 0)$, and that c^* is indeed optimal.

This is only a sketch of the general approach, with several assumptions made along the way. These assumptions can be replaced by strong technical regularity conditions on the primitives $(A, \mathcal{Y}, \mathcal{C}, g, h, f, F)$, but the known general conditions are too restrictive for most applications in finance. Instead, one typically uses the Bellman equation (3)–(4) as a means of guessing an explicit solution that, if correct, can often be validated by some variation of the above procedure. In some cases, the Bellman equation can also be used as the basis for a finite-difference numerical solution, as indicated in sources cited in Chapter 11.

9B Merton's Problem

We now apply the stochastic control approach to the solution of a "classic" optimal-consumption and investment problem in continuous time.

Suppose $X = (X^{(0)}, X^{(1)}, \ldots, X^{(N)})$ is an Ito process in $\mathbb{R}^{N+1}$ for the prices of $N+1$ securities. For each $i \geq 1$, we assume that

$$dX_t^{(i)} = \mu_i X_t^{(i)}\, dt + X_t^{(i)} \sigma^{(i)}\, dB_t; \qquad X_0^{(i)} > 0, \tag{9}$$

where σ^i is the i-th row of a matrix σ in $\mathbb{R}^{N \times d}$ with linearly independent rows, and where μ_i is a constant. This implies, in particular, that each process $X^{(i)}$ is a geometric Brownian motion of the sort used in the Black-Scholes model of option pricing. We suppose that $\sigma^{(0)} = 0$, so that $r = \mu_0$ is the short rate.

Utility is defined over the space D of *consumption* pairs (c, Z), where c is an adapted nonnegative consumption-rate process with $\int_0^T c_t\, dt < \infty$ almost surely and Z is an $\mathcal{F}_T$-measurable nonnegative random variable describing terminal lump-sum consumption. Specifically, $U : D \to \mathbb{R}$ is defined by

$$U(c, Z) = E\left[\int_0^T u(c_t, t)\, dt + F(Z)\right], \tag{10}$$

where

- $F : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave with $F(0) = 0$;
- $u : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ is continuous and, for each t in $[0, T]$, $u(\,\cdot\,, t) : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave, with $u(0, t) = 0$;
- F is strictly concave or zero, or for each t in $[0, T\,]$, $u(\,\cdot\,, t)$ is strictly concave or zero.

- At least one of u and F is non-zero.

A trading strategy is a process $\theta = (\theta^{(0)}, \ldots, \theta^{(N)})$ in $\mathcal{L}(X)$. (As defined in Chapter 5, this means merely that the stochastic integral $\int \theta \, dX$ exists.) Given an initial wealth $w > 0$, we say that (c, Z, θ) is *budget-feasible*, denoted $(c, Z, \theta) \in \Lambda(w)$, if (c, Z) is a consumption choice in D and $\theta \in \mathcal{L}(X)$ is a trading strategy satisfying

$$\theta_t \cdot X_t = w + \int_0^t \theta_s \, dX_s - \int_0^t c_s \, ds \geq 0, \qquad t \in [0, T], \tag{11}$$

and

$$\theta_T \cdot X_T \geq Z. \tag{12}$$

The first restriction (11) is that the current market value $\theta_t \cdot X_t$ of the trading strategy is non-negative and equal to its initial value w, plus any gains from security trade, less the cumulative consumption to date. The nonnegative wealth restriction can be viewed as a credit constraint, as in Section 6C. The second restriction (12) is that the terminal portfolio value is sufficient to cover the terminal consumption. We now have the problem, for each initial wealth w,

$$\sup_{(c,Z,\theta) \in \Lambda(w)} U(c, Z). \tag{13}$$

In order to convert this problem statement (13) into one that is more easily addressed within the stochastic-control formulation set up in Section 9A, we represent trading strategies in terms of the fractions $\varphi^{(1)}, \ldots, \varphi^{(N)}$ of total wealth held in the "risky" securities, those with price processes $X^{(1)}, \ldots, X^{(N)}$, respectively. Because wealth is restricted to be nonnegative, this involves no loss of generality. That is, for a given trading strategy θ in the original sense, we can let

$$\varphi_t^{(n)} = \frac{\theta_t^{(n)} X_t^{(n)}}{\theta_t \cdot X_t}, \qquad \theta_t \cdot X_t \neq 0, \tag{14}$$

with $\varphi_t^{(n)} = 0$ if $\theta_t \cdot X_t = 0$.

Problem (13) is converted into a standard control problem of the variety in Section 9A by first defining a state process. Using portfolio fractions to define trading strategies allows one to leave the security price process X out of the definition of the state process for the control problem. Instead, we can define a one-dimensional state process W for the investor's total wealth. As a notational convenience, we let $\lambda \in \mathbb{R}^N$ denote the vector in $\mathbb{R}^N$ with $\lambda_i = \mu_i - r$, the "excess expected rate of return" on security i. Given a consumption process c and an adapted process $\varphi = (\varphi^{(1)}, \dots, \varphi^{(N)})$ defining fractions of total wealth held in the risky securities, we can pose the question of existence of a nonnegative Ito process for wealth W satisfying

$$dW_t = [W_t(\varphi_t \cdot \lambda + r) - c_t]\, dt + W_t \varphi_t^\top \sigma \, dB_t; \qquad W_0 = w. \tag{15}$$

One may notice that in order for W to remain nonnegative, an admissible control (c, φ) has the property that $\varphi_t = 0$ and $W_t = c_t = 0$ for t larger than the stopping time $\inf\{s : W_s = 0\}$. Thus nonzero investment or consumption are ruled out once there is no remaining wealth.

The control problem associated with (13) is thus fixed by defining the primitives $(A, \mathcal{Y}, \mathcal{C}, g, h, f, F)$ as follows:

- $A = \mathbb{R}_+ \times \mathbb{R}^N$, with typical element $(\overline{c}, \overline{\varphi})$ representing the current consumption rate $\overline{c}$ and the fractions $\overline{\varphi}_1, \dots, \overline{\varphi}_N$ of current wealth invested in the risky securities.
- $\mathcal{Y} = \mathbb{R}_+$, with typical element w representing current wealth.
- $\mathcal{C}$ is the set of adapted processes (c, φ) valued in $\mathbb{R}_+$ and $\mathbb{R}^N$, respectively, with $\int_0^T c_t \, dt < \infty$ almost surely and $\int_0^T \varphi_t \cdot \varphi_t \, dt < \infty$ almost surely.
- $g[(\overline{c}, \overline{\varphi}), w] = w\overline{\varphi} \cdot \lambda + rw - \overline{c}$.
- $h[(\overline{c}, \overline{\varphi}), w] = w\overline{\varphi}^\top \sigma$.
- $f[(\overline{c}, \overline{\varphi}), w, t] = u(\overline{c}, t)$, where u is as given by (10).
- $F(w)$ is as given by (10).

An admissible control given initial wealth w is a control (c, φ) in $\mathcal{C}$ for which there is a unique nonnegative Ito process W satisfying (16).

The control problem $(A, \mathcal{Y}, \mathcal{C}, g, h, f, F)$ is equivalent to the original problem (13), with the small exception that the control formulation forces the investor to consume all terminal wealth, whereas the original budget constraint (12) allows the investor to leave some terminal wealth unconsumed. Because the terminal utility function F is increasing, however, this distinction is not important. Indeed, one can see that (c, Z, θ) is in $\Lambda(w)$ if and only if the wealth process $\{W_t = \theta_t \cdot X_t : t \in [0, T\,]\}$ is non-negative, satisfies (16), and $W_T \geq Z$, where φ is defined from θ by (15).

9C Solution to Merton's Problem

The Bellman equation (3) for Merton's problem, at a current level $w > 0$ of wealth, is

$$\sup_{(\overline{c}, \overline{\varphi}) \in A} \left\{ \mathcal{D}^{\overline{c}, \overline{\varphi}} J(w, t) + u(\overline{c}, t) \right\} = 0, \tag{16}$$

where

$$\mathcal{D}^{\overline{c}, \overline{\varphi}} J(w, t) = J_w(w, t)(w\overline{\varphi} \cdot \lambda + rw - \overline{c}) + J_t(w, t) + \frac{w^2}{2}\, \overline{\varphi}^\top \sigma \sigma^\top \overline{\varphi} J_{ww}(w, t),$$

with the boundary condition

$$J(w, T) = F(w), \qquad w \geq 0. \tag{17}$$

For any t, one may think of $J(\,\cdot\,, t)$ as the investor's *indirect utility function for wealth* at time t. We note that

$$J(0, t) = \int_t^T u(0, s)\, ds + F(0), \qquad t \in [0, T\,], \tag{18}$$

from our remark regarding the nonnegativity of solutions to (16).

Assuming that for each t, $u(\,\cdot\,, t)$ is strictly concave and twice continuously differentiable on $(0, \infty)$, the first-order condition for interior optimal choice of $\overline{c}$ in (16) implies that

$$\overline{c} = C(w, t) \equiv I[\, J_w(w, t), t],$$

where $I(\,\cdot\,, t)$ inverts $u_c(\,\cdot\,, t)$, meaning that $I[u_c(x, t), t] = x$ for all x and t. We let $I = 0$ if $u = 0$. Assuming that the indirect utility function $J(\,\cdot\,, t)$ for wealth is strictly concave, the first-order condition for optimal choice of $\overline{\varphi}$ in (16) implies that

$$\overline{\varphi} = \Phi(w, t) \equiv \frac{-J_w(w, t)}{wJ_{ww}(w, t)} (\sigma\sigma^\top)^{-1} \lambda. \tag{19}$$

We remark that the optimal portfolio fractions are given by a fixed vector $\tilde{\varphi} = (\sigma\sigma^\top)^{-1}\lambda$ of portfolio weights multiplied by the Arrow-Pratt measure of relative risk tolerance (reciprocal of relative risk aversion) of the indirect utility function $J(\,\cdot\,, t)$. This means effectively that, in this setting of time-homogeneous Gaussian returns and additive utility, it is enough for purposes of optimal portfolio choice for any investor to replace all risky investments with a single investment in a single "mutual fund," a security that invests in the given risky securities, trading among them so as to maintain the given proportions $\tilde{\varphi}$ of total asset value in each. Every investor would be content to invest in only this mutual fund and in riskless borrowing and lending, although different investors would have different fractions of wealth in the mutual fund, depending on the risk aversion of their indirect utility for wealth. Of course, this would not necessarily be consistent with market clearing, and therefore with equilibrium.

We focus for now on the special case of $u = 0$ and $F(w) = w^\alpha/\alpha$ for some coefficient $\alpha \in (0, 1)$. The associated *relative risk aversion* is $1-\alpha$. This is an example of a utility function that is often called *hyperbolic absolute risk averse (HARA)*. Because we take $u = 0$, only terminal consumption Z is optimally nonzero. An explicit value function J for this HARA utility example is conjectured as follows. Suppose Z is the optimal terminal consumption for initial wealth level 1. Now consider a new level w of initial wealth, for some fixed $w \in (0, \infty)$. One may see that wZ must be the associated optimal terminal consumption. Certainly, wZ can be obtained in a budget-feasible manner, for if $Z = \theta_T \cdot X_T$ can be obtained from initial wealth 1 with a trading strategy θ then, given the linearity of stochastic integrals, $w\theta$ is a budget-feasible strategy for initial wealth w, and we have $w\theta_T \cdot X_T = wZ$. If there were some alternative terminal consumption $\widehat{Z}$ that is budget-feasible with initial wealth w and with the higher utility

$$E\left(\frac{\widehat{Z}^\alpha}{\alpha}\right) > E\left[\frac{(wZ)^\alpha}{\alpha}\right], \tag{20}$$

then we would have a contradiction, as follows. First, dividing (20) through by w^α/α leaves

$$E\left[\left(\frac{\widehat{Z}}{w}\right)^\alpha\right] > E(Z^\alpha). \tag{21}$$

By the above reasoning, $\widehat{Z}/w$ can be financed with initial wealth 1, but then (21) contradicts the optimality of Z for initial wealth 1. Thus, as asserted, for any initial level of wealth w, the optimal terminal consumption is wZ. It follows that

$$J(w,0) = E\left[\frac{(wZ)^\alpha}{\alpha}\right] = \frac{w^\alpha}{\alpha}K,$$

where $K = E(Z^\alpha)$. Up to a constant K, which depends on all of the basic parameters $(\lambda, r, \sigma, \alpha, T)$ of the model, we have a reasonable conjecture for the initial indirect utility function $J(\,\cdot\,, 0)$. For $t \in (0,T)$, the optimal remaining utility can be conjectured in exactly the above manner, treating the problem as one with a time horizon of $T-t$ rather than T. We therefore conjecture that $J(w,t) = k(t)w^\alpha/\alpha$ for some function $k : [0,T] \to \mathbb{R}$. In order for J to be sufficiently differentiable for an application of Ito's Formula, we conjecture that k is itself continuously differentiable.

With this conjecture for J in hand, we can solve (19) to get

$$\overline{\varphi} = \frac{(\sigma\sigma^\top)^{-1}\lambda}{1-\alpha}, \tag{22}$$

or fixed portfolio fractions. The total fraction of wealth invested in risky assets is therefore decreasing in the relative risk aversion $1-\alpha$. Because $u = 0$, we have

$$\overline{c} = C(w,t) \equiv 0. \tag{23}$$

We can substitute (22)–(23) into the Bellman equation (16), using our conjecture $J(w,t) = k(t)w^\alpha/\alpha$ to obtain the ordinary differential equation

$$k'(t) = -\epsilon k(t), \tag{24}$$

where

$$\epsilon = \frac{\alpha\lambda^\top(\sigma\sigma^\top)^{-1}\lambda}{2(1-\alpha)} + r\alpha, \tag{25}$$

with the boundary condition from (17):

$$k(T) = 1. \tag{26}$$

Solving (24)–(26), we have

$$k(t) = e^{\epsilon(T-t)}, \qquad t \in [0,T].$$

We have found that the function J defined by $J(w,t) = e^{\epsilon(T-t)}w^\alpha/\alpha$ solves the Bellman equation (16) and the boundary condition (17), so J is therefore a logical candidate for the value function.

We now verify this candidate for the value function, and also that the conjectured optimal control (c^*, φ^*), which is given by $c_t^* = 0$ and $\varphi_t^* = (\sigma\sigma^\top)^{-1}\lambda/(1-\alpha)$, is indeed optimal. Let (c, φ) be an arbitrary admissible control for initial wealth w, and let W be the associated wealth process solving (16). From the Bellman equation (16), Ito's Formula, and the boundary condition (17), we have

$$J(w,0) + \int_0^T \beta_t \, dB_t \geq F(W_T), \tag{27}$$

where $\beta_t = J_w(W_t, t)W_t\varphi_t^\top \sigma$. Because J is nonnegative, the Bellman equation (16) and Ito's Formula also imply that a nonnegative process M is defined by $M_t = J(w, 0) + \int_0^t \beta_s \, dB_s$. We also know that M is a local martingale (as defined in Appendix D). A nonnegative local martingale is a supermartingale (a fact also stipulated in Appendix D). By taking expectations of each side of (27), this implies that for an arbitrary admissible control, $J(w, 0) \geq E[F(W_T)]$.

It remains to show that for the candidate optimal control (c^*, φ^*), with associated wealth process W^*, we have $J(w, 0) = E[F(W_T^*)]$, verifying optimality. For the candidate optimal control (c^*, φ^*), the Bellman equation and the same calculations leave us with

$$J(w,0) + \int_0^T \beta_t \, dB_t = F(W_T^*), \tag{28}$$

where equality appears in place of the inequality in (27) because the proposed optimal control achieves the supremum in the Bellman equation. Because $J_w(W_t^*, t)W_t^* = e^{\epsilon(T-t)}(W_t^*)^\alpha$, it can be seen as an exercise that for the candidate optimal control, $E\left(\int_0^T \beta_t^2 \, dt\right) < \infty$, implying by Proposition 5B that $\int \beta \, dB$ is a martingale. Taking expectations through (28) then leaves $J(w, 0) = E[F(W_T^*)]$, verifying the optimality of (c^*, φ^*) and confirming that the problem has optimal initial utility $J(w, 0) = e^{\epsilon T} w^\alpha/\alpha$.

9D The Infinite-Horizon Case

The primitives $(A, \mathcal{Y}, \mathcal{C}, g, h, f)$ of an infinite-horizon control problem are just as described in Section 9A, dropping the terminal reward F. The running reward function $f : A \times \mathcal{Y} \times [0, \infty) \to \mathbb{R}$ is usually defined, given a *discount* *rate* $\rho \in (0, \infty)$, by $f(a, y, t) = e^{-\rho t} v(a, y)$, for some $v : A \times \mathcal{Y} \to \mathbb{R}$. Given an initial state y in $\mathcal{Y}$, the value of an admissible control c in $\mathcal{C}$ is

$$V^c(y) = E\left[\int_0^\infty e^{-\rho t} v(c_t, Y_t^c)\, dt\right], \tag{29}$$

assuming that the expectation exists, where Y^c is given by (1). The supremum value $V(y)$ is as defined by (2). The finite-horizon Bellman equation (3) is replaced with

$$\sup_{a \in A} \{\mathcal{D}^a J(y) - \rho J(y) + v(a, y)\} = 0, \qquad y \in \mathcal{Y}, \tag{30}$$

for J in $C^2(\mathcal{Y})$, where

$$\mathcal{D}^a J(y) = J_y(y) g(a, y) + \frac{1}{2} \operatorname{tr}\left[h(a, y) h(a, y)^\top J_{yy}(y)\right].$$

Rather than the boundary condition (4), one can add technical conditions yielding the so-called *transversality condition*

$$\lim_{T \to \infty} E\left(e^{-\rho T} |\, J(Y_T^c)|\right) = 0, \tag{31}$$

for any given initial state $Y_0^c = y$ in $\mathcal{Y}$ and any admissible control c. With this, the same arguments and technical assumptions applied in Section 9A imply that a solution J to the Bellman equation (30) defines the value $J(y) = V(y)$ of the problem. The essential difference is the replacement of (7) with

$$J(y) \geq E\left[\int_0^T e^{-\rho t} v(c_t, Y_t^c)\, dt + e^{-\rho T} J(Y_T^c)\right], \qquad T > 0,$$

from which $J(y) \geq V^c(y)$ for an arbitrary admissible control c by taking the limit of the right-hand side as $T \to \infty$, using (31). Similarly, a candidate optimal control is defined in feedback form by a function $C : \mathcal{Y} \to A$ with the property that for each y, the action $C(y)$ solves the Bellman equation (30). Once again, technical conditions on the primitives guarantee the existence of an optimal control, but such conditions are often too restrictive in practice, and the Bellman equation is frequently used more as an aid in conjecturing a solution.

In Merton's problem, for example, with $v(\bar{c}, w) = \bar{c}^\alpha / \alpha$, $\alpha \in (0, 1)$, it is natural to conjecture that $J(w) = K w^\alpha / \alpha$ for some constant K. With some calculations, the Bellman equation (30) for this candidate value function J leads to $K = \gamma^{\alpha-1}$, where

$$\gamma = \frac{\rho - r\alpha}{1-\alpha} - \frac{\alpha\lambda^\top(\sigma\sigma^\top)^{-1}\lambda}{2(1-\alpha)^2}. \tag{32}$$

The associated consumption-portfolio policy (c^*, φ^*) is given by $\varphi_t^* = (\sigma\sigma^\top)^{-1}\lambda/(1-\alpha)$ and $c_t^* = \gamma W_t^*$, where W^* is the wealth process generated by (c^*, φ^*). In order to confirm the optimality of this policy, the transversality condition (31) must be checked, and is satisfied provided $\gamma > 0$. This verification is left as an exercise.

9E The Martingale Formulation

The objective now is to use the martingale results of Chapter 6 as the basis of a new method for solving Merton's problem (13). First, we state a corollary numeraire invariance, in the form of Lemma 6L.

Lemma. *Let Y be any deflator. Given an initial wealth $w \geq 0$, a strategy (c, Z, θ) is budget-feasible given price process X if and only*

$$\theta_t \cdot X_t^Y = wY_0 + \int_0^t \theta_s \, dX_s^Y - \int_0^t Y_s c_s \, ds \geq 0, \qquad t \in [0, T], \tag{33}$$

and

$$\theta_T \cdot X_T^Y \geq ZY_T. \tag{34}$$

For simplicity, we take the case $N = d$. By Girsanov's Theorem (Appendix D), there is an equivalent martingale measure Q for the deflated price process $\hat{X}$, defined by $\hat{X}_t = e^{-rt}X_t$, and Q is defined by the density process ξ, in that

$$E_t^Q(Z) = \frac{1}{\xi_t} E_t^P(Z\xi_T), \tag{35}$$

for any random variable Z with $E^Q(|Z|) < \infty$, where

$$\xi_t = \exp\left(-\eta^\top B_t - \frac{t}{2}\eta^\top\eta\right), \qquad t \in [0, T], \tag{36}$$

and where η is the market price of risk defined by

$$\eta = \sigma^{-1}\lambda. \tag{37}$$

(The invertibility of σ is implied by our earlier assumption that the rows of σ are linearly independent. The fact that ξ satisfies the Novikov condition and that $\mathrm{var}(\xi_T) < \infty$, as demanded by the definition of an equivalent martingale measure, are easily shown as an exercise.) As shown in Proposition 6F, the associated state-price deflator π is defined by $\pi_t = \xi_t e^{-rt}$.

Proposition. *Given a consumption choice (c, Z) in D and some initial wealth w, there exists a trading strategy θ such that (c, Z, θ) is budget-feasible if and only if*

$$E\left(\pi_T Z + \int_0^T \pi_t c_t \, dt\right) \leq w. \tag{38}$$

Proof: Suppose (c, Z, θ) is budget-feasible. Applying the previous numeraire-invariance lemma to the state-price defaltor π, and using the fact that $\pi_0 = \xi_0 = 1$, we have

$$w + \int_0^T \theta_t \, dX_t^\pi \geq \pi_T Z + \int_0^T \pi_t c_t \, dt. \tag{39}$$

Because X^π is a martingale under Q, the process M, defined by $M_t = w + \int_0^t \theta_s \, dX_s^\pi$, is a local martingale under Q. Moreover, by (33), M is nonnegative, and therefore a supermartingale (Appendix D). Taking expectations through (39) leaves (38).

Conversely, suppose (c, Z) satisfies (38), and let M be the Q-martingale defined by

$$M_t = E_t^Q\left(e^{-rT} Z + \int_0^T e^{-rt} c_t \, dt\right), \qquad t \in [0, T].$$

By Girsanov's Theorem (Appendix D), a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q is defined by $B_t^Q = B_t + \eta t$, and B^Q has the martingale representation property. We thus know there there is some $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(d)})$ with components in $\mathcal{L}^2$ such that

$$M_t = M_0 + \int_0^t \varphi_s \, dB_s^Q, \qquad t \in [0, T\,],$$

where $M_0 \leq w$. For the deflator Y defined by $Y_t = e^{-rt}$, we also know that $\hat{X} = X^Y$ is a Q-martingale. From the definitions of the market price of risk η and of B^Q,

$$d\hat{X}_t^{(i)} = \hat{X}_t^{(i)} \sigma^{(i)} \, dB_t^Q, \qquad 1 \leq i \leq N.$$

Because σ is invertible and $\hat{X}$ is strictly positive with continuous sample paths, we can choose adapted processes $(\theta^{(1)}, \ldots, \theta^{(N)})$ such that

$$(\theta_t^{(1)} \hat{X}_t^{(1)}, \ldots, \theta_t^{(N)} \hat{X}_t^{(N)})\sigma = \varphi_t^\top, \qquad t \in [0, T].$$

This implies that

$$M_t = M_0 + \sum_{i=1}^{N} \int_0^t \theta_s^{(i)} \, d\hat{X}_s^{(i)}.$$

We can also let

$$\theta_t^{(0)} = w + \sum_{i=1}^{N} \int_0^t \theta_s^{(i)} \, d\hat{X}_s^{(i)} - \sum_{i=1}^{N} \theta_t^i \hat{X}_t^{(i)} - \int_0^t e^{-rs} c_s \, ds. \tag{40}$$

Because φ has components in $\mathcal{L}^2$, we know that $\theta = (\theta^{(0)}, \ldots, \theta^{(N)})$ is in $\mathcal{L}(\hat{X})$. From (38), Fubini's Theorem, and the fact that $\xi_t = \pi_t e^{rt}$ defines the density process for Q, we have

$$M_0 = E^Q \left(e^{-rT} Z + \int_0^T e^{-rt} c_t \, dt \right) \le w. \tag{41}$$

From (40), for any $t \le T$,

$$\begin{aligned} \theta_t \cdot \hat{X}_t &= w + \int_0^t \theta_s \, d\hat{X}_s - \int_0^t e^{-rs} c_s \, ds \\ &= w + M_t - M_0 - \int_0^t e^{-rs} c_s \, ds \\ &= w - M_0 + E_t^Q \left(\int_t^T e^{-rs} c_s \, ds + e^{-rT} Z \right) \ge 0, \end{aligned}$$

using from (41) the fact that $w \ge M_0$. Restating this inequality in terms of the deflator, $Y_t = e^{-rt}$, we have (33). We can also use the same inequality for $t = T$ and the fact that $E_T^Q(e^{-rT} Z) = e^{-rT} Z$ to obtain (34). Thus, by Lemma 9E, (c, Z, θ) is budget-feasible. ▮

Corollary. *Given a consumption choice (c^*, Z^*) in D and some initial wealth w, there exists a trading strategy θ^* such that (c^*, Z^*, θ^*) solves Merton's problem (13) if and only if (c^*, Z^*) solves the problem*

$$\sup_{(c,Z) \in D} U(c, Z) \quad \text{subject to} \quad E\left(\int_0^T \pi_t c_t \, dt + \pi_T Z \right) \le w. \tag{42}$$

9F Martingale Solution

We are now in a position to obtain a relatively explicit solution to Merton's problem (13) by applying the previous Corollary.

By the Saddle Point Theorem (which can be found in Appendix B) and the strict monotonicity of U, the control (c^*, Z^*) solves (42) if and only if there is a scalar Lagrange multiplier $\gamma^* \geq 0$ such that, *first:* (c^*, Z^*) solves the unconstrained problem

$$\sup_{(c,Z)\in D} \quad \mathcal{L}(c, Z; \gamma^*), \tag{43}$$

where, for any $\gamma \geq 0$,

$$\mathcal{L}(c, Z; \gamma) = U(c, Z) - \gamma E\left(\pi_T Z + \int_0^T \pi_t c_t \, dt - w\right), \tag{44}$$

and *second,* using the fact that at least one of u and F is strictly increasing, $\gamma^* > 0$ and the complementary-slackness condition can be stated as

$$E\left(\pi_T Z^* + \int_0^T \pi_t c_t^* \, dt\right) = w. \tag{45}$$

Applying Corollary 9E, we can thus summarize our progress on Merton's problem (13) as follows.

Proposition. *Given some (c^*, Z^*) in D, there is a trading strategy θ^* such that (c^*, Z^*, θ^*) solves Merton's problem (13) if and only if there is a constant $\gamma^* > 0$ such that (c^*, Z^*) solves (43), and $E\left(\pi_T Z^* + \int_0^T \pi_t c_t^* \, dt\right) = w$.*

In order to obtain an intuition for the solution of (43), we begin with some arbitrary $\gamma > 0$ and treat $U(c, Z) = E[\int_0^T u(c_t) \, dt + F(Z)]$ intuitively by thinking of "E" and "$\int$" as finite sums, in which case the first-order conditions for optimality of $(c^*, Z^*) \gg 0$ for the problem $\sup_{(c,Z)} \mathcal{L}(c, Z; \gamma)$, assuming differentiability of u and F, are

$$u_c(c_t^*, t) - \gamma \pi_t = 0, \qquad t \in [0, T], \tag{46}$$

and

$$F'(Z^*) - \gamma \pi_T = 0. \tag{47}$$

Solving, we have

$$c_t^* = I(\gamma\pi_t, t), \qquad t \in [0, T], \tag{48}$$

and

$$Z^* = I_F(\gamma\pi_T), \tag{49}$$

where, as we recall, $I(\,\cdot\,, t)$ inverts $u_c(\,\cdot\,, t)$ and where $I_F = 0$ if $F = 0$ and otherwise I_F inverts F'. We will confirm these conjectured forms (48) and (49) of the solution in the next theorem. Under strict concavity of u or F, the inversions $I(\,\cdot\,, t)$ and I_F, respectively, are continuous and strictly decreasing. A decreasing function $\hat{w} : (0, \infty) \to \overline{\mathbb{R}}$ is therefore defined by

$$\hat{w}(\gamma) = E\left[\int_0^T \pi_t I(\gamma\pi_t, t)\, dt + \pi_T I_F(\gamma\pi_T)\right]. \tag{50}$$

(We have not yet ruled out the possibility that the expectation may be $+\infty$.) All of this implies that (c^*, Z^*) of (48)-(49) maximizes $\mathcal{L}(c, Z; \gamma)$ provided the required initial investment $\hat{w}(\gamma)$ is equal to the endowed initial wealth w. This leaves an equation $\hat{w}(\gamma) = w$ to solve for the "correct" Lagrange multiplier γ^*, and with that an explicit solution to the optimal consumption policy for Merton's problem.

We can be a little more systematic about the properties of u and F in order to guarantee that $\hat{w}(\gamma) = w$ can be solved for a unique $\gamma^* > 0$. A strictly concave increasing function $F : \mathbb{R}_+ \to \mathbb{R}$ that is differentiable on $(0, \infty)$ satisfies *Inada conditions* if $\inf_x F'(x) = 0$ and $\sup_x F'(x) = +\infty$. If F satisfies these Inada conditions, then the inverse I_F of F' is well defined as a strictly decreasing continuous function on $(0, \infty)$ whose image is $(0, \infty)$.

Condition A. *Either F is zero or F is differentiable on $(0, \infty)$, strictly concave, and satisfies Inada conditions. Either u is zero or, for all t, $u(\,\cdot\,, t)$ is differentiable on $(0, \infty)$, strictly concave, and satisfies Inada conditions. (Recall, at least one of u and F is nonzero.) For each $\gamma > 0$, $\hat{w}(\gamma)$ is finite.*

The asumption of finiteness of $\hat{w}(\,\cdot\,)$ can be shown under regularity conditions on utility cited in the Notes.

Theorem. *Under Condition A, for any $w > 0$, Merton's problem (13) has a solution (c^*, Z^*, θ^*), where (c^*, Z^*) is given by (48)-(49) for a unique $\gamma \in (0, \infty)$.*

Proof: Under Condition A, the Dominated Convergence Theorem implies that $\hat{w}(\,\cdot\,)$ is continuous. Because one or both of $I(\,\cdot\,, t)$ and $I_F(\,\cdot\,)$ have

$(0, \infty)$ as their image and are strictly decreasing, $\hat{w}(\,\cdot\,)$ inherits these two properties. From this, given any initial wealth $w > 0$, there is a unique γ^* with $\hat{w}(\gamma^*) = w$. Let (c^*, Z^*) be defined by (48)-(49), taking $\gamma = \gamma^*$. Proposition 9E tells us there is a trading strategy θ^* such that (c^*, Z^*, θ^*) is budget-feasible. Let (θ, c, Z) be any budget-feasible choice. Proposition 9E also implies that (c, Z) satisfies (38). The first-order conditions (46) and (47) are sufficient (by concavity of u and F) for optimality of $c^*(\omega, t)$ and $Z^*(\omega)$ in the problems

$$\sup_{\overline{c} \in [0,\infty)} u(\overline{c}, t) - \gamma^* \pi(\omega, t)\overline{c}$$

and

$$\sup_{\overline{Z} \in [0,\infty)} F(\overline{Z}) - \gamma^* \pi(\omega, T)\overline{Z},$$

respectively. Thus,

$$u(c_t^*, t) - \gamma^* \pi_t c_t^* \geq u(c_t, t) - \gamma^* \pi_t c_t, \quad 0 \leq t \leq T, \tag{51}$$

and

$$F(Z^*) - \gamma^* \pi_T Z^* \geq F(Z) - \gamma^* \pi_T Z. \tag{52}$$

Integrating (51) from 0 to T, adding (52), and taking expectations, and then applying the complementary slackness condition (45) and the budget constraint (38), leaves $U(c^*, Z^*) \geq U(c, Z)$. As (c, Z, θ) is arbitrary, this implies the optimality of (c^*, Z^*, θ^*). ∎

This result, giving a relatively explicit consumption solution to Merton's problem, has been extended in many directions, including relaxing the regularity conditions on utility in Condition A, and indeed even generalizing the assumption of additive utility to allow for habit-formation or recursive utility, as indicated in the Notes. In the next section, we will show that one can also extend to allow general Ito security price processes, under technical conditions.

For a specific example, we once again consider terminal consumption only, taking $u \equiv 0$ and $F(w) = w^\alpha/\alpha$ for $\alpha \in (0,1)$. Then $c^* = 0$ and the calculations above imply that $\hat{w}(\gamma) = E\left[\pi_T(\gamma\pi_T)^{1/(\alpha-1)}\right]$. Solving $\hat{w}(\gamma^*) = w$ for γ^* leaves

$$\gamma^* = w^{\alpha-1} E\left(\pi_T^{\alpha/(\alpha-1)}\right)^{\alpha-1}.$$

It is left as an exercise to check that (49) can be reduced explicitly to

$$Z^* = I_F(\gamma^* \pi_T) = W_T,$$

where

$$dW_t = W_t(r + \overline{\varphi} \cdot \lambda)\, dt + W_t \overline{\varphi}^\top \sigma \, dB_t; \qquad W_0 = w,$$

where $\overline{\varphi} = (\sigma\sigma^\top)^{-1}\lambda/(1-\alpha)$ is the vector of fixed optimal portfolio fractions found previously from the Bellman equation.

9G A Generalization

We generalize the security-price process $X = (X^{(0)}, X^{(1)}, \ldots, X^{(N)})$ to be of the form

$$dX_t^{(i)} = \mu_t^{(i)} X_t^{(i)}\, dt + X_t^{(i)} \sigma_t^{(i)}\, dB_t; \qquad X_0^{(i)} > 0, \tag{53}$$

where $\mu = (\mu^{(0)}, \ldots, \mu^{(N)})$ the $\mathbb{R}^{N \times d}$-valued process σ whose i-th row is $\sigma^{(i)}$ are bounded adapted processes. We suppose that $\sigma^{(0)} = 0$, so that again $r = \mu^{(0)}$ is the short-rate process. We assume for simplicity that $N = d$. That this is without loss of generality is shown in an exercise, as a consequence of Lévy's characterization of Brownian motion. The excess expected returns of the "risky" securities are defined by an $\mathbb{R}^N$-valued process λ given by $\lambda_t^{(i)} = \mu_t^{(i)} - r_t$. We assume that σ is invertible (almost everywhere) and that the market-price-of-risk process η, defined by $\eta_t = \sigma_t^{-1}\lambda_t$, is bounded. It follows from the results of Chapter 6 that markets are complete (at least in the sense of Proposition 6I) and that there are no arbitrages that are reasonably well behaved.

In this setting, a state-price deflator π is defined by

$$\pi_t = \exp\left(-\int_0^t r_s \, ds\right) \xi_t, \tag{54}$$

where

$$\xi_t = \exp\left(-\frac{1}{2}\int_0^t \eta_s \cdot \eta_s \, ds - \int_0^t \eta_s \, dB_s\right).$$

The reformulation of Merton's problem given by Proposition 9F and the form (48)-(49) of the solution (when it exists) still apply, substituting only the state-price deflator π of (54) for that given in the earlier special case of constant r, λ, and σ. Once again, the only difficulty to overcome is a solution γ^* to $\hat{w}(\gamma^*) = w$, where $\hat{w}$ is again defined by (53). This is guaranteed by the same Condition A of the previous section. The proof of Theorem 9F thus suffices for the following extension.

Proposition. *Suppose μ and σ, are bounded adapted processes,* $\operatorname{rank}(\sigma) = d$ *almost everywhere, and $\eta = \sigma^{-1}\lambda$ is bounded. Under Condition A, for any $w > 0$, Merton's problem has the optimal consumption policy given by (48)-(49) for a unique scalar $\gamma > 0$.*

Although this approach generates an explicit solution for the optimal consumption policy up to an unknown scalar γ, it does not say much about the form of the optimal trading strategy, beyond its existence. The Notes cite sources in which an optimal strategy is represented in terms of the *Malliavin calculus*. The original stochastic-control approach, in a Markov setting, gives explicit solutions for the optimal trading strategy in terms of the derivatives of the value function. Although there are few examples in which these derivatives are known explicitly, they can be approximated by a numerical solution of the Hamilton-Jacobi-Bellman equation, by extending the finite-difference methods given in Chapter 11.

9H The Utility-Gradient Approach

The martingale approach can be simplified, at least under technical conditions, by adopting the utility-gradient approach of Chapter 2. Although conceptually easy, this theory has only been developed to the point of theorems under restrictive conditions, and with proofs beyond the scope of this book, so we shall merely sketch out the basic ideas and refer to the Notes for sources with proofs and more details.

We let L_+ denote the set of nonnegative adapted consumption processes satisfying the integrability condition $E\left(\int_0^T c_t\, dt\right) < \infty$. We adopt a concave utility function $U : L_+ \to \mathbb{R}$, not necessarily of the additive form

$$U(c) = E\left[\int_0^T u(C_t, t)\, dt\right]. \tag{55}$$

We fix the security-price process X of Section 9G. Fixing also the initial wealth w, we say that a consumption process c in L_+ is budget-feasible if there is some trading strategy θ such that

$$\theta_t \cdot X_t = w + \int_0^t \theta_s\, dX_s - \int_0^t c_s\, ds \geq 0, \qquad t \in [0, T],$$

with $\theta_T \cdot X_T = 0$. A budget-feasible consumption process c is optimal if it solves the problem

$$\sup_{c \in \mathcal{A}} U(c),$$

where $\mathcal{A}$ denotes the set of budget-feasible consumption processes. If c^* is budget-feasible and the gradient $\nabla U(c^*)$ (defined in Appendix A) of U at c^* exists, the gradient approach to optimality reviewed in Appendix A leads to the first-order condition for optimality:

$$\nabla U(c^*; c^* - c) \leq 0, \qquad c \in \mathcal{A}.$$

We suppose that c^* is budget-feasible and that $\nabla U(c^*)$ exists, with a Riesz representation π that is an Ito process. That is,

$$\nabla U(c^*; c - c^*) = E\left[\int_0^T (c_t - c_t^*)\pi_t \, dt\right], \qquad c \in \mathcal{A}.$$

As shown in Appendix G, this is true for the additive model (55), under natural conditions, taking $\pi_t = u_c(c_t^*, t)$. Appendix G also gives Riesz representations of the utility gradients of other forms of utility functions, such as continuous-time versions of the habit-formation and recursive utilities considered in Exercises 2.8 and 2.9.

Based on Proposition 2D and the results of Section 9G, it is natural to conjecture that c^* is optimal if and only if the Riesz representation π of $\nabla U(c^*)$ is in fact a state-price deflator. In order to explore this conjecture, we suppose that π is indeed a state-price deflator; that is, X^π is a martingale. Numeraire Invariance implies that c is budget-feasible if and only if there is some trading strategy θ with

$$\theta_t \cdot X_t^\pi = w\pi_0 + \int_0^t \theta_s \, dX_s^\pi - \int_0^t c_s \, \pi_s \, ds \geq 0, \qquad t \in [0, T], \tag{56}$$

and $\theta_T \cdot X_T^\pi = 0$. For any budget-feasible strategy (c, θ), because $\int \theta \, dX^\pi$ is a local martingale that is bounded below, and therefore a supermartingale, (56) implies that

$$E\left(\int_0^T c_t \, \pi_t \, dt\right) \leq w\pi_0.$$

Applying this in particular to c^*, where we expect it to hold with strict equality under technical conditions, we have

$$\nabla U(c^*; c - c^*) = E\left[\int_0^T \pi_t (c_t - c_t^*) \, dt\right] \leq 0, \qquad c \in \mathcal{A}. \tag{57}$$

Thus, assuming some technical conditions along the way, we have shown that the first-order conditions for optimality of a budget-feasible choice c is essentially that the Riesz Representation π of the utility gradient at c is a state-price deflator. (This was precisely what we found in the finite-dimensional setting of Chapter 2.) We would next like to see how to deduce an optimal choice c^* from this first-order condition (57). We may have a significant amount of structure with which to determine c^* on this basis. First, from Chapter 6, we know that a state-price deflator π is given, under regularity, by

$$\pi_t = \pi_0 \exp\left(-\int_0^t r_s\, ds\right) \xi_t, \tag{58}$$

where ξ is the density process for some equivalent martingale measure, which implies that $d\xi_t = \xi_t \eta_t\, dB_t$ for a market-price-of-risk process η. Second, U may be one of the popular utility functions for which we can calculate the gradient $\nabla U(c)$ at any c. Finally, we can attempt to invert for an optimal c^* by matching the Riesz representation of $\nabla U(c^*)$ to one of the state-price deflators that we can calculate from (58).

In the case of additive utility, for example, if c^* is optimal then a state-price deflator π can be chosen, for some scalar $k > 0$, by $k\pi_t = u_c(c^*_t, t)$, so that $c^*_t = I(k\pi_t, t)$, where $I(\,\cdot\,, t)$ inverts $u_c(\,\cdot\,, t)$. Finally, we need to choose k so that c^* is budget-feasible. For the case of complete markets, it suffices, by the same numeraire-invariance argument made earlier, that

$$)\pi_0\, w = E\left(\int_0^T c^*_t\, \pi_t\, dt\right) = E\left[\int_0^T I(k\pi_t, t)\pi_t\, dt\right]. \tag{59}$$

Provided $I(\,\cdot\,, t)$ has range $(0, \infty)$ for all t, the arguments used in Section 9F can be applied for the existence of some scalar $k > 0$ satisfying (59). It is enough, for instance, that a market price of risk process η can be chosen to be bounded, and that I satisfies a uniform growth condition in its first argument. The Notes cite examples of a nonadditive utility function U with the property that for each deflator π in a suitably general class, one can recover a unique consumption process c^* with the property that $\nabla U(c^*)$ has π as its Riesz representation. Subject to regularity conditions, the habit-formation and recursive-utility functions have this property.

For the case of incomplete markets (for which it is not true that $\text{rank}(\sigma) = d$ almost everywhere), all of the above steps can be carried out in the absence of arbitrage, except that there need not be a trading strategy θ^* that finances the candidate solution c^*. Papers cited in the Notes have taken the following approach. With incomplete markets, there is a family of different market-price-of-risk processes. The objective is to choose a market-price-of-risk process η^* with the property that, when matching the Riesz representation of the utility gradient to $k\eta^*$, we can choose k so that c^* can be financed. This can be done under technical regularity conditions.

Exercises

Exercise 9.1 For the candidate optimal portfolio control $\varphi_t^* = \overline{\varphi}$ given by (22), verify that (28) is indeed a martingale as asserted.

Exercise 9.2 Solve Merton's problem in the following cases. Add any regularity conditions that you feel are appropriate.

(A) Let T be finite, $F = 0$, and $u(c,t) = e^{-\rho t} c^\alpha / \alpha, \alpha \in (0,1)$.

(B) Let T be finite, $F = 0$, and $u(c,t) = \log c$.

(C) Let $T = +\infty$ and $u(c,t) = e^{-\rho t} c^\alpha / \alpha$, $\alpha \in (0,1)$. Verify the solution given by $c_t^* = \gamma W_t^*$ and $\varphi_t^* = (\sigma\sigma^\top)^{-1}\lambda/(1-\alpha)$, where γ is given by (32). Verify the so-called transversality condition (31) with $\gamma(1-\alpha) > 0$.

Exercise 9.3 Extend the example in Section 9D, with $v(\overline{c}, w) = \overline{c}^\alpha/\alpha$, to the case without a riskless security. Add regularity conditions as appropriate.

Exercise 9.4 The rate of growth of capital stock in a given production technology is determined by a "random shock" process Y solving the stochastic differential equation

$$dY_t = (b - \kappa Y_t)\, dt + k\sqrt{Y}_t\, dB_t; \qquad Y_0 = y \in \mathbb{R}_+, \qquad t \geq 0,$$

where b, κ, and k are strictly positive scalars with $2b > k^2$, and where B is a standard Brownian motion. Let $\mathcal{C}$ be the space of nonnegative adapted consumption processes satisfying $\int_0^T c_t\, dt < \infty$ almost surely for all $T \geq 0$. For each c in $\mathcal{C}$, a capital stock process K^c is defined by

$$dK_t^c = (K_t^c h Y_t - c_t)\, dt + K_t^c \epsilon \sqrt{Y}_t\, dB_t; \qquad K_0^c = x > 0,$$

where h and ϵ are strictly positive scalars with $h > \epsilon^2$. Consider the control problem

$$V(x,y) = \sup_{c \in \mathcal{C}} \quad E\left[\int_0^T e^{-\rho t} \log(c_t)\, dt\right],$$

subject to $K_t^c \geq 0$ for all t in $[0, T]$.

(A) Let $C : \mathbb{R}_+ \times [0, T] \to \mathbb{R}_+$ be defined by

$$C(x, t) = \frac{\rho x}{1 - e^{-\rho(T-t)}},$$

and let K be the solution of the SDE

$$dK_t = [K_t h Y_t - C(K_t, t)]\, dt + K_t \epsilon \sqrt{Y}_t\, dB_t; \qquad K_0 = x > 0.$$

Finally, let c^* be the consumption process defined by $c_t^* = C(K_t, t)$. Show that c^* is the unique optimal consumption control. Hint: Verify that $V(x, y) = J(x, y, 0)$, where J is of the form

$$J(x, y, t) = A_1(t) \log(x) + A_2(t) y + A_3(t), \qquad (x, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times [0, T),$$

where A_1, A_2, and A_3 are (deterministic) real-valued functions of time. State the function A_1 and differential equations for A_2 and A_3.

(B) State the value function and the optimal consumption control for the infinite-horizon case. Add regularity conditions as appropriate.

Exercise 9.5 An agent has the objective of maximizing $E[u(W_T)]$, where W_T denotes wealth at some future time T and $u : \mathbb{R} \to \mathbb{R}$ is increasing and strictly concave. The wealth W_T is the sum of the market value of a fixed portfolio of assets and the terminal value of the margin account of a futures trading strategy, as elaborated below. This problem is one of characterizing optimal futures hedging. The first component of wealth is the spot market value of a fixed portfolio $p \in \mathbb{R}^M$ of M different assets whose price processes $S^1, \ldots, S^M$ satisfy the respective stochastic differential equations

$$dS_t^m = \mu_m(t)\, dt + \sigma_m(t)\, dB_t; \qquad t \geq 0; \qquad S_0^m = 1, \qquad m \in \{1, \ldots, M\},$$

where, for each m, $\mu_m : [0, T\,] \to \mathbb{R}$ and $\sigma_m : [0, T\,] \to \mathbb{R}^d$ are continuous. There are futures contracts for K assets with delivery at some date $\tau > T$, having futures-price processes $F^1, \ldots, F^k$ satisfying the stochastic differential equations

$$dF_t^k = m_k(t)\, dt + v_k(t)\, dB_t; \qquad t \in [0, T\,], \qquad 1 \leq k \leq K,$$

where m_k and v_k are continuous on $[0, T]$ into $\mathbb{R}$ and $\mathbb{R}^d$, respectively. For simplicity, we assume that there is a constant short rate r for borrowing or lending. One takes a futures position merely by committing oneself to mark a margin account to market. Conceptually, that is, if one holds a long (positive) position of, say, ten futures contracts on a particular asset and the price of the futures contract goes up by a dollar, then one receives ten dollars from the short side of the contract. (In practice, the contracts are largely insured against default by the opposite side, and it is normal to treat the contracts as default-free for modeling purposes.) The margin account earns interest at the riskless rate (or, if the margin account balance is negative, one loses interest at the riskless rate). We ignore margin calls or borrowing limits. Formally, as described in Section 8C, the futures-price process is actually the cumulative-dividend process of a futures contract; the true price process is zero. Given any bounded adapted process $\theta = (\theta^{(1)}, \dots, \theta^{(K)})$ for the agent's futures-position process, the agent's wealth at time T is $p \cdot S_T + X_T$, where X is the Ito process for the agent's margin account value, defined by $X_0 = 0$ and $dX_t = rX_t \, dt + \theta_t \, dF_t$.

(A) Set up the agent's dynamic hedging problem for choice of futures-position process θ in the framework of continuous-time stochastic control. State the Bellman equation and first-order conditions. Derive an explicit expression for the optimal futures position θ_t involving the (unknown) value function. Make regularity assumptions such as differentiability and nonsingularity. Hint: Let $W_t = p \cdot S_t + e^{r(T-t)} X_t$, $t \in [0, T]$.

(B) Solve for the optimal policy θ in the case $m \equiv 0$, meaning no expected futures-price changes. Add any regularity conditions needed.

(C) Solve the problem explicitly for the case $u(w) = -e^{-\alpha w}$, where $\alpha > 0$ is a scalar risk aversion coefficient. Add any regularity conditions needed.

Exercise 9.6 In the setting of Section 9B, consider the special case of the utility function

$$U(c, Z) = E\left[\int_0^T \log(c_t) \, dt + \sqrt{Z}\right].$$

Obtain a closed-form solution for Merton's problem (14). Hint: The mixture of logarithm and power function in the utility makes this a situation in which the martingale approach has an advantage over the Bellman approach, from which it might be difficult to conjecture a value function. Once the optimal consumption policy is found, do not forget to calculate the optimal portfolio trading strategy.

Exercise 9.7 (Utility-Gradient Example) Suppose B is a standard Brownian motion and there are two securities with price processes S and β given by

$$\begin{aligned} dS_t &= \mu_t S_t\,dt + \sigma_t S_t\,dB_t; \qquad S_0 > 0 \\ d\beta_t &= r_t \beta_t\,dt; \qquad \beta_0 > 0, \end{aligned}$$

where μ, σ, and r are bounded adapted processes with $\mu_t > r_t$ for all t. We take the infinite horizon case, with utility function U defined by

$$U(c) = E\left(\int_0^\infty e^{-\rho t} c_t^\alpha\,dt\right),$$

where $\alpha \in (0,1)$ and $\rho \in (0,\infty)$. Taking the utility-gradient approach of Section 9H, c^* is, in principle, an optimal choice if and only if

$$E\left(\int_0^\infty \pi_t\,c_t^*\,dt\right) = w,$$

where $\nabla U(c^*)$ has Riesz representation π, and where S^π and β^π are martingales. Assuming that the solution c^* is an Ito process with

$$dc_t^* = c_t^*\,\mu_t^*\,dt + c_t^*\,\sigma_t^*\,dB_t,$$

we can write

$$d\pi_t = \pi_t\,\mu_\pi(t)\,dt + \pi_t\,\sigma_\pi(t)\,dB_t$$

for processes μ_π and σ_π that can be solved explicitly in terms of μ^* and σ^* from Ito's Formula and the fact that $\pi_t = \alpha e^{-\rho t} c_t^{\alpha-1}$. Assuming that S^π and β^π are indeed martingales, solve for μ^* and σ^* explicitly.

Exercise 9.8 Verify that, as defined by (35) and (36), Q is indeed an equivalent martingale measure, including the property that $\text{var}(\xi_T) < \infty$.

Exercise 9.9 (Constrained Investment Behavior) Security markets are characterized by price processes $S = (S^{(1)}, \ldots, S^{(N)})$ and β, with

$$d\beta_t = \beta_t r_t\,dt, \quad \beta_0 > 0,$$

for a bounded adapted process r, and with

$$dS_t^{(i)} = S_t^{(i)} \mu_t^{(i)}\,dt + S_t^{(i)} \sigma_t^{(i)}\,dB_t,$$

where, for each i, $\mu^{(i)}$ and $\sigma^{(i)}$ are bounded adapted processes in $\mathbb{R}$ and $\mathbb{R}^d$ respectively. We also assume that $\Gamma_t = (\sigma_t \sigma_t^\top)^{-1}$ is well defined and bounded. With a trading strategy specified in terms of a bounded adapted process φ valued in $\mathbb{R}^N$ with $\int_0^T \varphi_t \cdot \varphi_t \, dt < \infty$, and with a non-negative consumption process c, the wealth process $W^{(\varphi,c)}$ of an investor is defined by

$$dW_t^{(\varphi,c)} = [W_t^{\varphi,c)}(\varphi_t \cdot \lambda_t + r_t) - c_t] \, dt + W_t^{(\varphi,c)} \varphi_t^\top \sigma_t \, dB_t; \quad W_0^{(\varphi,c)} = w,$$

where $\lambda_t^i = \mu_t^i - r_t$ and where $w > 0$ is a given constant. The investment-consumption strategy (φ, c) is admissible if $W_t^{(\varphi,c)} \geq 0$ for all t. Consider an investor with the utility criterion

$$U(c) = E\left[\int_0^T e^{-\rho t} \log c_t \, dt\right],$$

where $\rho > 0$ is a constant, and c is chosen from the set of non-negative adapted processes such that $U(c)$ is well defined.

(A) (Unconstrained case) Let $\mathcal{C}^*(r, \mu, \sigma)$ denote the set of solutions to the investor's optimization problem,

$$\sup_{(\varphi,c) \in \mathcal{C}(w)} U(c),$$

where $\mathcal{C}(w)$ is the set of admissible strategies. Calculate $\mathcal{C}^*(r, \mu, \sigma)$.

(B) (Leverage Constraints) Let $\mathcal{C}(w, \ell) = \{(\varphi, c) \in \mathcal{C}(w) : \sum_{i=1}^N \varphi_t^i \leq \ell_t\}$, where ℓ is a non-negative bounded adapted process that sets an upper bound on the leverage of the investment strategy. The investor's problem is now

$$\sup_{(\varphi,c) \in \mathcal{C}(w,\ell)} U(c).$$

Solve this leverage-constrained problem. Hint: Reduce the problem to that of the un-constrained case by an adjustment of the interest rate r, to reflect the shadow price of the leverage constraint, so that the problem may be effectively solved unconstrained.

(C) (Leverage and shortsales constraints) Let

$$\mathcal{C}(w, \ell, b) = \{(\varphi, c) \in \mathcal{C}(w, \ell) \, : \, \varphi_t \geq -b_t\},$$

where ℓ is as above and $b = (b^{(1)}, \dots, b^{(N)})$, where b^i is for each i a non-negative bounded adapted process that places, in addition to a leverage constraint, a bound on short sales as a fraction of portfolio value. Now, solve:

$$\sup_{(\varphi,c)\in\mathcal{C}(w,\ell,b)} U(c).$$

Hint: Again, reduce to the unconstrained case, this time by suitable adjustment of all of the return coefficients (r, μ) to reflect the shadow prices of the constraints.

Exercise 9.10 (Investment and Price Behavior with Jumps) The objective of this exercise is to extend the basic results for consumption-investment models and asset pricing to an economy in which the volatilities and expected rates of return of the available securities may change suddenly, and depend on a "regime" state process defined by a two-state Markov chain Z. When a regime switch occurs, the price may jump as well, for example to reflect the sudden change in the distribution of returns on equilibrium asset values.

We fix a probability space on which is defined a standard Brownian motion B in $\mathbb{R}^d$ and the two-state continuous-time Markov chain Z, as defined in Appendix F, with states 0 and 1, and transition intensities $\lambda(0)$ and $\lambda(1)$. We let M denote the compensated version of Z.

Suppose $X = (X^{(0)}, X^{(1)}, \dots, X^{(N)})$ is an adapted process in $\mathbb{R}^{N+1}$ for the prices of $N+1$ securities. For each $i \geq 1$, we assume that

$$dX_t^{(i)} = \mu_i(Z_t)X_t^{(i)}\,dt + X_t^{(i)}\sigma^{(i)}(Z_t)\,dB_t + X_{t-}^{(i)}\beta_i(Z_{t-})\,dM_t; \quad X_0 > 0,$$

where, for each $\ell \in \{0, 1\}$,

$\sigma^{(i)}(\ell)$ is the i-th row of a constant matrix $\sigma(\ell)$ in $\mathbb{R}^{N\times d}$.

$\mu_i(\ell)$ is a constant.

$\beta_i(\ell)$ is a constant strictly less than 1 in absolute value.

This implies, in particular, that within a "regime," each process $X^{(i)}$ behaves as a geometric Brownian motion of the sort used in the Black-Scholes model of option pricing. When the regime changes, the price process jumps and its drift and diffusion parameters change. When the regime changes from 0 to 1, for example, the price of risky security i jumps by a multiplicative factor of $\beta_i(0)$. When the regime changes from 1 to zero, the price of risky security i jumps by a multiplicative factor of $-\beta_i(1)$.

Given a short rate process $\{r(Z_t) : t \geq 0\}$, for given constants $r(0)$ and $r(1)$, the market value of an investment rolled over at the short rate defines a value process $X^{(0)}$ by

$$dX_t^{(0)} = r(Z_t)X_t^{(0)}\,dt; \quad X_0^{(0)} > 0.$$

The filtration of tribes generated by (B, Z), and augmented with null sets, as above, defines the information available to investors. A trading strategy is a predictable $\mathbb{R}^{N+1}$-valued process θ such that the stochastic integral $\int \theta\, dX$ exists. A trading strategy θ is self-financing if

$$\theta_t \cdot X_t = \theta_0 \cdot X_0 + \int_0^t \theta_s\, dX_s, \quad 0 \leq t \leq T.$$

(A) (Complete Markets and Equivalent Martingale Measure) Find conditions on the primitive functions $(\sigma, \mu, r, \beta, \lambda)$ defining asset returns that are sufficient for the existence of a unique equivalent martingale measure Q for the deflated price process $X_t/X_t^{(0)}$. (Do not assume that λ or β are trivial.) Show, under these conditions, that for any bounded $\mathcal{F}_T$-measurable random variable Y there is a self-financing trading strategy θ with bounded market value $\theta_t \cdot X_t$, with $\theta_T \cdot X_T = Y$, and with

$$\theta_0 \cdot X_0 = E^Q\left[\exp\left(\int_0^T -r(Z_t)\,dt\right)Y\right].$$

Also, given the existence of an equivalent martingale measure, show that any self-financing trading strategy θ satisfying $\theta_t \cdot X_t \geq 0$ for all t cannot be an arbitrage.

Hint: For a given equivalent probability measure Q, let

$$\xi_t = E_t\left(\frac{dQ}{dP}\right), \quad t \leq T.$$

Because ξ is a martingale, it has a martingale representation in the form

$$d\xi_t = \xi_t\sigma_\xi(t)\,dB_t + \xi_{t-}\beta_\xi(t)\,dM_t,$$

for some adapted processes σ_ξ and β_ξ. For Q to be an equivalent martingale measure under the numeraire deflator $X^{(0)}$, it must be the case that a P-martingale is defined by the process whose value at time t is

$$\frac{\xi_t X_t^{(i)}}{X_t^{(0)}}.$$

Using Ito's Formula from Appendix G for the case of a product of processes of this form, compute restrictions on σ_ξ and β_ξ in terms of the primitives, and then provide restrictions on the asset-return coefficient functions μ, r, σ, λ, and β under which ξ, and therefore Q, are uniquely well defined. Please don't forget that ξ should be of finite variance.

(B) (Martingale Formulation of Optimality) Utility is defined over the space L_+ of non-negative adapted consumption processes satisfying the restriction that $\int_0^T c_t\, dt < \infty$ almost surely. Specifically, $U : L_+ \to \mathbb{R}$ is defined by

$$U(c) = E\left[\int_0^T e^{-\rho t} u(c_t)\, dt\right].$$

For this part of the problem, T is a fixed finite time and $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave and increasing.

Given an initial wealth $w > 0$, we say that (c, θ) is *budget-feasible*, denoted $(c, \theta) \in \Lambda(w)$, if $c \in L_+$ and if $\theta \in \mathcal{L}(X)$ is a trading strategy satisfying

$$\theta_t \cdot X_t = w + \int_0^t \theta_s\, dX_s - \int_0^t c_s\, ds \geq 0, \quad t \in [0, T],$$

where $w > 0$ is a given constant and

$$\theta_T \cdot X_T \geq 0.$$

We now have the problem, for each initial wealth w and each initial regime $i \in \{0, 1\}$,

$$J(i, w, 0) = \sup_{(c,\theta)\in\Lambda(w)} U(c). \tag{60}$$

Using the martingale approach, compute the optimal consumption process up to a missing scalar Lagrange multipler. Justify your answer, making technical assumptions as needed.

(C) (Parametric Example) We change the formulation in Part (B) by letting $T = +\infty$ and by considering the special case $u(c) = c^\alpha$, for some

α in $(0, 1)$. A consumption process c is non-negative and adapted, with $\int_0^t c_s \, ds < \infty$ for each $t > 0$. We also simplify the problem by assuming that $\beta_i = 0$ for all i, meaning that a regime shift causes no jump in asset prices, but may cause a change in expected rates of returns, volatilities, and correlations. State the solution to (60), proving its optimality by reducing the supremum value function J defined by (60) to 2 unknown coefficients, k_0 and k_1, one for each initial regime. Using your extension of the Hamilton-Jacobi-Bellman equation for optimal control in this setting, obtain two non-linear restrictions on these 2 unknown coefficients. Assume existence of a solution to this system of equations for k_0 and k_1. Compute the candidate optimal consumption and portfolio fraction policies as explicitly as possible. Verify your candidate solution, under additional explicit regularity conditions on the primitive parameters as well as k_0 and k_1. Please be extremely careful to provide a complete proof of optimality given k_0 and k_1.

(D) (Robinson Crusoe) Let $d = 1$ (one-dimensional Brownian motion). Robinson must consume at each time t from a physical stock K_t of consumption commodity, satisfying the production equation

$$dK_t = [\eta(Z_t)K_t - c_t] \, dt + K_t \zeta(Z_t) \, dB_t + K_{t-}\delta(Z_{t-}) \, dZ_t, \tag{61}$$

where η, ζ, and δ are real-valued functions on $\{0, 1\}$ and c is a non-negative adapted consumption process to be chosen by Robinson. We assume that $|\delta| < 1$. Robinson's utility for consumption is defined by

$$U(c) = E\left[\int_0^\infty e^{-\rho t} c_t^\alpha \, dt\right],$$

for a given $\alpha \in (0, 1)$. Robinson's problem is

$$\sup_{c \in \mathcal{A}} \quad U(c), \tag{62}$$

where $\mathcal{A}$ is the space containing any consumption process c such that the stock K_t of commodity solving (61) remains non-negative for all t. Solve (62). Hint: Conjecture the form of the value function. Now, for each initial state $i \in \{0, 1\}$, consider the stopping time τ of first transition to the other state. Begin your calculation of the unknown coefficients of the solution by conditioning on this stopping time τ.

(E) (Incomplete Information and Filtering) We now consider the special case of a single risky asset ($N = 1$) for which, with each change in regime there is no jump in the risky asset price (that is, $\beta = 0$), no change in the interest rate (that is, $r(1) = r(0) = \overline{r}$ for some constant $\overline{r}$), and no jumps in the volatility (that is, $\sigma(1) = \sigma(2) = q$ for some constant q). With a change in regime, however, there is a change in the mean-rate-of-return coefficient. That is, $\mu(1) \neq \mu(0)$. For simplicity, we will assume that $\lambda(0) = \lambda(1) = \overline{\lambda}$, for some constant $\overline{\lambda} > 0$, so that the arrival intensity of a change in regime is the same in both regimes.

Suppose, however, that the investor is *not able to observe the regime state process Z,* but can only observe the risky-asset's price process $S = X^{(1)}$. This means that, for the investor, the relevant filtration of tribes describing the available information is $\{\mathcal{F}_t^S = \sigma(\{S_u : u \leq t\})\}$.

Now solve the optimal portfolio investment strategy for an investor with the utility criterion

$$U(c) = E\left[\int_0^\infty e^{-\rho t} \log c_t \, dt\right],$$

where $\rho > 0$ is a constant, and c is chosen from the set of non-negative adapted processes such that $U(c)$ is well defined.

Hint: It may assist you to work with a stochastic differential model for asset price behavior given the limited information available. For this, let $p_t = P(Z_t = 1 \mid \mathcal{F}_t^S)$ denote the conditional probability at time t that $Z_t = 1$, given the observed asset prices to that time. By adding and subtracting the same thing from the stochastic differential expression for S, we have

$$dS_t = S_t m(p_t) \, dt + S_t q \, d\overline{B}_t,$$

where

$$d\overline{B}_t = \frac{1}{q}[\mu(Z_t) - m(p_t)] \, dt + dB_t,$$

and where, for any $\alpha \in [0, 1]$,

$$m(\alpha) = \alpha\mu(1) + (1 - \alpha)\mu(0)$$

defines the conditional expectation of the mean-rate of return on the risky asset given probability α that the unknown regime is 1. It can be shown, for the probability space $(\Omega, \mathcal{F}, P)$ and the limited filtration $\{\mathcal{F}_t^S : t \geq 0\}$ available to the investor, that $\overline{B}$ is a standard Brownian motion. It turns out, moreover, that

$$dp_t = \overline{\lambda}(1 - 2p_t) \, dt + kp_t(1 - p_t) \, d\overline{B}_t,$$

where $k = [\mu(1) - \mu(0)]/q$. The initial condition p_0 is the investor's prior probability assesment that $Z(0) = 1$. We have therefore effectively reduced the original investment problem to one of complete observation, with a stochastic mean-rate of return $m(p_t)$ determined by a separate Markov process p satisfying its own stochastic differential equation.

Notes

Standard treatments of stochastic control in this setting are given by Fleming and Rishel [1975], Krylov [1980], Bensoussan [1983], Lions [1981], Lions [1983], Fleming and Soner [1993], and Davis [1993]. Fleming and Soner (1993) develop the notion of *viscosity solutions* of the Hamilton-Jacobi-Bellman equation. Among other advantages of this approach, it allows one to characterize the continuous-time stochastic control problem as the limit of discrete Markov control problems of the sort considered in Chapter 3.

Merton [1969] and Merton [1971], in perhaps the first successful application of stochastic control methods in an economics application, formulated and solved the problem described in Section 9B. (Another early example is Mirrlees [1974].) Extensions and improvements of Merton's result have been developed by Aase [1984], Fleming and Zariphopoulou [1991], Karatzas, Lehoczky, Sethi, and Shreve [1986], Lehoczky, Sethi, and Shreve [1983], Lehoczky, Sethi, and Shreve [1985], Sethi and Taksar [1988], Fitzpatrick and Fleming [1991], Richard [1975], Jacka [1984], Ocone and Karatzas [1991] (who apply the Malliavin calculus), and Merton [990b].

The martingale approach to optimal investment described in Section 9E has been developed in a series of papers. Principle among these are Cox and Huang [1989] and, subsequently, Karatzas, Lehoczky, and Shreve [1987]. This literature includes Cox [1983], Pliska [1986], Cox and Huang [1991], Back [1986], Back and Pliska [1987], Huang and Pagès [1992], **?**], **?**], Jeanblanc and Pontier [1990], Richardson [1989], and **?**]. For applications of duality techniques to markets with constraints, see Cvitanić and Karatzas [1992], Cvitanić and Karatzas [1993], Cvitanić and Karatzas [1994], He and es [1993], He and Pearson [991a], He and Pearson [991b], Karatzas, Lehoczky, Shreve, and Xu [1991], and El Karoui and Quenez [1991].

For problems with mean-variance-criteria in a continuous-time setting, see Ansel and Stricker [1993], Bajeux-Besnainou and Portait [1993], Bossaerts and Hillion [1994], Bouleau and Lamberton [1993], Duffie and Jackson [1990], Duffie and Richardson [1991], Föllmer and Schweizer [1990], Föllmer and Sondermann [1986], Lakner [994a], and Schweizer [993a], Schweizer [993b], Schweizer [994a], and Schweizer [994b].

For optimality under various habit-formation utilities, see Constantinides [1990], Detemple and Zapatero [1992], Ingersoll [1992], Ryder and Heal [1973], and Sundaresan [1989]. A model involving local substitution for consumption was developed by Hindy and Huang [1992], Hindy and Huang [1993a], and Hindy, Huang, and Kreps [1992]. See, also, Hindy, Huang, and Zhu [993b].

Ekern [1993] is an example of a model of irreversible investment. Dixit and Pindyck [1993] review many other models of optimal production under uncertainty using stochastic control methods.

For developments of recursive utility in continuous-time settings, and the associated technical problem of backward stochastic differential equations, see Ahn [1993], **?**], **?**], Alvarez and Tourin [1994], Darling [1994], Duffie and Epstein (with Skiadas) (1992a), Duffie and Epstein [992b], Duffie and Lions [1990], Duffie and Skiadas [1994], **?**], Karoui, Peng, and Quenez [1994], Ma [991a], Ma [991b], Ma [1993], **?**], Pardoux and Peng [1990], Pardoux and Peng [1994], Peng [1992], Peng [1993], Svensson [1989], and Uzawa [1968]. For the related problem of backward-forward stochastic differential equations see Antonelli [1993], Cvitanić and Ma [1994], Duffie, Geoffard, and Skiadas [1994], **?**], Ma, Protter, and Yong [1993], and Ma and Yong [1993].

The utility-gradient approach to optimal investment of Section 9H is based on work by Harrison and Kreps [1979], Kreps [1981], Huang [1985b], Foldes [1978], Foldes [1979], Foldes [1990], Foldes [991a], Foldes [991b], Foldes [991c], Back [991a], and **?**], and is extended in these sources to an abstract setting with more general information and utility functions.

For optimal investment problems in the case of transactions costs, see Akian, Menaldi, and Sulem [1993], Alvarez [1991], Arntzen [1994], Chang [1993], Constantinides [1986], Davis and Norman [1990], Davis and Panas [1991], Duffie and Sun [1990], Dumas and Luciano [1989], Edirisinghe, Naik, and Uppal [1991], Fleming, Grossman, Vila, and Zariphopoulou [1989], Jouini and Kallal [993a], Jouini and Kallal [993b], Pliska and Selby [1994], Schroder [1993], Shreve and Soner [1994], Shreve, Soner, and Xu [1991], Vayanos and Vila [1992], and Zariphopoulou [1992]. See also the references cited in the Notes of Chapter 6.

For results with incomplete markets, see Adler and Detemple [1988], Cuoco [1994], Cvitanić and Karatzas [1992], Cvitanić and Karatzas [1994], **?**], Duffie and Zariphopoulou [1993], Dybvig [1989], Karoui and Jeanblanc-Picqué [1994], He and es [1993], Koo [1991], Koo [994a], Koo [994b], Scheinkman and Weiss [1986], and Svensson and Werner [1993].

For the case of short-sales constraints and other forms of portfolio retrictions, see Back and Pliska [1986], Brennan, Schwartz, and Lagnado [1993], Cuoco [1994], Cvitanić Karatzas (1992, 1994), Dybvig [1994], Fleming and Zariphopoulou [1991], He and es [1993], Hindy [1995], Shirakawa [1994], Vila and Zariphopoulou [1991], Xu and Shreve [992a], Xu and Shreve [992b], Zariphopoulou (1992), and Zariphopoulou [1994].

On the existence of additive or other particular forms of utility consistent with given asset prices, sometimes called *integrability*, see Bick [1986], He and Huang [1994], He and Leland [1993], Hodges and Carverhill [1992], and Wang [993a]. On turnpike problems, see Cox and Huang (1991, 1992) and Huang and Zariphopoulou [1994]. For problems in settings with incomplete information, usually requiring filtering of the state, see Dothan and Feldman [1986], Gennotte [1984], Detemple [1991], Föllmer and Schweizer (1990), Karatzas [1991], Karatzas and Xue [1990], Kuwana [1994], Lakner [994b], **?**], Ocone and Karatzas (1991), and Schweizer (1993b).

Exercise 9.9 is based in part on Cvitanìc and Karatzas (1992, 1994) and on Tepla (1997). Exercise 9.10 is based on Honda (1997). For the results on filtering used in Exercise 9.10 of **?**].

Quenez [1992] and Karatzas [1989] survey some of the topics in this chapter. Exercise 9.4 is from Cox, Ingersoll, and Ross [1985].

Detemple [1986]127 Aase [1993]128 Akian, Sulem, and Taksar [1999]129 Altay, Nageswaran, and Kazemi [1995]130 Amendinger, Imkeller, and Schweizer [1998]131 Avellaneda and Levy [1995]132 Avellaneda [1994]133 Balduzzi and Lynch [1998]134 Balduzzi and Lynch [1997]135 Bally [1995]136 Barles, Buckdahn, and Pardoux [1995]137 Bergman [985b]138 Boyle and Yang [1995]139 Broadie, Cvitanić, and Soner [1996]140 Buckdahn [995a]141 Buckdahn [995b]142 Cadenillas and Pliska [1996]143 Chapman [1995]144 Clelow and Hodges [1996]145 Cox and Huang [1992]146 Cuoco [1995]147 Cuoco and Cvitanić [1995]148 Cuoco and Liu [1997]149 Cuoco and Zapatero [1997]150 Cvitanić [1994]151 Cvitanić [1998]152 Cvitanić [1995]153 Cvitanić and Karatzas [1995]154 Cvitanić and Karatzas [1996]155 Cvitanić, Karatzas, and Soner [1998]156 Davis and List [1996]157 Derviz [1996]158 Detemple and Murthy [994a]159 Duffie and Skiadas) [992a]160 Duffie, Fleming, Soner, and Zariphopoulou [1997]161 Dybvig, Rogers, and Back [1995]162 Karoui, Kapoudjian, Pardoux, Peng, and Quenez [1995]163 Fisher and Gilles [1998]164 Foldes [1996]165 Gegout-Petit and Pardoux [1995]166 Glasserman [1996]167 Gourieroux, Laurent, and Pham [1995]168 Grossman and Zhou [1996]169 Honda [1996]170 Honda [997b]171 Honda [997c]172 Huang [1992]173 Huang [1995]174 Iyengar and

Cover [1997]175 Kuwana [995b]176 Lakner [1988]177 Lakner [1995]178 Lakner and Slud [1991]179 Lioui [1995]180 Liu [1999]181 Lynch and Balduzzi [1998]182 Pagès. [1987]183 Pikovsky and Karatzas [1995]184 Radner and Shepp [1995]185 Ruegg [1996]186 Schroder and Skiadas [1998]187 Schroder and Skiadas [1999]188 Schroder and Skiadas [1997]189 Sekine [1998]190 Sethi [1995]191 Sethi and Taksar [1992]192 Sethi, Taksar, and Presman [1992]193 Sircar [1996]194 Taksar, Klass, and Assaf [1988]195 Tenney [1995]196 Tepla [1995]197 Tepla [996a]198 Tepla [996b]199 Weerasinghe [1996]200

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