SABR Volatility Surface Notes

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1 SABR Model

The SABR model's main purpose is to supply a volatility for any level of forward swap rate or strike. SABR is short for stochastic alpha-beta-rho, but the model has four parameters (alpha, beta, rho and nu). Each parameter controls different aspects of SABR vol surface. The model is described by the following SDE equations [1].

$$\begin{aligned} dF &= \sigma_t F^{\beta} dW_1 \\ d\sigma_t &= \nu \sigma_t dW_2 \\ dW_1 dW_2 &= \rho dt \end{aligned} \tag{1}$$

Where

- F is the underlying forward rate, and $F_0 = f$ is the current forward rate (a constant).
- β (beta) is a constant exponent telling us what mixture of normal and lognormal dynamics to use. When $\beta = 0$ and $\beta = 1$, the forward rate follows pure normal(Gaussian) and pure lognormal, respectively. This parameter controls the skewness of the vol surface, with range $0 \leq \beta \leq 1$.
- σ_t is volatility, and $\sigma_0 = \alpha$ (alpha) is the current volatility (a constant). The parameter α controls the overall height of the curve.
- ν (nu) is the volatility of volatility. This parameter controls the smile of the vol surface, increasing ν will increase the vol smile.
- ρ (rho)is the correlation between forward value and volatility. Similar as β, it also controls the skewness of the vol surface, with range $0 \leq \rho \leq 1$.

2 SABR Implied Volatility

The classic SABR implied volatility is based on lognormal (black) model. To handle negative forward rates/strikes, shifted lognormal model and normal model can be used. In the option formulas, we will use ϕ as an indicator function for option price, with 1 represents a call and -1 a put option.

2.1 Lognormal(Black) Volatility

The prices of European call options in the SABR model are given by Black model. The forward rate SDE, and the option price for a current forward rate f, strike K, maturity T, and volatility σ_L are as follows.

$$
\begin{aligned}
dF &= \sigma_L F dW \\
V &= \phi e^{-rT}[(fN(\phi d_1) - KN(\phi d_2)] \\
d_1 &= \frac{\log(\frac{f}{K}) + \frac{1}{2}\sigma_L^2 T}{\sigma_L \sqrt{T}}, d_2 = d_1 - \sigma_L \sqrt{T}
\end{aligned} \tag{2}
$$

With the estimation of α, β, ν, ρ parameters, the SABR volatility can be implied by the following formulas, from (2.17a) - (2.18) of [1] :

$$
\begin{aligned}
\sigma_L(K, f) &= \frac{\alpha}{(fK)^{(1-\beta)/2}\left[1 + \frac{(1-\beta)^2}{24}\log^2(f/K) + \frac{(1-\beta)^4}{1920}\log^4(f/K) + \cdots\right]} \cdot \frac{z}{x(z)} \\
&\cdot \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]T + \cdots\right\}
\end{aligned} \tag{3}
$$

where

$$
z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log(f/K), \; x(z) = \log\left\{\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1-\rho}\right\}
$$

For the special case of at-the-money options $f = K$, the formula reduces to

$$
\sigma_L^{ATM}(f, f) = \frac{\alpha}{f^{(1-\beta)}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]T + \cdots\right\} \tag{4}
$$

2.2 Shifted Lognormal(Black) Volatility

The prices of European call options in the SABR model are given by shifted Black model. In order to handle negative forward/strike rates, a positive shift s is added to the classic Black model, so that F and K can become positive again. In the shifted version, the forward rate SDE, and the option price for a current forward rate $f + s$, strike $K + s$, maturity T, and shifted volatility σ_L are as follows.

$$
\begin{aligned}
dF &= \sigma_{SL}(F + s)dW \\
V &= \phi e^{-rT}[(f + s)N(\phi d_1) - (K + s)N(\phi d_2)] \\
d_1 &= \frac{\log(\frac{f+s}{K+s}) + \frac{1}{2}\sigma_{SL}^2 T}{\sigma_{SL}\sqrt{T}}, d_2 = d_1 - \sigma_{SL}\sqrt{T}
\end{aligned} \tag{5}
$$

Note that the shift s should be chosen so that $(F + s) > 0$ for all $0 \leq t \leq T$, and $(K + s) > 0$ for all strikes. The SABR shifted lognormal volatility can be implied using the lognormal vol formulas (3)(4), with forward rate and strike replaced by the shifted forward rate and strike.

$$
\sigma_{SL}(K, f) = \sigma_L(K + s, f + s) \tag{6}
$$

$$
\sigma_{SL}^{ATM}(f, f) = \sigma_L^{ATM}(f + s, f + s) \tag{7}
$$

2.3 Normal(Bachelier) Volatility

The prices of European call options in the SABR model are given by Normal model. the forward rate SDE, and the option price for a current forward rate f , strike K, maturity T, and implied volatility σ_N are as follows [2].

$$\begin{aligned} dF &= \sigma_N dW \quad\quad (8) \\ V &= \phi(f-K)N(\phi d_1) + \sigma_N \sqrt{T} n(d_1)] \\ d_1 &= \frac{f-K}{\sigma_N \sqrt{T}} \end{aligned}$$

Where $N(d_1)$ is the cumulative normal distribution function, and $n(d_1)$ is the standard normal density function. The SABR normal vol is implied by the following formula, from (B.67a) of [1].

$$\sigma_N(K,f) = \frac{\alpha(1-\beta)(f-K)}{f^{1-\beta}-K^{1-\beta}} \cdot \frac{z}{x(z)} \cdot \left\{ 1 + \left[\frac{-\beta(2-\beta)\alpha^2}{24 f_{av}^{2-2\beta}} + \frac{\rho\alpha\nu\beta}{4 f_{av}^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2 \right] T + \cdots \right\} \quad (9)$$

where

$$f_{av} = \sqrt{fK}, z = \frac{\nu}{\alpha}\frac{f-K}{f_{av}^{\beta}}, x(z) = \log\left\{ \frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho} \right\}$$

For the special case of at-the-money options $f = K$, this formula reduces to

$$\sigma_N^{ATM}(f,f) = \alpha f^{\beta} \left\{ 1 + \left[\frac{\beta(\beta-2)\alpha^2}{24 f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2 \right] T \right\} \quad (10)$$

In the normal model, in order to allow negative rates, β needs to be set to zero [4].

3 Calibration of SABR

For SABR calibration, we need the market swaption volatility quotations for different strikes, both at-the-money and out-of-the-money. It is important to choose the right market implied vol for the SABR model to calibrate against, i.e. If choosing either Black model, Shifted Black model, or Normal model, then the quoted market implied vol should be lognormal volatilities, shifted lognormal volatilities, or normal volatilities, correspondingly. See details in [4].

The parameter β needs to be estimated first. The value of β can be picked mainly in two ways:

- Predetermined user input: by choosing a value deemed appropriate for the market (a value such as 0.5).
- Historical estimation: by looking at the correlation between volatility and forward rates.

After β is fixed, there are two methods to estimate other parameters of SABR model, as follows [5].

3.1 Calibrate Alpha, Rho, and Nu Directly

Once β is set, it remains to estimate α , ρ, and ν. This is accomplished by minimizing the squared errors between the market volatilities σ_{market} and SABR implied volatilities σ.

$$MSE(\alpha,\rho,\nu) = \min \sum \left\{ \sigma^i_{market} - \sigma^i(f_i, K_i, \alpha, \rho, \nu) \right\}^2 \quad (11)$$

3.2 Calibrate Rho and Nu by Implying Alpha from At-The-Money Volatility

After fixing the value of β, we estimate ρ and ν, and derive α from ρ and ν by inverting the ATM volatility equation. For Lognormal volatility, by inverting (4), the following cubic polynomial is solved for α, and the smallest positive real root is selected [3] [5].

$$\left[\frac{(1-\beta)^2 T}{24 f^{2-2\beta}}\right]\alpha^3 + \left[\frac{\rho\beta\nu T}{4 f^{1-\beta}}\right]\alpha^2 + \left[1 + \frac{2-3\rho^2}{24}\nu^2 T\right]\alpha - \sigma_L^{ATM} f^{1-\beta} \quad = \quad 0 \tag{12}$$

For Normal volatility, by inverting (10), the following cubic polynomial is solved for α, and the smallest positive real root is selected [6].

$$\left[\frac{\beta(\beta-2) T}{24 f^{2-2\beta}}\right]\alpha^3 + \left[\frac{\rho\beta\nu T}{4 f^{1-\beta}}\right]\alpha^2 + \left[1 + \frac{2-3\rho^2}{24}\nu^2 T\right]\alpha - \sigma_N^{ATM} f^{-\beta} \quad = \quad 0 \tag{13}$$

In the minimization algorithm, at every iteration we find an α in terms of ρ and ν by solving the above equation (12) or (13), so the parameter alpha now is denoted as a function $\alpha(\rho, \nu)$. With this alpha function, the minimum squared errors between the market volatilities σ_{market} and SABR implied volatilities σ becomes:

$$MSE(\alpha, \rho, \nu) \quad = \quad \min \sum \left\{\sigma^i_{market} - \sigma^i(f_i, K_i, \alpha(\rho, \nu), \rho, \nu)\right\}^2 \tag{14}$$

Models calibrated using this method produce ATM volatilities that are equal to market quotes. This approach is widely used in swaptions, where ATM volatilities are quoted most frequently and are important to match.

4 Implementation

Currently Mlib supports SABR calibration to market Swaption vols, which can be lognormal, shifted lognormal, or normal vols, using Hagan's analytical formulas. The first model calibration method is supported, where alpha, rho, and nu are calibrated directly. Users can add weights to the calibration points. For instance, we can choose 100% weight on the swaption ATM vols, so that the model ATM vols can better match the market ones.

Some notes on Bloomberg terminal: VCUB command is used for market swaption volatilities. Premium is quoted in basis points. For OTM Vol Spread, if the vol is in percent, the spread is in percent; if the vol is in bps, the spread is in bps.

References

[1] Hagan, P. S., Kumar, D., Lesniewski, A. S. and Woodward, D. E. *Managing Smile Risk, 2002.*

[2] Espen Gaarder Haug. *The Complete Guide to Option Pricing Formulas, 1997.*

[3] Fabrice Douglas Rouah. *The SABR Model.*

[4] Mathworks. *Work with Negative Interest Rates.*

[5] Mathworks. *Calibrate the SABR Model.*

[6] Mathworks. *Calibrate the SABR Model using Normal (Bachelier) Volatilities with Negative Strikes.*