The GARCH linear SDE:
Explicit formulas and the pricing of a quanto CDS

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Abstract

We derive an efficient closed-form approximation for the moment generating function of the integral of a mean-reverting stochastic process, which follows a linear SDE that we call GARCH. We then consider a financial application, namely the pricing of a quanto CDS under stochastic intensity of default and an FX devaluation model. Numerical results are finally showcased.

1 Introduction

The explicit calculation of the moment generating function of the integral of a mean-reverting stochastic process is a problem that arises in several Mathematical Finance applications such as: i) zero-coupon bond pricing in a short-rate model; ii) calculation of survival probability in a reduced-form model with stochastic intensity of default; iii) efficient simulation of the volatility (or variance) process in a stochastic (or stochastic-local) volatility model; iv) the pricing of options on realized variance, including timer options.[1]

The most common mean-reverting processes, for which this moment generating function can be calculated in closed form, are the Ornstein-Uhlenbeck and square-root processes, with applications in interest-rate, default as well as volatility modeling. In this article, we focus on an alternative mean-reverting process, which, following Lewis (2000), we call GARCH. A GARCH process is described by a linear SDE, whose coefficients are affine functions of the underlying stochastic process, but one with mean-reverting drift and linear diffusion coefficient. As proven by Nelson (1990), any such SDE is the continuous-time limit diffusion of the variance in a Bollerslev (1986) GARCH discrete-time equation. This

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[1]When the stochastic process is non mean-reverting, as is the case of an equity asset or an FX rate, another financial application is the closed-form pricing of Asian options.

motivates using the term GARCH also for the continuous-time limit process and its corresponding SDE.

Contrary to the Ornstein-Uhlenbeck or square-root processes, however, the continuous-time GARCH process does not allow for the explicit calculation of the moment generating function of its integral. Nevertheless, we will derive accurate approximations in closed form using chaos expansions or, equivalently, an efficient recursive procedure that can easily be implemented in a software such as Mathematica.[2]

The calculation of the above moment generating function for alternative dynamics has been addressed by Tourrucoo, Hagan and Schleiniger (2007) for the generalized Black-Karasinski (1991) model, by Antonov and Spector (2011) in the context of a general short-rate model, both using perturbation methods, and by Stehlikova and Capriotti (2014) for the Black-Karasinski (1991) model using an exponent-expansion procedure.[3] Compared to these works, the GARCH process has the advantage of simpler and more explicit formulas.

The continuous-time mean-reverting GARCH process has been used in the financial literature mostly to model volatility or variance of asset returns, see for instance Lewis (2000), Paulot (2009) or Bloomberg (2015). Thanks to the approximations and numerical procedures we introduce, this process could also be used for interest-rate as well as default-intensity modeling. To this end, the financial application we consider is the pricing of CDS and quanto CDS under stochastic intensity of default and an FX devaluation model. We will derive closed-form approximations for both, as well as a simple rule-of-thumb formula for their ratio. This formula was introduced by Mercurio (2015) with no explicit proof. In this paper, we will provide a formal justification for it.

2 The GARCH linear SDE

A time-homogeneous GARCH process λ is a continuous-time diffusion process that satisfies the following linear SDE

$$d\lambda_t = \kappa(\vartheta - \lambda_t)\,\mathrm{d}t + \sigma\lambda_t\,\mathrm{d}W_t^\lambda \tag{1}$$

with initial condition λ_0, and where κ, ϑ and σ are positive constants, and W^λ is a standard Brownian motion under a given measure Q.

This SDE can be solved explicitly, see for instance Kloeden and Platen (1992). We have:

$$\lambda_t = \lambda_0 Y_t\, e^{-\kappa t} + \kappa\vartheta Y_t \int_0^t e^{-\kappa(t-u)} \frac{1}{Y_u}\,\mathrm{d}u \tag{2}$$

where $\mathrm{d}Y_t = \sigma Y_t\,\mathrm{d}W_t^\lambda$ with $Y_0 = 1$.

[2]In this article, we study the time-homogeneous case, namely that where the SDE has constant coefficients. Adding a time-dependent mean-reversion rate would not complicate the analysis, but would make the notation heavier. A possible, simpler extension that allows to calibrate an initial term structure, be it of rates or default probabilities, is obtained by shifting the time-homogeneous case with a time-dependent parameter, along the lines suggested by Brigo and Mercurio (2001).

[3]Capriotti (2018) extended his own approach to more general dynamics.

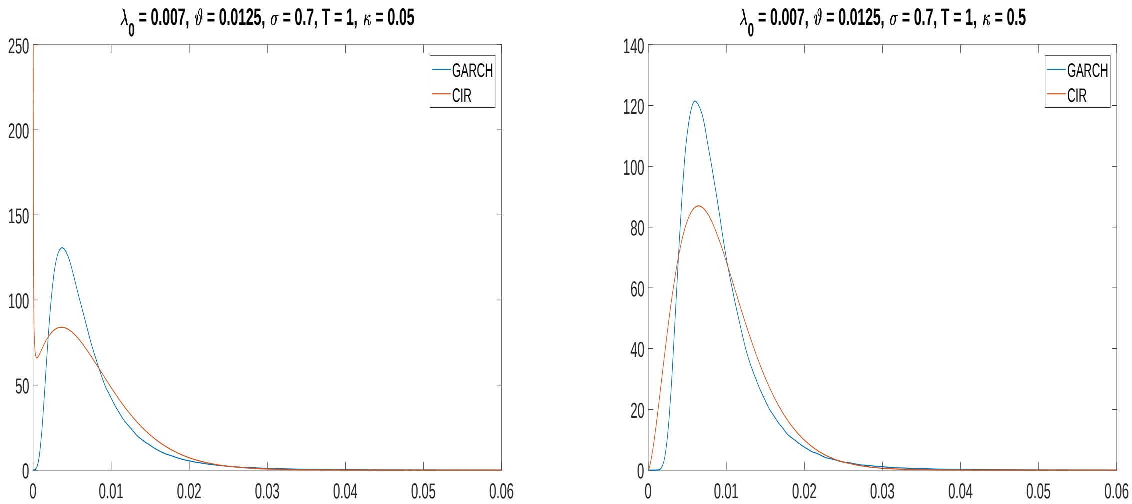

Figure 1: Comparison of the density of λ_T, T=1y, with the non-central chi-square density obtained by matching the first two moments of λ_T. Model parameters: $\lambda_0 = 0.007$, $\vartheta = 0.0125$, $\sigma = 0.7$, $\kappa = 0.05$ (left) and $\kappa = 0.5$ (right).

The GARCH process λ has the following additional properties: i) it is strictly positive, that is, thanks to (2), $\lambda_t > 0$ for all t when $\lambda_0 > 0$; ii) it does not explode in finite time; iii) positive moments of $\sup\{\lambda_u : 0 \leq u \leq t\}$ are finite; iv) moments of all orders can be calculated using an exact recursive formula based on matrix algebra; v) it admits an asymptotic (stationary) density

$$f_\infty(z) = \frac{q^\mu}{\Gamma(\mu)} z^{-\mu-1}\, e^{-q/z}$$

which is Inverse Gamma with parameters $\mu := 1 + \frac{2\kappa}{\sigma^2}$ and $q := \frac{2\kappa\vartheta}{\sigma^2}$. Furthermore, a GARCH process has more reasonable density profiles than those implied by the widely used square-root process, see Figure 1. In Figure 2, we compare the density of a GARCH process at different times with its stationary density.

In this paper, we want to calculate, for any $t \leq T$,

$$S(t,T) = \mathbb{E}\left[e^{-\int_t^T \lambda_u \,\mathrm{d}u} | \mathcal{F}_t\right] \tag{3}$$

where $\mathbb{E}$ denotes expectation under Q and $\mathcal{F}_t$ is the sigma-algebra generated by market risk factors up to time t. This expectation represents a zero-coupon bond price when λ is a short-rate process, or a survival probability when λ is a stochastic intensity of default. Hereafter, $S(t,T)$ will generically referred to as survival probability, since the financial application we will consider is based on a credit model.

When $\vartheta = 0$, process (1) reduces to a geometric Brownian motion, and the corresponding calculation of (3) was done by Dothan (1978) assuming that λ represents an

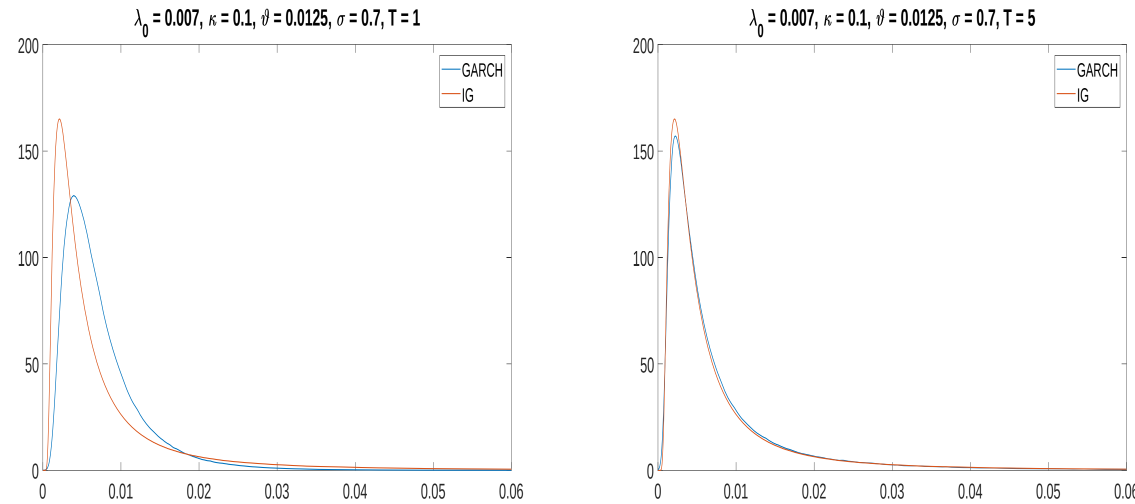

Figure 2: Comparison of the density of λ_T with its stationary Inverse Gamma density. Model parameters: $\lambda_0 = 0.007$, $\kappa = 0.1$, $\vartheta = 0.0125$, $\sigma = 0.7$, T=1y, (left) and $T = 5$y (right).

instantaneous short-rate process. His formulas have been corrected by Pintoux and Privault (2017) among others. In general, that is for $\vartheta \neq 0$, no semi-analytic formula, however, is available.

Since λ is time-homogeneous, then $S(t,T) = S(0, T-t)$. So, it will be enough to compute (3) at $t = 0$. With some abuse of notation, we will write $S(\lambda_0, \tau)$ to denote $S(0,\tau)$ while stressing the dependence on the initial condition.

By the Feynman-Kac theorem, see for instance Karatzas and Shreve (1991), functional S satisfies the following PDE:

$$\mathcal{L}S := -\frac{\partial S}{\partial \tau} + \kappa(\vartheta - \lambda)\frac{\partial S}{\partial \lambda} + \frac{1}{2}\sigma^2\lambda^2\frac{\partial^2 S}{\partial \lambda^2} - \lambda S = 0, \tag{4}$$

with the initial boundary condition

$$S(\lambda, 0) = 1. \tag{5}$$

3 Chaos expansions

Our closed-form approximation for (3) is based on expanding the exponent in its RHS using a Wiener-Ito chaos expansion, see for instance Di Nunno, Øksendal and Proske (2009). This can be achieved thanks to the linearity of the SDE (1), which allows for a simple iterative calculation leading to the desired expansion.

We prove in Appendix A that the following expansion holds for any given T:

$$-\int_0^T \lambda_u \,\mathrm{d}u = \sum_{n=0}^{\infty} \sigma^n I_n \tag{6}$$

where

$$
\begin{aligned}
I_0 &:= (\vartheta - \lambda_0)\frac{1 - e^{-\kappa T}}{\kappa} - \vartheta T \\
I_n &:= \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_T(t_1, t_n)\,\mathrm{d}W^\lambda_{t_1} \cdots \mathrm{d}W^\lambda_{t_n}, \quad n \geq 1
\end{aligned}
$$

and

$$
f_T(t, s) := \frac{e^{-\kappa T} - e^{-\kappa s}}{\kappa}\big(\lambda_0 + \vartheta(e^{\kappa t} - 1)\big) \tag{7}
$$

and where convergence of the series in (6) is in mean-square, and hence in probability.

We obtain an approximation for $S(\lambda_0, T)$ by taking a formal exponential of the power series (6), truncating it at some order N, and taking the expectation of the remaining terms. We get:

$$
S(\lambda_0, T) \approx e^{I_0}\mathbb{E}\Big[1 + \sigma I_1 + \sigma^2\Big(\frac{1}{2}I_1^2 + I_2\Big) + \sigma^3\Big(\frac{1}{6}I_1^3 + I_1 I_2 + I_3\Big) + \cdots\Big] \tag{8}
$$

From (4), which shows that $S(\lambda_0, T)$ is an even function of σ, we deduce that odd-power terms in (8) must have zero expected value. So, the formula for $S(\lambda_0, T)$ only contains even-power terms in σ. For instance, to sixth order in σ, that is for $N = 6$, we have:

$$
S(\lambda_0, T) = e^{I_0}\big[1 + C_1\sigma^2 + C_2\sigma^4 + C_3\sigma^6\big] + O(\sigma^8) \tag{9}
$$

where, reporting only terms with non-zero expectation,

$$
\begin{aligned}
C_1 &:= \frac{1}{2}\mathbb{E}(I_1^2) \\
C_2 &:= \frac{1}{24}\mathbb{E}(I_1^4) + \frac{1}{2}\mathbb{E}(I_2^2) + \frac{1}{2}\mathbb{E}(I_1^2 I_2) \\
C_3 &:= \frac{1}{720}\mathbb{E}(I_1^6) + \frac{1}{24}\mathbb{E}(I_1^4 I_2) + \frac{1}{6}\mathbb{E}(I_1^3 I_3) + \frac{1}{4}\mathbb{E}(I_1^2 I_2^2) + \mathbb{E}(I_1 I_2 I_3) + \frac{1}{6}\mathbb{E}(I_2^3) + \frac{1}{2}\mathbb{E}(I_3^2)
\end{aligned}
$$

We can show that all the expectations in C_i, $i = 1, 2, 3$, can be written as integrals of deterministic functions expressed in terms of f_T. Details are given in Appendix B. Therefore, based on (7), the resulting formula for $S(\lambda_0, T)$ is given by e^{I_0} times a linear combination of terms of the form $T^m e^{n\kappa T}$ with integer m and n.

4 A singular perturbation expansion

An alternative approach is based on a singular perturbation technique for the PDE (4). To this end, we seek a solution in the form of an asymptotic power series

$$
S(\lambda, \tau) = \sum_{i=0}^{\infty} \sigma^{2i}\, S_i(\lambda, \tau), \tag{10}
$$

meaning that, for any $N \in \mathbb{N}$,

$$S(\lambda, \tau) = \sum_{i=0}^{N} \sigma^{2i} S_i(\lambda, \tau) + O(\sigma^{2N+2})$$

as $\sigma \to 0$.

Substituting the ansatz (10) into (4), and collecting terms of the same order in the powers of σ^2, we obtain that the initial term $S_0(\lambda, \tau)$ satisfies the linear first-order PDE

$$-\frac{\partial S_0}{\partial \tau} + \kappa\,(\vartheta - \lambda)\,\frac{\partial S_0}{\partial \lambda} - \lambda\, S_0 = 0 \tag{11}$$

whereas terms $S_i(\lambda, \tau)$, $i = 1, 2, \ldots$, satisfy the recursive PDEs

$$-\frac{\partial S_i}{\partial \tau} + \kappa\,(\vartheta - \lambda)\,\frac{\partial S_i}{\partial \lambda} + \frac{1}{2}\,\lambda^2\,\frac{\partial^2 S_{i-1}}{\partial \lambda^2} - \lambda\, S_i = 0 \tag{12}$$

Since the boundary value $S(\lambda, 0) = 1$ does not depend on σ, the boundary conditions for the terms of the expansion (10) are $S_0(\lambda, 0) = 1$ and $S_i(\lambda, 0) = 0$, $i \geq 1$.

The closed-form solution of the PDE (11) is then given by

$$S_0(\lambda, \tau) = \exp\Big\{(\vartheta - \lambda)\,\frac{1 - e^{-\kappa\,\tau}}{\kappa} - \vartheta\,\tau\Big\}, \tag{13}$$

which shows that

$$S_0(\lambda_0, T) = e^{I_0} \tag{14}$$

As per the other terms, it is convenient to rescale them by $S_0(\lambda, \tau)$. We thus define functions $Q_i(\lambda, \tau)$ by

$$S_i(\lambda, \tau) = S_0(\lambda, \tau) Q_i(\lambda, \tau), \qquad i = 1, 2, \ldots, \tag{15}$$

so, the asymptotic power series for $S(\lambda, \tau)$ takes the form

$$S(\lambda, \tau) = S_0(\lambda, \tau)\left[1 + \sum_{i=1}^{N} \sigma^{2i}\, Q_i(\lambda, \tau) + O(\sigma^{2N+2})\right]$$

as $\sigma \to 0$.

Substituting (15) into (12), we obtain the following recursion relations for $Q_i(\lambda, \tau)$, $i \geq 1$:

$$\dot{Q}_{i+1} - \kappa(\vartheta - r) Q'_{i+1} - f_i(\lambda, \tau) = 0, \tag{16}$$

where

$$f_i(\lambda, \tau) = \frac{\lambda^2}{2\kappa^2}\Big[(1 - e^{-\kappa\tau})^2 Q_i + 2\kappa(e^{-\kappa\tau} - 1) Q'_i + \kappa^2 Q''_i\Big]. \tag{17}$$

and where $\dot{Q}$ denotes derivative with respect to τ, while Q' and Q'' denote, respectively, the first and second-order derivatives with respect to λ.

The first-order PDE (16) can be solved by integration. We get:

$$Q_{i+1}(\lambda, \tau) = \int_0^\tau f_i\Big(\vartheta + e^{-\kappa(\tau-u)}(\lambda - \vartheta), u\Big)\,\mathrm{d}u. \tag{18}$$

Therefore, starting from $Q_0 = 1$, we can recursively compute f_0, Q_1, f_1, Q_2, f_2, Q_3, etc. For example,

$$Q_1(\lambda, \tau) = \int_0^\tau \Big(\vartheta + e^{-\kappa(\tau-u)}(\lambda - \vartheta)\Big)^2 \, \frac{(1 - e^{-\kappa u})^2}{2\kappa^2}\,\mathrm{d}u.$$

so, in principle, we can compute Q_j in closed form to arbitrary order j.

It is tedious but easy to check that this expansion result agrees with that of the previous section. In fact, besides (14), we also have that $Q_i(\lambda_0, T) = C_i$, $i = 1, 2, 3$.

Remark 4.1. *The expansion technique outlined in this section can be applied to any short-rate models and not just the GARCH process, see also Liang (2017). In particular, the bond prices for the Cox-Ingersoll-Ross model or the Vasicek model can easily be approximated. We can then use the exact bond-price formulas in these two models to gauge the accuracy of the corresponding approximations.*

5 The implied average intensity

Given the survival probability $S(\lambda_0, t)$, we define the associated intensity $R(t)$ as follows:

$$S(\lambda_0, t) = e^{-R(t)t}$$

Therefore, the average implied intensity from time 0 to time t is given by

$$R(t) := -\frac{\ln S(\lambda_0, t)}{t} \tag{19}$$

Besides having a clear economic meaning, this quantity allows us to better gauge the quality of the derived approximation for the GARCH survival probability. In fact, small approximation errors in $S(\lambda_0, t)$ can lead to more noticeable discrepancies when R coordinates are used.

Based on (19), and using the specific relationship between T and σ in the approximation of $S(\lambda_0, T)$ to different orders, we can show that:

$$R(T) := \lambda_0 + \kappa(\vartheta - \lambda_0)\frac{T}{2} + \left[\kappa^2(\lambda_0 - \vartheta) - \sigma^2\lambda_0^2\right]\frac{T^2}{6} + \dots$$

Higher order terms up to the sixth are reported in Appendix C. The advantage of this approximation is that it is much simpler and much more compact than the corresponding higher-order expansions in σ. However, being it an expansion in T, we can only expect it to work for small maturities, see also the numerical examples below.

6 Numerical examples

We test the goodness of our approximation of $S(\lambda_0, T)$ for different orders N and different model parameters.

In Figure 3, we show the approximations we get for even orders up to the tenth, and for maturities up to ten years, and compare them with the corresponding Monte Carlo values based on simulating dynamics (1). The accuracy of the approximations depends on the chosen model parameters, on the approximation order and the maturity being considered. However, already the sixth-order, or even the fourth-order, expansion is typically very accurate, with errors below one bp, for maturities up to five years. Smaller values of σ clearly improve the accuracy of the approximation. Larger values of κ produce a similar effect, while larger values of T tend to decrease the accuracy. Notice that, being our approximation an asymptotic expansion in σ, it is not necessarily true that higher orders produce a lower error, see for instance the lower left plot.

We also test the accuracy of the small-time approximation for $R(t)$ given in Appendix C. Results are shown in Figure 4, where we compare Monte Carlo values with approximations up to sixth order. The approximations from the third order on appear to be very accurate for maturities up to five years. However, not surprisingly, they tend to deteriorate as the maturity increases, but also when κ increases.

7 A financial application: the pricing of a quanto CDS

A CDS is a credit derivative containing two legs: the premium leg and the protection leg. The protection buyer pays the protection seller a periodic fee, equal to the CDS rate $\mathbb{S}$ multiplied by the notional, in exchange for protection at the time of default of some reference asset. If the reference asset defaults at time τ before maturity T, then the protection buyer receives a payment equal to the loss given default L multiplied by the notional. The payments of the two legs are made in the same currency, dubbed the standard currency.

A quanto CDS is very similar, but with the difference that running premium and protection at default are paid in a non-standard currency, see Elizalde, Doctor and Singh (2010) for more details. The loss given default L depends on the recovery nature of the reference asset and is the same for both CDS contracts.

For simplicity, we hereafter assume that premia are paid continuously, and that the risk-free rates r_d for the standard currency and r_f for the non-standard currency are constant. The CDS rate $\mathbb{S}$ and the quanto CDS rate $\mathbb{S}_q$ are defined so that premium and protection

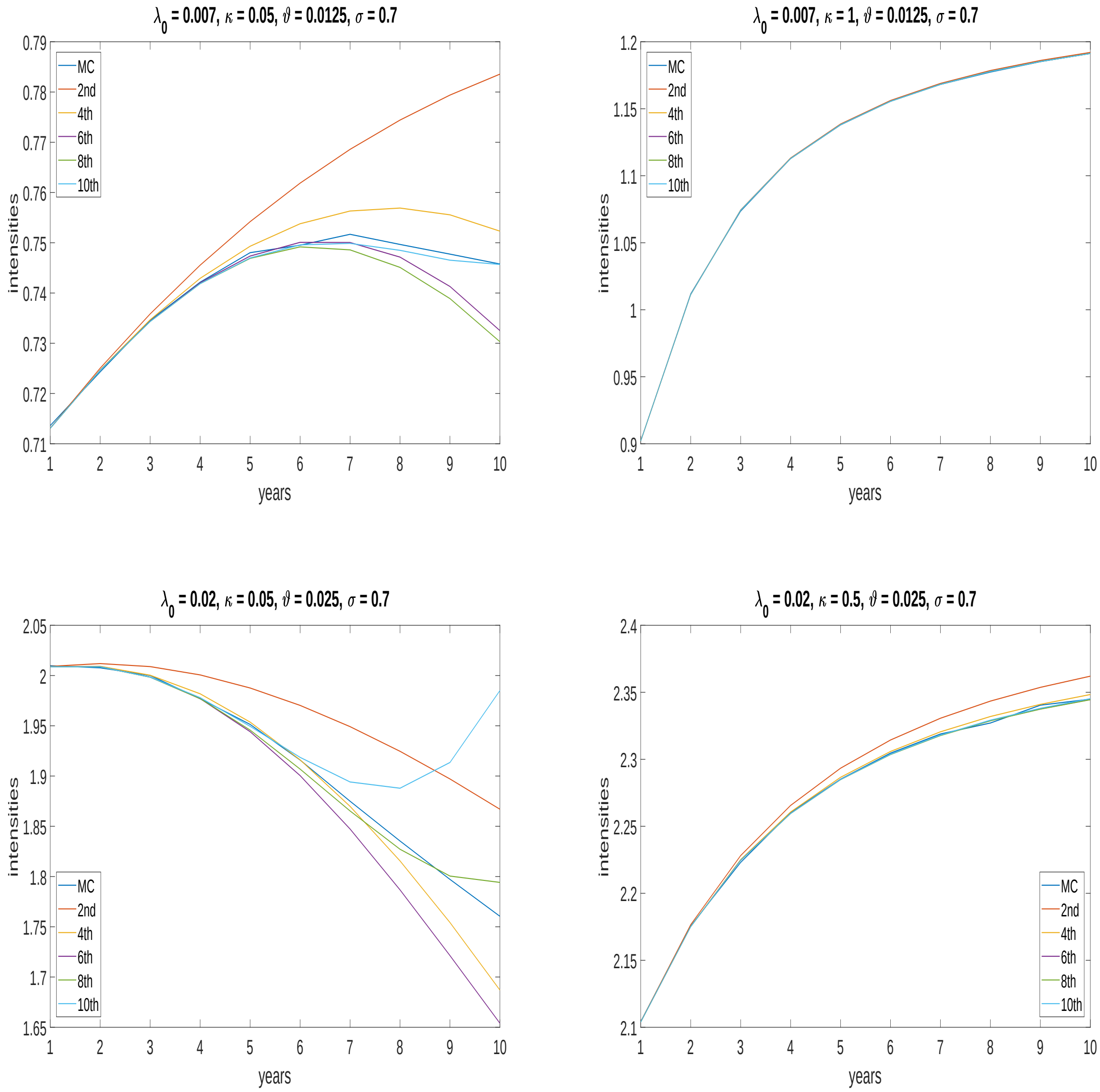

Figure 3: Plots of Monte Carlo values of $R(t)$, for different maturities t, along with corresponding approximations of different orders in σ. Model parameters: $\lambda_0 = 0.007$, $\vartheta = 0.0125$, $\sigma = 0.7$, $\kappa = 0.05$ (top left) and $\kappa = 1$ (top right); $\lambda_0 = 0.02$, $\vartheta = 0.025$, $\sigma = 0.7$, $\kappa = 0.05$ (bottom left) and $\kappa = 0.5$ (bottom right).

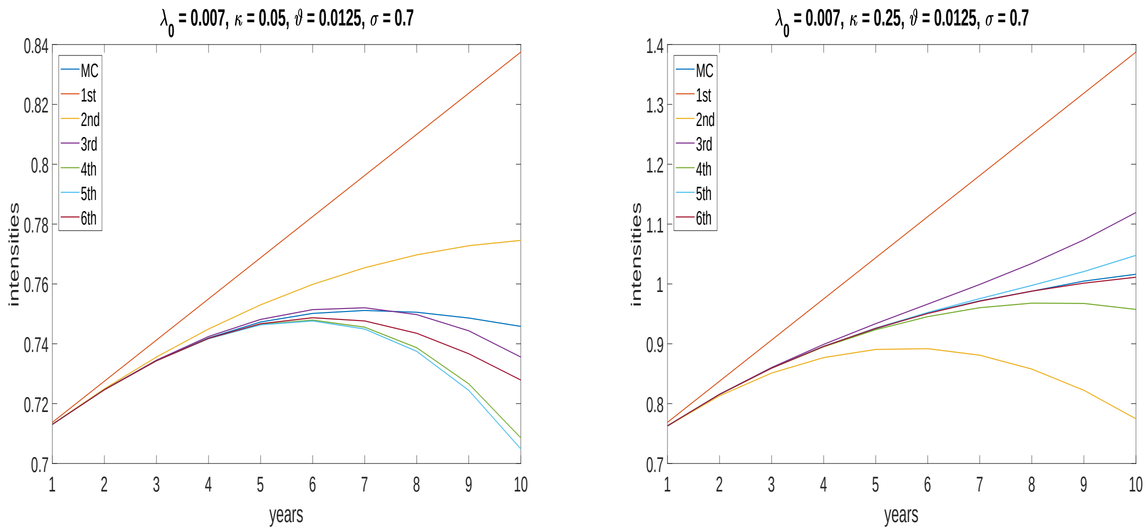

Figure 4: Plots of Monte Carlo values of $R(t)$, for different maturities t, along with corresponding approximations of different orders in t. Model parameters: $\lambda_0 = 0.007$, $\vartheta = 0.0125$, $\sigma = 0.7$, $\kappa = 0.05$ (left) and $\kappa = 0.25$ (right).

legs have the same value in their respective contracts:

$$\mathbb{S} = L \frac{\mathbb{E}\left[D(0,\tau)\, 1_{\{\tau \le T\}}\right]}{\mathbb{E}\left[\int_0^T D(0,t) 1_{\{\tau > t\}}\, \mathrm{d}t\right]} \tag{20}$$

$$\mathbb{S}_q = L \frac{\mathbb{E}\left[D(0,\tau)\, X_\tau 1_{\{\tau \le T\}}\right]}{\mathbb{E}\left[\int_0^T D(0,t) 1_{\{\tau > t\}} X_t\, \mathrm{d}t\right]} \tag{21}$$

where we set $D(0,t) := e^{-r_d t}$, and X_t is the value at time t of one unit of non-standard currency in standard currency.

We then assume that default is modeled using a Cox process N with stochastic intensity of default given by the GARCH process (1). The default time τ is the first time $N_t = 1$, so by conditioning on the realization of λ_t:

$$\mathbb{E}[1_{\{\tau > t\}}] = \mathbb{E}\left[e^{-\int_0^t \lambda_s\, \mathrm{d}s}\right] \tag{22}$$

The calculation of the quanto CDS rate requires modeling the exchange rate as well. To this end, we assume that X follows a geometric Brownian motion with a jump at default:

$$\mathrm{d}X_t = (r_d - r_f - \lambda_t J_t) X_t\, \mathrm{d}t + \sigma_X X_t\, \mathrm{d}W_t^X + J_t X_{t-}\, \mathrm{d}N_t, \tag{23}$$

where $J_t := J 1_{\{t \le \tau\}}$ and J is a constant proportional jump size. So, X_t can only jump once and exactly at the default time τ, see also Li and Mercurio (2015) for details. We assume that the Brownian motions W^λ and W^X are correlated with a constant correlation coefficient ρ.

Under the Cox process assumption, the CDS rate $\mathbb{S}$ becomes

$$\mathbb{S} = L\frac{\int_0^T D(0,t)\ \mathbb{E}\left[\lambda_t e^{-\int_0^t \lambda_s \,\mathrm{d}s}\right] \mathrm{d}t}{\int_0^T D(0,t)\ \mathbb{E}\left[e^{-\int_0^t \lambda_s \,\mathrm{d}s}\right] \mathrm{d}t} \tag{24}$$

Since we can write

$$\mathbb{E}\left[\lambda_t e^{-\int_0^t \lambda_s \,\mathrm{d}s}\right] = -\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[e^{-\int_0^t \lambda_s \,\mathrm{d}s}\right] \tag{25}$$

this implies that

$$\mathbb{S} = -L\frac{\int_0^T D(0,t)\,\mathrm{d}S(\lambda_0,t)}{\int_0^T D(0,t)S(\lambda_0,t)\,\mathrm{d}t} = L\frac{1 - D(0,T)S(\lambda_0,T)}{\int_0^T D(0,t)S(\lambda_0,t)\,\mathrm{d}t} - r_d \tag{26}$$

by integration by parts. Therefore, $\mathbb{S}$ can be calculated using the approximation for $S(\lambda_0,t)$ outlined in the previous sections.

The quanto CDS rate $\mathbb{S}_q$ can be calculated in a similar fashion. In fact, denoting by $\mathbb{E}_f$ the expectation in the risk-neutral measure Q_f of the non-standard currency, and setting $D_f(0,t) := e^{-r_f t}$, we have:

$$\mathbb{S}_q = L\frac{\mathbb{E}_f\left[D_f(0,\tau)\,1_{\{\tau\le T\}}\right]}{\mathbb{E}_f\left[\int_0^T D_f(0,t)1_{\{\tau>t\}}\,\mathrm{d}t\right]} \tag{27}$$

Using measure-change results for jump diffusions, see also Lando (1998), we can show that the intensity λ^f of N under Q_f is given by

$$\mathrm{d}\lambda_t^f = \kappa_f(\vartheta_f - \lambda_f)\,\mathrm{d}t + \sigma_f\lambda_t^f\,\mathrm{d}W_t^{\lambda,f} \tag{28}$$

where $W^{\lambda,f}$ is a standard Brownian motion under Q_f and

$$\begin{aligned}
\kappa_f &= \kappa - \rho\sigma\sigma_X \\
\vartheta_f &= (1+J)\frac{\kappa\vartheta}{\kappa - \rho\sigma\sigma_X} \\
\sigma_f &= \sigma \\
\lambda_0^f &= (1+J)\lambda_0
\end{aligned}$$

Therefore, when changing the measure, the intensity of default is still given by a GARCH process, with the same volatility but different drift parameters. This allows us to also calculate $\mathbb{S}_q$ using our approximation of survival probability since we can write:

$$\mathbb{S}_q = L\frac{1 - D_f(0,T)S_f(\lambda_0^f,T)}{\int_0^T D_f(0,t)S_f(\lambda_0^f,t)\,\mathrm{d}t} - r_f$$

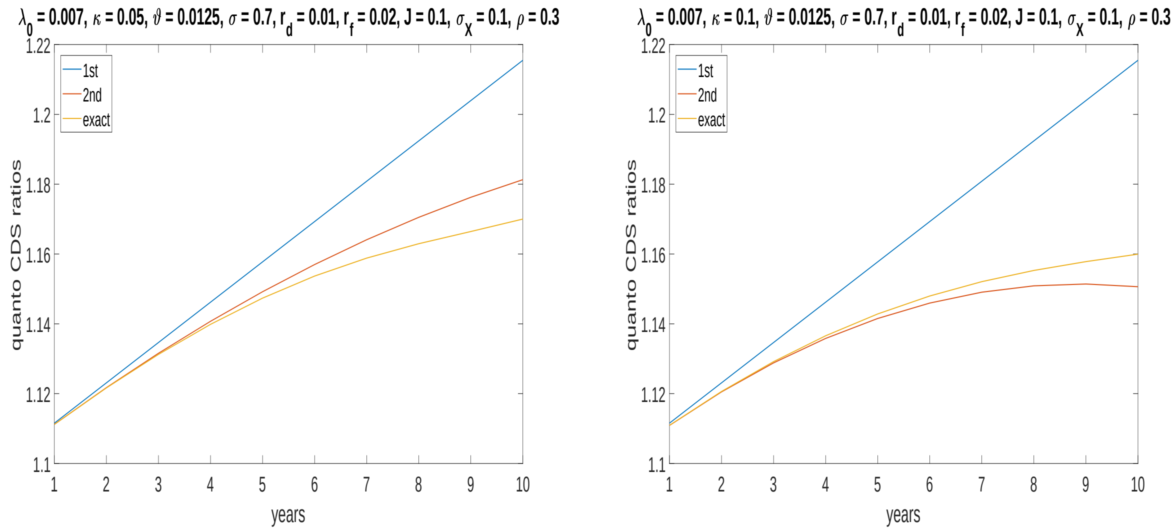

Figure 5: Exact quanto CDS ratios, for different maturities t, compared with first- and second-order approximations in t. Model parameters: $\lambda_0 = 0.007$, $\vartheta = 0.0125$, $\sigma = 0.7$, $r_d = 0.01$, $r_f = 0.02$, $J = 0.1$, $\sigma_X = 0.1$, $\rho = 0.3$, $\kappa = 0.05$ (left) and $\kappa = 0.1$ (right).

where $S_f(\lambda_0^f, t) = \mathbb{E}_f[1_{\{\tau>t\}}] = \mathbb{E}_f\left[e^{-\int_0^t \lambda_s^f \, \mathrm{d}s}\right]$.

Finally, we can derive a small-time approximation for the quanto CDS ratio by using the small-time expansion for the survival probability. To first order in T, we have:

$$\frac{\mathbb{S}_q}{\mathbb{S}} = (1+J)\left[1 + \frac{1}{2}\rho\sigma\sigma_X T\right] + o(T) \tag{29}$$

which gives a simple formula for deriving quanto CDS rates from quoted CDS rates, or *vice versa*, at least for maturities that are not too large. From this formula, we see that the FX devaluation, as measured by J, defines the CDS ratio for small maturities. But, as soon as T increases, stochastic intensity kicks in, and its contribution becomes increasingly sizeable.[4] A second-order expansion is also easy to derive, but is here omitted for brevity.

The accuracy of this approximation can be tested using Monte Carlo or higher-order approximation formulas for $S(\lambda_0, t)$ and $S_f(\lambda_0^f, t)$. Results are shown in Figure 5, where we compare first- and second-order expansion ratios to the corresponding exact values.

8 Conclusions

We derived closed-form approximations for the survival probability and the implied average intensity associated to a GARCH process. We then applied our results to the pricing of a

[4]The pricing of a quanto CDS under a devaluation FX model was also considered by Brigo et al. (2015), who assumed the same default intensity model of Stehlikova and Capriotti (2014). However, they could only derive a zero-th order formula for $\mathbb{S}_q/\mathbb{S}$, which agrees with (29) in the limit $T \to 0$. An extension of Brigo et al. (2015) model was proposed by Itkin et al. (2017), who used a radial-basis-function method to solve their CDS pricing problem.

Model	Strictly positive	$S(\lambda_0, T)$	Invariant dynamics
Vasicek	No	Exact	Yes
CIR	Yes/No	Exact	No
Exponential Vasicek	Yes	Approximation	Yes
GARCH	Yes	Approximation	Yes
Inverse GARCH	Yes	Approximation	Yes

Table 1: Comparison of different model dynamics. By Inverse GARCH, we denote the process obtained by taking the reciprocal of a GARCH process.

quanto CDS and derived a closed-form approximation for the quanto CDS ratio.

Compared to other dynamics, the GARCH model has several advantages. It is strictly positive when the initial condition is, it leads to a relatively simple approximation for survival probabilities, and it has invariant dynamics when changing the measure from domestic to foreign. A summary of properties for a number of mean-reverting processes is given in Table 8.

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9 Appendix A: Proof of the chaos expansion (6)

Omitting the superscript λ in W_t^λ for ease of notation, we integrate both sides of (1) from $t = 0$ to $t = T$, and get:

$$\lambda_T = \lambda_0 + \kappa\vartheta T - \kappa \int_0^T \lambda_t \, \mathrm{d}t + \sigma \int_0^T \lambda_t \, \mathrm{d}W_t \tag{30}$$

Similarly, integrating both sides of the SDE followed by $\widetilde{\lambda}_t = e^{\kappa t}\lambda_t$, we have:

$$\lambda_T = \lambda_0 \, e^{-\kappa T} + \vartheta\,(1 - e^{-\kappa T}) + \sigma \int_0^T \lambda_t \, e^{-\kappa(T-t)} \, \mathrm{d}W_t \tag{31}$$

Subtracting (31) from (30), and rearranging terms, we get:

$$\begin{aligned} -\int_0^T \lambda_t \, \mathrm{d}t &= (\vartheta - \lambda_0)\,\frac{1 - e^{-\kappa T}}{\kappa} - \vartheta T - \sigma \int_0^T \lambda_t \, \frac{1 - e^{-\kappa(T-t)}}{\kappa} \, \mathrm{d}W_t \\ &= (\vartheta - \lambda_0)\,\frac{1 - e^{-\kappa T}}{\kappa} - \vartheta T + \sigma \int_0^T \widetilde{\lambda}_t \, \frac{e^{-\kappa T} - e^{-\kappa t}}{\kappa} \, \mathrm{d}W_t \end{aligned} \tag{32}$$

where we assume $\kappa \neq 0$.

A chaos expansion for (32) can be derived by first deriving a chaos expansion for $\widetilde{\lambda}_t$. To this end, we rewrite (31) as:

$$\widetilde{\lambda}_T = \widetilde{m}(T) + \sigma \int_0^T \widetilde{\lambda}_t \, \mathrm{d}W_t \tag{33}$$

where $\widetilde{m}(t) := \mathbb{E}[\widetilde{\lambda}_t] = \lambda_0 + \vartheta(e^{\kappa t} - 1)$.

A chaos expansion for process $\widetilde{\lambda}_t$ can be derived by repeatedly using equation (33) to replace occurrences of $\widetilde{\lambda}$ on its own right-hand-side:

$$\begin{aligned} \widetilde{\lambda}_t &= \widetilde{m}(t) + \sigma \int_0^t \left[\widetilde{m}(s) + \sigma \int_0^s \widetilde{\lambda}_u \, \mathrm{d}W_u \right] \mathrm{d}W_s \\ &= \widetilde{m}(t) + \sigma \int_0^t \widetilde{m}(s) \, \mathrm{d}W_s + \sigma^2 \int_0^t \int_0^s \left[\widetilde{m}(u) + \sigma \int_0^u \widetilde{\lambda}_v \, \mathrm{d}W_v \right] \mathrm{d}W_u \, \mathrm{d}W_s \\ &= \widetilde{m}(t) + \sum_{n=1}^{N-1} \sigma^n \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \widetilde{m}(t_1) \, \mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_n} + R_N(t) \end{aligned}$$

where $N \geq 2$, and we set

$$R_N(t) := \sigma^N \int_0^t \int_0^{t_N} \cdots \int_0^{t_2} \widetilde{\lambda}(t_1) \, \mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_N}$$

Since, for each given t,

$$\begin{aligned}\mathbb{E}[R_N^2(t)] &= \sigma^{2N} \int_0^t \int_0^{t_N} \cdots \int_0^{t_2} \mathbb{E}[\widetilde{\lambda}^2(t_1)]\,\mathrm{d}t_1 \cdots \mathrm{d}t_N \\ &= \sigma^{2N} \int_0^t \frac{(t-s)^{N-1}}{(N-1)!}\,\mathbb{E}[\widetilde{\lambda}^2(s)]\,\mathrm{d}s \end{aligned} \tag{34}$$

and

$$\lim_{N\to\infty} \mathbb{E}[R_N^2(t)] = 0 \tag{35}$$

thanks to the dominated convergence theorem, then we can write

$$\widetilde{\lambda}_t = \widetilde{m}(t) + \sum_{n=1}^{\infty} \sigma^n \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \widetilde{m}(t_1)\,\mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_n} \tag{36}$$

where convergence of the series is in mean-square.

Plugging (36) into (32) then leads to (6). The mean-square convergence of the series in (6) can be proven by showing that

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\int_0^T R_N(t)\,\frac{e^{-\kappa T} - e^{-\kappa t}}{\kappa}\,\mathrm{d}W_t\right)^2\right] = 0$$

which follows again from the dominated convergence theorem and from (35), since

$$\mathbb{E}\left[\left(\int_0^T R_N(t)\,\frac{e^{-\kappa T} - e^{-\kappa t}}{\kappa}\,\mathrm{d}W_t\right)^2\right] = \int_0^T \mathbb{E}[R_N^2(t)]\left(\frac{e^{-\kappa T} - e^{-\kappa t}}{\kappa}\right)^2 \mathrm{d}t$$

10 Appendix B: Closed-form formulas for C_1, C_2 and C_3

Repeated application of Ito's lemma in one and two dimensions, along with isometries of iterated Ito integrals, leads to:

$$
\begin{aligned}
\mathbb{E}(I_1^2) &= \int_0^T f_T^2(t,t)\,\mathrm{d}t \\
\mathbb{E}(I_1^4) &= 3\left[\int_0^T f_T^2(t,t)\,\mathrm{d}t\right]^2 \\
\mathbb{E}(I_2^2) &= \int_0^T \int_0^t f_T^2(u,t)\,\mathrm{d}u\,\mathrm{d}t \\
\mathbb{E}(I_1^2 I_2) &= 2\int_0^T \int_0^t f_T(u,u) f_T(u,t) f_T(t,t)\,\mathrm{d}u\,\mathrm{d}t \\
\mathbb{E}(I_1^6) &= 15\left[\int_0^T f_T^2(t,t)\,\mathrm{d}t\right]^3 \\
\mathbb{E}(I_1^4 I_2) &= 12\int_0^T f_T^2(t,t)\,\mathrm{d}t \int_0^T \int_0^t f_T(u,u) f_T(u,t) f_T(t,t)\,\mathrm{d}u\,\mathrm{d}t \\
\mathbb{E}(I_1^3 I_3) &= 6\int_0^T \int_0^t \int_0^s f_T(u,u) f_T(s,s) f_T(t,t) f_T(u,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t \\
\mathbb{E}(I_1^2 I_2^2) &= 4\int_0^T \int_0^t \int_0^s f_T(u,t) f_T(s,s) f_T(u,s) f_T(t,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + 4\int_0^T \int_0^t \int_0^s f_T(u,u) f_T(s,t) f_T(u,s) f_T(t,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + 4\int_0^T \int_0^t \int_0^s f_T(u,u) f_T(s,s) f_T(u,t) f_T(s,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^T f_T^2(t,t)\,\mathrm{d}t \int_0^T \int_0^t f_T^2(u,t)\,\mathrm{d}u\,\mathrm{d}t \\
\mathbb{E}(I_1 I_2 I_3) &= \int_0^T \int_0^t \int_0^s f_T(u,s) f_T(t,t) f_T(u,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^T \int_0^t \int_0^s f_T^2(u,t) f_T(s,s)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t + \int_0^T \int_0^t \int_0^s f_T(u,t) f_T(u,u) f_T(s,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t \\
\mathbb{E}(I_2^3) &= 6\int_0^T \int_0^t \int_0^s f_T(u,s) f_T(u,t) f_T(s,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t \\
\mathbb{E}(I_3^2) &= \int_0^T \int_0^t \int_0^s f_T^2(u,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t
\end{aligned}
$$

Accordingly, we can write:

$$C_1 := \int_0^T c_1(t)\,\mathrm{d}t$$
$$C_2 := \int_0^T \int_0^t c_2(u,t)\,\mathrm{d}u\,\mathrm{d}t$$
$$C_3 := \int_0^T \int_0^t \int_0^s c_3(u,s,t)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t$$

for suitably defined functions c_i, $i = 1,2,3$.

All of these integrals can be calculated explicitly using the definition of f_T in (7).

11 Appendix C: Small-time approximation for the implied intensity

The implied intensity $R(T)$ admits the following short-term expansion:

$$R(T) = \lambda_0 + \sum_{n=1}^{6} A_n \frac{T^n}{(n+1)!} + O(T^7)$$

where we report terms up to the sixth order, and

$$\begin{aligned}
A_1 &:= \kappa(\vartheta - \lambda_0) \\
A_2 &:= \kappa^2(\lambda_0 - \vartheta) - \sigma^2\lambda_0^2 \\
A_3 &:= \kappa^3(\vartheta - \lambda_0) + \sigma^2\kappa\lambda_0(5\lambda_0 - 2\vartheta) - \sigma^4\lambda_0^2 \\
A_4 &:= \kappa^4(\lambda_0 - \vartheta) - \sigma^2\kappa^2(17\lambda_0^2 - 12\vartheta\lambda_0 + 2\vartheta^2) + \sigma^4\lambda_0(8\lambda_0^2 + 7\kappa\lambda_0 - 2\kappa\vartheta) - \sigma^6\lambda_0^2 \\
A_5 &:= \kappa^5(\vartheta - \lambda_0) + \sigma^2\kappa^3(49\lambda_0^2 - 46\vartheta\lambda_0 + 12\vartheta^2) - \sigma^4\kappa(94\lambda_0^3 - 34\vartheta\lambda_0^2 \\
&\quad + 31\kappa\lambda_0^2 - 16\kappa\vartheta\lambda_0 + 2\kappa\vartheta^2) + \sigma^6\lambda_0(34\lambda_0^2 + 9\kappa\lambda_0 - 2\kappa\vartheta) - \sigma^8\lambda_0^2 \\
A_6 &:= \kappa^6(\lambda_0 - \vartheta) - \sigma^2\kappa^4(129\lambda_0^2 - 144\vartheta\lambda_0 + 46\vartheta^2) + \sigma^4\kappa^2(676\lambda_0^3 - 452\vartheta\lambda_0^2 \\
&\quad + 68\vartheta^2\lambda_0 + 111\kappa\lambda_0^2 - 78\kappa\vartheta\lambda_0 + 16\kappa\vartheta^2) - \sigma^6(184\lambda_0^4 + 498\kappa\lambda_0^3 - 148\kappa\vartheta\lambda_0^2 \\
&\quad + 49\kappa^2\lambda_0^2 - 20\kappa^2\vartheta\lambda_0 + 2\kappa^2\vartheta^2) + \sigma^8\lambda_0(114\lambda_0^2 + 11\kappa\lambda_0 - 2\kappa\vartheta) - \sigma^{10}\lambda_0^2
\end{aligned}$$