The Garch linear SDE: explicit formulas and the pricing of a quanto CDS

Minqiang Li, Fabio Mercurio and Serge Resnick derive an efficient closed-form approximation for the moment-generating function of the integral of a mean-reverting stochastic process, which follows a linear stochastic differential equation that we call generalised autoregressive conditional heteroscedasticity. We then consider a financial application, namely the pricing of a quanto credit default swap under stochastic intensity of default and a foreign exchange devaluation model. Numerical results are finally showcased

The explicit calculation of the moment-generating function of the integral of a mean-reverting stochastic process is a problem that arises in several mathematical finance applications, such as (i) zero-coupon bond pricing in a short-rate model, (ii) the calculation of survival probability in a reduced-form model with stochastic intensity of default, (iii) efficient simulation of the volatility (or variance) process in a stochastic (or stochastic-local) volatility model, and (iv) the pricing of options on realised variance, including timer options.[1]

The most common mean-reverting processes for which this moment-generating function can be calculated in closed form are the Ornstein-Uhlenbeck and square-root processes, with applications in interest rate, default and volatility modelling. In this article, we focus on an alternative mean-reverting process, which, following Lewis (2000), we call generalised autoregressive conditional heteroscedasticity (Garch). A Garch process is described by a linear stochastic differential equation (SDE), with coefficients that are affine functions of the underlying stochastic process, but one with mean-reverting drift and a linear diffusion coefficient. Any such SDE is the continuous-time diffusion limit of the variance in a Bollerslev Garch discrete-time equation. This motivates using the term 'Garch' for the continuous-time limit process and its corresponding SDE as well. In the financial literature, this process is also referred to as an inhomogeneous geometric Brownian motion (see, for instance, Zhao (2009) or the recent work of Capriotti *et al* (2018), who derived analytical formulas for the transition probabilities and Arrow-Debreu prices for such a process).

Contrary to the Ornstein-Uhlenbeck or square-root processes, however, the continuous-time Garch process does not allow for the explicit calculation of the moment-generating function of its integral. Nevertheless, we will derive accurate approximations in closed form using chaos expansions or, equivalently, an efficient recursive procedure that can easily be implemented in software such as Mathematica.[2]

The calculation of the above moment-generating function for alternative dynamics has been addressed by Tourrucoo *et al* (2007) for the generalised Black-Karasinski model, by Antonov and Spector (2011) in the context of a general short-rate model (both of these studies use perturbation methods), and by Stehlikova and Capriotti (2014) for the Black-Karasinski model using an exponent-expansion procedure. In comparison with the processes employed in these works, the Garch process has the advantage of simpler and more explicit formulas.

The continuous-time mean-reverting Garch process has mostly been used in the financial literature to model the volatility or variance of asset returns. Thanks to the approximations and numerical procedures we introduce, this process could also be used for interest-rate as well as default-intensity modelling. To this end, the financial application we consider is the pricing of credit default swaps (CDSs) and quanto CDSs under stochastic intensity of default and a forex devaluation model. We will derive closed-form approximations for both as well as a simple rule-of-thumb formula for their ratio. More references can be found in Li *et al* (2018).

The Garch linear SDE

A time-homogeneous Garch process λ is a continuous-time diffusion process that satisfies the following linear SDE:

$$d\lambda_t = \kappa(\vartheta - \lambda_t)\,dt + \sigma\lambda_t\,dW_t^{\lambda} \tag{1}$$

with initial condition λ_0, and where κ, ϑ and σ are positive constants, and W^{λ} is a standard Brownian motion under a given measure Q.

This SDE can be solved explicitly. We have:

$$\lambda_t = \lambda_0 Y_t e^{-\kappa t} + \kappa\vartheta Y_t \int_0^t e^{-\kappa(t-u)}\frac{1}{Y_u}\,du \tag{2}$$

where $dY_t = \sigma Y_t\,dW_t^{\lambda}$ with $Y_0 = 1$.

The Garch process λ has the following additional properties:

(i) it is strictly positive, that is, thanks to (2), $\lambda_t > 0$ for all t when $\lambda_0 > 0$;

(ii) it does not explode in finite time;

(iii) positive moments of $\sup\{\lambda_u : 0 \leqslant u \leqslant t\}$ are finite;

(iv) moments of all orders can be calculated using an exact recursive formula based on matrix algebra;

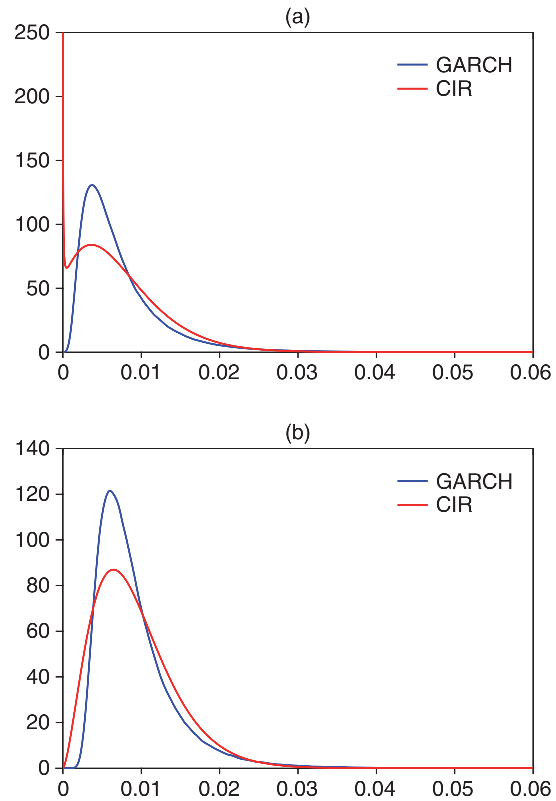
(v) it admits an asymptotic (stationary) density:

$$f_\infty(z) = \frac{q^{\mu}}{\Gamma(\mu)} z^{-\mu-1} e^{-q/z}$$

[1] *When the stochastic process is non-mean-reverting, as is the case with an equity asset or a foreign exchange rate, another financial application is the closed-form pricing of Asian options.*

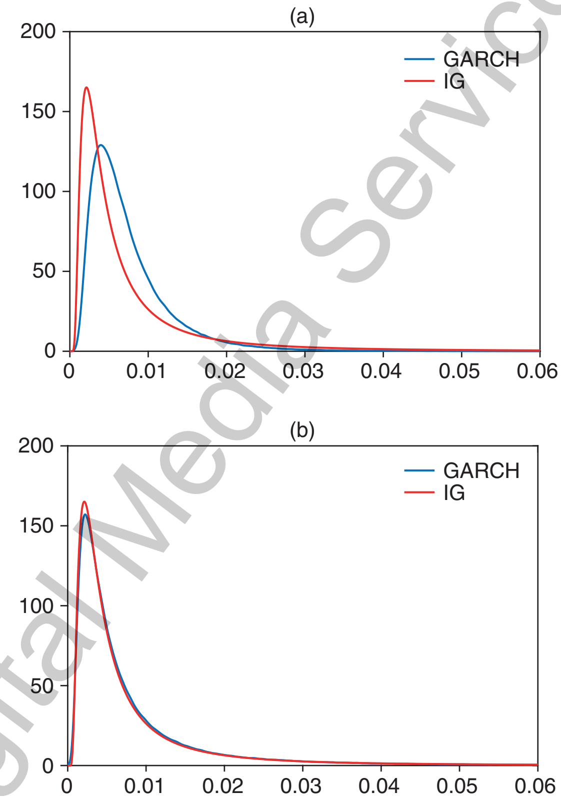
[2] *In this article, we study the time-homogeneous case, namely that in which the SDE has constant coefficients. Adding a time-dependent mean-reversion rate would not complicate the analysis, but it would make the notation heavier. A possible, simpler extension that allows us to calibrate an initial term structure, be it of rates or default probabilities, is obtained by shifting the time-homogeneous case with a time-dependent parameter, along the lines suggested by Brigo and Mercurio (2001).*

1 Comparison of the density of λ_T, $T = 1$y, with the non-central chi-square density obtained by matching the first two moments of λ_T



Model parameters: $\lambda_0 = 0.007$, $\vartheta = 0.0125$, $\sigma = 0.7$, (a) $\kappa = 0.05$ and (b) $\kappa = 0.5$

2 Comparison of the density of λ_T with its stationary inverse Gamma density



Model parameters: $\lambda_0 = 0.007$, $\kappa = 0.1$, $\vartheta = 0.0125$, $\sigma = 0.7$, (a) $T = 1$y and (b) $T = 5$y

which is inverse Gamma with parameters:

$$\mu := 1 + \frac{2\kappa}{\sigma^2} \quad \text{and} \quad q := \frac{2\kappa\vartheta}{\sigma^2}$$

Furthermore, a Garch process has more reasonable density profiles than those implied by the widely used square-root process (see figure 1). In figure 2, we compare the density of a Garch process at different times with its stationary density. More properties and closed-form results for the Garch process can be found in Zhao (2009).

In this article, we want to calculate, for any $t \leqslant T$:

$$S(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_u \,\mathrm{d}u\right) \,\middle|\, \mathcal{F}_t\right] \tag{3}$$

where $\mathbb{E}$ denotes expectation under Q and $\mathcal{F}_t$ is the sigma-algebra generated by market risk factors up to time t. This expectation represents a zero-coupon bond price when λ is a short-rate process, or a survival probability when λ is a stochastic intensity of default. Hereafter, $S(t,T)$ will generically be referred to as survival probability, since the financial application we will consider is based on a credit model.

When $\vartheta = 0$, (1) reduces to a geometric Brownian motion, and (3) can be calculated in closed form. In general, however – that is, for $\vartheta \neq 0$ – no semi-analytic formula is available.

Since λ is time homogeneous, $S(t,T) = S(0, T-t)$. So, it will be enough to compute (3) at $t = 0$. With some abuse of notation, we will write $S(\lambda_0, \tau)$ to denote $S(0,\tau)$, while stressing the dependence on the initial condition.

By the Feynman-Kac theorem, functional S satisfies the following partial differential equation (PDE):

$$\mathcal{L}S := -\frac{\partial S}{\partial \tau} + \kappa(\vartheta - \lambda)\frac{\partial S}{\partial \lambda} + \tfrac{1}{2}\sigma^2\lambda^2\frac{\partial^2 S}{\partial \lambda^2} - \lambda S = 0 \tag{4}$$

with the initial boundary condition:

$$S(\lambda, 0) = 1 \tag{5}$$

Chaos expansions

Our closed-form approximation for (3) is based on expanding the exponent on its right-hand side using a Wiener-Ito chaos expansion. This can be achieved thanks to the linearity of SDE (1), which allows for a simple iterative calculation that leads to the desired expansion.

We prove in the appendix of Li *et al* (2018) that the following expansion holds for any given T:

$$-\int_0^T \lambda_u \,\mathrm{d}u = \sum_{n=0}^{\infty} \sigma^n I_n \tag{6}$$

where:

$$I_0 := (\vartheta - \lambda_0)\frac{1 - e^{-\kappa T}}{\kappa} - \vartheta T$$

$$I_n := \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_T(t_1, t_n)\, dW^{\lambda}_{t_1} \cdots dW^{\lambda}_{t_n}, \quad n \geqslant 1$$

and:

$$f_T(t, s) := \frac{e^{-\kappa T} - e^{-\kappa s}}{\kappa}(\lambda_0 + \vartheta(e^{\kappa t} - 1)) \tag{7}$$

and where convergence of the series in (6) is in mean square, and hence in probability.

We obtain an approximation for $S(\lambda_0, T)$ by taking a formal exponential of the power series (6), truncating it at some order N and taking the expectation of the remaining terms. We get:

$$S(\lambda_0, T) \approx e^{I_0}\mathbb{E}[1 + \sigma I_1 + \sigma^2(\tfrac{1}{2}I_1^2 + I_2) + \sigma^3(\tfrac{1}{6}I_1^3 + I_1 I_2 + I_3) + \cdots] \tag{8}$$

From (4), which shows $S(\lambda_0, T)$ is an even function of σ, we deduce that odd-power terms in (8) must have zero expected value. So, the formula for $S(\lambda_0, T)$ only contains even-power terms in σ. For instance, to sixth order in σ, that is, for $N = 6$, we have:

$$S(\lambda_0, T) = e^{I_0}[1 + C_1\sigma^2 + C_2\sigma^4 + C_3\sigma^6] + O(\sigma^8) \tag{9}$$

where, reporting only terms with non-zero expectation:

$$C_1 := \tfrac{1}{2}\mathbb{E}(I_1^2)$$

$$C_2 := \tfrac{1}{24}\mathbb{E}(I_1^4) + \tfrac{1}{2}\mathbb{E}(I_2^2) + \tfrac{1}{2}\mathbb{E}(I_1^2 I_2)$$

$$C_3 := \tfrac{1}{720}\mathbb{E}(I_1^6) + \tfrac{1}{24}\mathbb{E}(I_1^4 I_2) + \tfrac{1}{6}\mathbb{E}(I_1^3 I_3) + \tfrac{1}{4}\mathbb{E}(I_1^2 I_2^2) + \mathbb{E}(I_1 I_2 I_3) + \tfrac{1}{6}\mathbb{E}(I_2^3) + \tfrac{1}{2}\mathbb{E}(I_3^2)$$

We can show that all the expectations in C_i, $i = 1, 2, 3$, can be written as integrals of deterministic functions expressed in terms of f_T. Details are given in Li *et al* (2018). Because of (7), the resulting formula for $S(\lambda_0, T)$ is given by e^{I_0} times a linear combination of terms of the form $T^m e^{n\kappa T}$ with integers m and n.

A recursive algorithm

An alternative approach is based on a perturbation method applied to the solution of PDE (4). Again, details are given in Li *et al* (2018). Here, we summarise the method as follows.

We define:

$$S_j(\lambda, \tau) := S_0(\lambda, \tau)\sum_{i=0}^{j} \sigma^{2i} Q_i(\lambda, \tau) \tag{10}$$

where the expansion is in powers of σ^2, as this is the quantity that appears in the PDE, and $Q_0 = 1$. We want:

$$\mathcal{L}S_j(\lambda, \tau) = O(\sigma^{2(j+1)}) \tag{11}$$

Plugging S_j into PDE (4), we see that in order to cancel all terms with orders $\leqslant j$, we need to have the following recursion for $i > 0$:

$$\dot{Q}_{i+1} - \kappa(\vartheta - \lambda)Q'_{i+1} - f_i(\lambda, \tau) = 0 \tag{12}$$

where:

$$f_i(\lambda, \tau) = \frac{\lambda^2}{2\kappa^2}[(1 - e^{-\kappa\tau})^2 Q_i + 2\kappa(e^{-\kappa\tau} - 1)Q'_i + \kappa^2 Q''_i] \tag{13}$$

and where $\dot{Q}$ denotes the derivative with respect to τ, while Q' and Q'' denote, respectively, the first- and second-order derivatives with respect to λ.

The first-order PDE (12) can be solved by integration. We get:

$$Q_{i+1}(\lambda, \tau) = \int_0^{\tau} f_i(\vartheta + e^{-\kappa(\tau - u)}(\lambda - \vartheta), u)\, du \tag{14}$$

Therefore, starting from $Q_0 = 1$, we can recursively compute f_0, Q_1, f_1, Q_2, f_2, Q_3, etc. For example:

$$Q_1(\lambda, \tau) = \int_0^{\tau} (\vartheta + e^{-\kappa(\tau - u)}(\lambda - \vartheta))^2 \frac{(1 - e^{-\kappa u})^2}{2\kappa^2}\, du$$

so, in principle, we can compute Q_j in closed form to arbitrary order j.

It is tedious but easy to check this expansion result agrees with that of the previous section. In fact, we have:

$$S_0(\lambda_0, T) = e^{I_0}$$

$$Q_i(\lambda_0, T) = C_i, \quad i = 1, 2, 3$$

A similar technique can be used to derive terms in the expansion of Arrow-Debreu prices or, equivalently, state-price density for the Garch process.

Remark 1 Contrary to the chaos method that leverages the linearity of the Garch SDE, the expansion technique outlined in this section can be applied to any short-rate models and not just the Garch process (see also Liang 2017). In particular, the bond prices for the Cox-Ingersoll-Ross model or the Vasicek model can be approximated easily. We can then use the exact bond-price formulas in these two models to gauge the accuracy of the corresponding approximations.

The implied average intensity

Given the survival probability $S(\lambda_0, t)$, we define the associated intensity $R(t)$ as follows:

$$S(\lambda_0, t) = e^{-R(t)t}$$

Therefore, the average implied intensity from time 0 to time t is given by:

$$R(t) := -\frac{\ln S(\lambda_0, t)}{t} \tag{15}$$

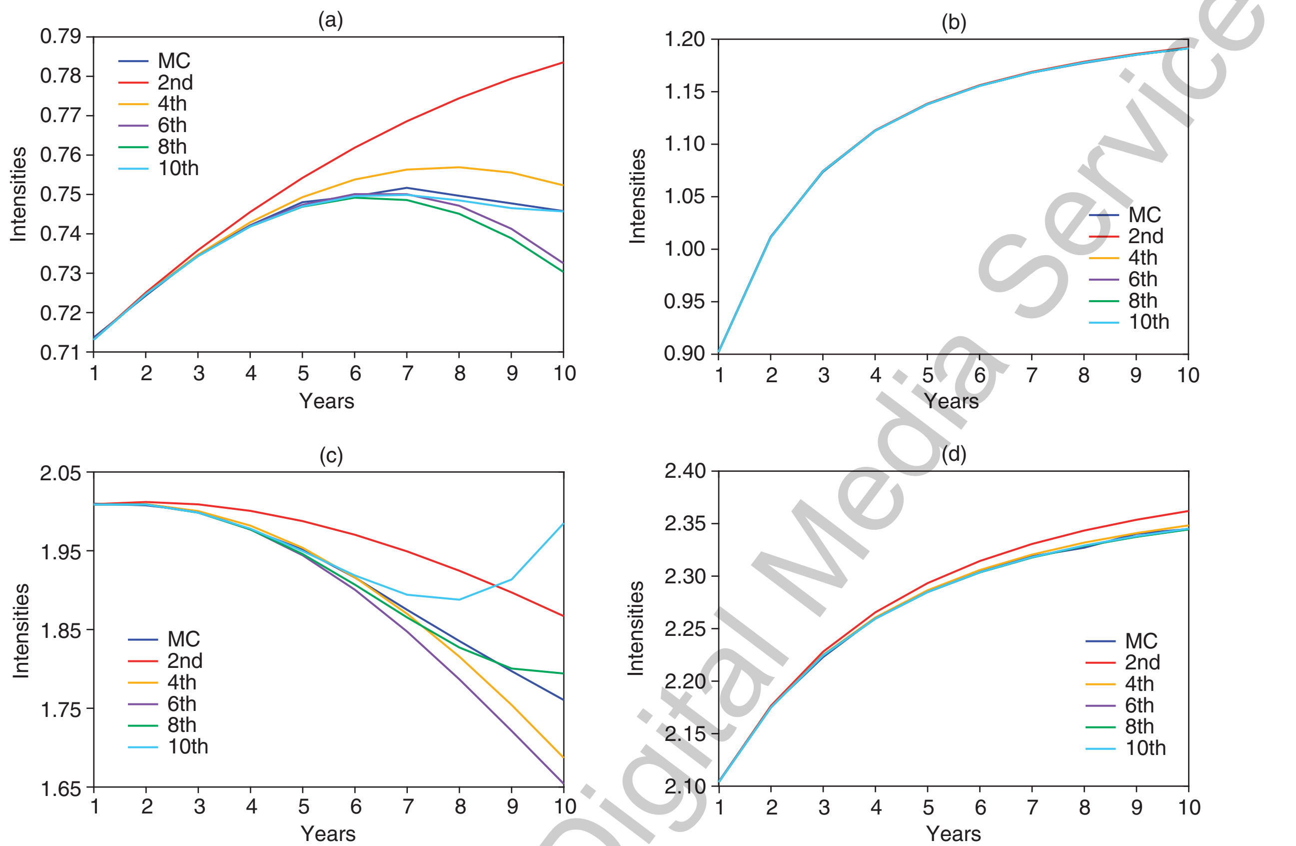
Besides having a clear economic meaning, this quantity allows us to better gauge the quality of the derived approximation for the Garch survival probability. In fact, small approximation errors in $S(\lambda_0, t)$ can lead to more noticeable discrepancies when R co-ordinates are used.

Based on (15), and using the specific relationship between T and σ in the approximation of $S(\lambda_0, T)$ to different orders, we can show:

$$R(T) := \lambda_0 + \kappa(\vartheta - \lambda_0)\frac{T}{2} + [\kappa^2(\lambda_0 - \vartheta) - \sigma^2\lambda_0^2]\frac{T^2}{6} + \cdots$$

Higher-order terms up to the sixth can be found in Li *et al* (2018). The advantage of this approximation is that it is much simpler and much more compact than the corresponding higher-order expansions in σ. However, it

3 Plots of Monte Carlo values of $R(t)$ for different maturities t, along with corresponding approximations of different orders in σ



Model parameters: $\lambda_0 = 0.007$, $\vartheta = 0.0125$, $\sigma = 0.7$, (a) $\kappa = 0.05$ and (b) $\kappa = 1$; $\lambda_0 = 0.02$, $\vartheta = 0.025$, $\sigma = 0.7$, (c) $\kappa = 0.05$ and (d) $\kappa = 0.5$

being an expansion in T, we can only expect it to work for small maturities (see also the numerical examples below).

Numerical examples

We test the goodness of our approximation of $S(\lambda_0, T)$ for different orders N and different model parameters.

In figure 3, we show the approximations we get for even orders up to the tenth and for maturities up to ten years, and we compare them with the corresponding Monte Carlo values based on simulating dynamics (1). The accuracy of the approximations depends on the chosen model parameters, on the approximation order and on the maturity being considered. However, the sixth-order, or even the fourth-order, expansion is typically very accurate, with errors below one basis point for maturities up to five years. Smaller values of σ clearly improve the accuracy of the approximation. Larger values of κ produce a similar effect, while larger values of T tend to decrease the accuracy. Notice that, because our approximation is an asymptotic expansion in σ, it is not necessarily true that higher orders produce a lower error (see, for instance, the lower left plot).

We also test the accuracy of the small-time approximation for $R(t)$. Results are shown in figure 4, where we compare Monte Carlo values with approximations up to the sixth order. The approximations from the third order on appear to be very accurate for maturities up to five years. However, unsurprisingly, they tend to deteriorate not only as the maturity increases but also when κ increases.

A financial application: the pricing of a quanto CDS

A CDS is a credit derivative containing two legs: the premium leg and the protection leg. The protection buyer pays the protection seller a periodic fee, equal to the CDS rate $\mathbb{S}$ multiplied by the notional, in exchange for protection at the time of default of some reference asset. If the reference asset defaults at time τ before maturity T, then the protection buyer receives a payment equal to the loss given default L multiplied by the notional. The payments of the two legs are made in the same currency, dubbed the standard currency.

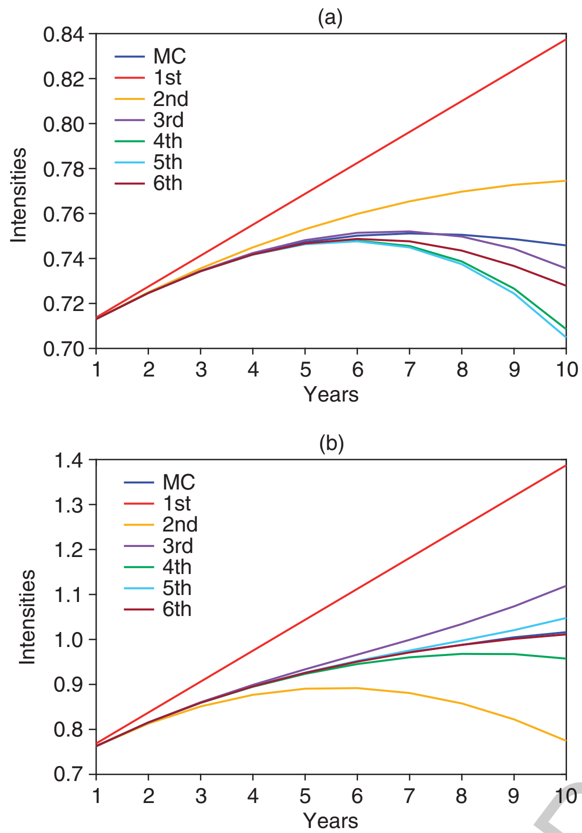
A quanto CDS is very similar, the difference being that running premium and protection at default are paid in a non-standard currency (see Elizalde *et al* (2010) for more details). The loss given default L depends on the recovery nature of the reference asset and is the same for both CDS contracts.

For simplicity, we hereafter assume premiums are paid continuously, and the risk-free rates r_d for the standard currency and r_f for the non-standard currency are constant. The CDS rate $\mathbb{S}$ and the quanto CDS rate $\mathbb{S}_q$ are defined such that the premium and protection legs have the same value in their respective contracts:

$$\mathbb{S} = L \frac{\mathbb{E}[D(0,\tau)1_{\{\tau \leqslant T\}}]}{\mathbb{E}[\int_0^T D(0,t)1_{\{\tau > t\}}\,\mathrm{d}t]} \tag{16}$$

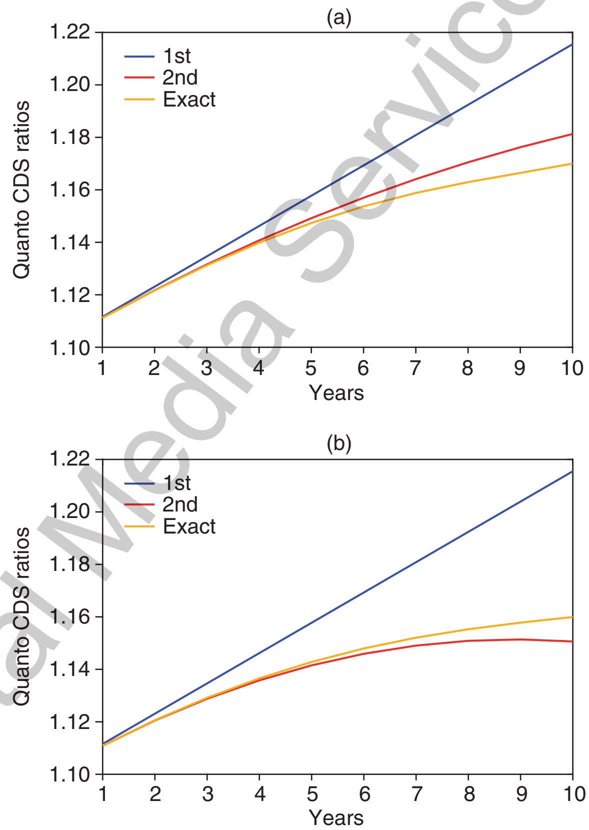
$$\mathbb{S}_q = L \frac{\mathbb{E}[D(0,\tau)X_\tau 1_{\{\tau \leqslant T\}}]}{\mathbb{E}[\int_0^T D(0,t)1_{\{\tau > t\}}X_t\,\mathrm{d}t]} \tag{17}$$

4 Plots of Monte Carlo values of $R(t)$ for different maturities t, along with corresponding approximations of different orders in t



Model parameters: $\lambda_0 = 0.007$, $\vartheta = 0.0125$, $\sigma = 0.7$, (a) $\kappa = 0.05$ and (b) $\kappa = 0.25$

5 Exact quanto CDS ratios for different maturities t, compared with first- and second-order approximations in t



Model parameters: $\lambda_0 = 0.007$, $\vartheta = 0.0125$, $\sigma = 0.7$, $r_d = 0.01$, $r_f = 0.02$, $J = 0.1$, $\sigma_X = 0.1$, $\rho = 0.3$, (a) $\kappa = 0.05$ and (b) $\kappa = 0.1$

where we set $D(0,t) := \mathrm{e}^{-r_d t}$, and X_t is the value at time t of one unit of non-standard currency in standard currency.

We then assume default is modelled using a Cox process N, with stochastic intensity of default given by the Garch process (1). The default time τ is the first time $N_t = 1$, so by conditioning on the realisation of λ_t:

$$\mathbb{E}[1_{\{\tau>t\}}] = \mathbb{E}\bigg[\exp\bigg(-\int_0^t \lambda_s \,\mathrm{d}s\bigg)\bigg] \tag{18}$$

The calculation of the quanto CDS rate requires modelling the exchange rate as well. To this end, we assume X follows a geometric Brownian motion with a jump at default:

$$\mathrm{d}X_t = (r_d - r_f - \lambda_t J_t) X_t \,\mathrm{d}t + \sigma_X X_t \,\mathrm{d}W_t^X + J_t X_{t-} \,\mathrm{d}N_t \tag{19}$$

where $J_t := J 1_{\{t \leqslant \tau\}}$ and J is a constant proportional jump size. So, X_t can only jump once and exactly at the default time τ (see also Li and Mercurio (2015) for details). We assume the Brownian motions W^λ and W^X are correlated with a constant correlation coefficient ρ.

Under the Cox process assumption, the CDS rate $\mathbb{S}$ becomes:

$$\mathbb{S} = L \frac{\int_0^T D(0,t)\mathbb{E}[\lambda_t \exp(-\int_0^t \lambda_s \,\mathrm{d}s)] \,\mathrm{d}t}{\int_0^T D(0,t)\mathbb{E}[\exp(-\int_0^t \lambda_s \,\mathrm{d}s)] \,\mathrm{d}t} \tag{20}$$

Since we can write:

$$\mathbb{E}\bigg[\lambda_t \exp\bigg(-\int_0^t \lambda_s \,\mathrm{d}s\bigg)\bigg] = -\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\bigg[\exp\bigg(-\int_0^t \lambda_s \,\mathrm{d}s\bigg)\bigg] \tag{21}$$

this implies:

$$\mathbb{S} = -L \frac{\int_0^T D(0,t) \,\mathrm{d}S(\lambda_0,t)}{\int_0^T D(0,t) S(\lambda_0,t) \,\mathrm{d}t} = L \frac{1 - D(0,T) S(\lambda_0,T)}{\int_0^T D(0,t) S(\lambda_0,t) \,\mathrm{d}t} - r_d \tag{22}$$

via integration by parts. Therefore, $\mathbb{S}$ can be calculated using the approximation for $S(\lambda_0,t)$ outlined in the previous sections.

The quanto CDS rate $\mathbb{S}_q$ can be calculated in a similar fashion. In fact, denoting by $\mathbb{E}_f$ the expectation in the risk-neutral measure Q_f of the non-standard currency, and setting $D_f(0,t) := \mathrm{e}^{-r_f t}$, we have:

$$\mathbb{S}_q = L \frac{\mathbb{E}_f[D_f(0,\tau) 1_{\{\tau \leqslant T\}}]}{\mathbb{E}_f[\int_0^T D_f(0,t) 1_{\{\tau>t\}} \,\mathrm{d}t]} \tag{23}$$

Using measure-change results for jump diffusions, we can show the intensity λ^f of N under Q_f is given by:

$$\mathrm{d}\lambda_t^f = \kappa_f(\vartheta_f - \lambda_f) \,\mathrm{d}t + \sigma_f \lambda_t^f \,\mathrm{d}W_t^{\lambda,f} \tag{24}$$

where $W^{\lambda,f}$ is a standard Brownian motion under Q_f and:

$$\kappa_f = \kappa - \rho\sigma\sigma_X$$
$$\vartheta_f = (1+J)\frac{\kappa\vartheta}{\kappa - \rho\sigma\sigma_X}$$
$$\sigma_f = \sigma$$
$$\lambda_0^f = (1+J)\lambda_0$$

Therefore, when changing the measure, the intensity of default is still given by a Garch process with the same volatility but different drift parameters. This allows us to calculate $\mathbb{S}_q$ using our approximation of the survival probability, since we can write:

$$\mathbb{S}_q = L\frac{1 - D_f(0,T)S_f(\lambda_0^f, T)}{\int_0^T D_f(0,t)S_f(\lambda_0^f, t)\,dt} - r_f$$

where:

$$S_f(\lambda_0^f, t) = \mathbb{E}_f[1_{\{\tau>t\}}] = \mathbb{E}_f\left[\exp\left(-\int_0^t \lambda_s^f\,ds\right)\right]$$

Finally, we can derive a small-time approximation for the quanto CDS ratio by using the small-time expansion for the survival probability. To first order in T, we have:

$$\frac{\mathbb{S}_q}{\mathbb{S}} = (1+J)[1 + \tfrac{1}{2}\rho\sigma\sigma_X T] + o(T) \qquad (25)$$

which gives a simple formula for deriving quanto CDS rates from quoted CDS rates, or vice versa, at least for maturities that are not too large. From this formula, we can see the forex devaluation, as measured by J, defines the CDS ratio for small maturities. However, as soon as T increases, stochastic intensity kicks in, and its contribution becomes increasingly sizeable.[3] A second-order expansion is also easy to derive but is omitted here for brevity.

[3] *The pricing of a quanto CDS under a devaluation forex model was also considered by Brigo et al (2015), who assumed the same default intensity model of Stehlikova and Capriotti (2014). However, they could only derive a zero-order formula for* $\mathbb{S}_q/\mathbb{S}$*, which agrees with (25) in the limit* $T \to 0$.

A. Comparison of different model dynamics

Model	Strictly positive	$S(\lambda_0, T)$	Invariant dynamics
Vasicek	No	Exact	Yes
Cox-Ingersoll-Ross	Yes/No	Exact	No
Exponential Vasicek	Yes	Approximation	Yes
Garch	Yes	Approximation	Yes
Inverse Garch	Yes	Approximation	Yes

By Inverse Garch, we mean the process obtained by taking the reciprocal of a Garch process

The accuracy of this approximation can be tested using Monte Carlo or higher-order approximation formulas for $S(\lambda_0, t)$ and $S_f(\lambda_0^f, t)$. Our results are shown in figure 5, where we compare first- and second-order expansion ratios with the corresponding exact values.

Conclusions

We derived closed-form approximations for the survival probability and the implied average intensity associated with a Garch process. We then applied our results to the pricing of a quanto CDS and derived a closed-form approximation for the quanto CDS ratio.

Compared with other dynamics, the Garch model has several advantages. It is strictly positive when the initial condition is likewise, it leads to a relatively simple approximation for survival probabilities, and it has invariant dynamics when changing the measure from domestic to foreign. A summary of properties for a number of mean-reverting processes is given in table A. ■

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