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Author(s): A. Itkin, V. Shcherbakov, A. Veygman

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International Journal of Theoretical and Applied Finance

New model for pricing Quanto Credit Default Swaps

A. ITKIN

Tandon School of Engineering, New York University,
12 Metro Tech Center, RH 517E, Brooklyn NY 11201, USA.
aitkin@nyu.edu

V.SHCHERBAKOV

Uppsala University, Department of Information Technology, Division of Scientific Computing,
Box 337, 751 05 Uppsala, Sweden.
victor.shcherbakov@it.uu.se

A. VEYGMAN

HSBS, New York, USA.
veygmana@gmail.com

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We propose a new model for pricing quanto credit default swaps (CDS) and risky bonds. The model operates with four stochastic factors, namely: the hazard rate, the foreign exchange rate, the domestic interest rate, and the foreign interest rate, and allows for jumps-at-default in both the foreign exchange rate and the foreign interest rate. Corresponding systems of partial differential equations are derived similar to how this is done by Bielecki et al. (2005). A localized version of the Radial Basis Function partition of unity method is used to solve these four-dimensional equations. The results of our numerical experiments qualitatively explain the discrepancies observed in the marked values of CDS spreads traded in domestic and foreign economies.

Keywords: Quanto credit default swaps; reduced form models; jump-at-default; stochastic interest rates; radial basis function method.

1. Introduction

A quanto CDS is a credit default swap (CDS) with the special feature that the swap premium payments, and/or the cashflows in the case of default, are paid in a different currency to that of the reference asset. A typical example would be a CDS that has its reference as a dollar-denominated bond for which the premium of the swap is payable in euros. And in case of default the payment equals the recovery rate on the dollar bond payable in euros. In other words, this CDS is written on a dollar bond, while its premium is payable in euros. These contracts are widely used to hedge holdings in bonds or bank loans that are denominated in a foreign currency (other than the investor home currency).

As mentioned in Hampden-Turner and Goves (2010), this product enables investors to take views on joint spread and foreign exchange (FX) rate moves with value a function of

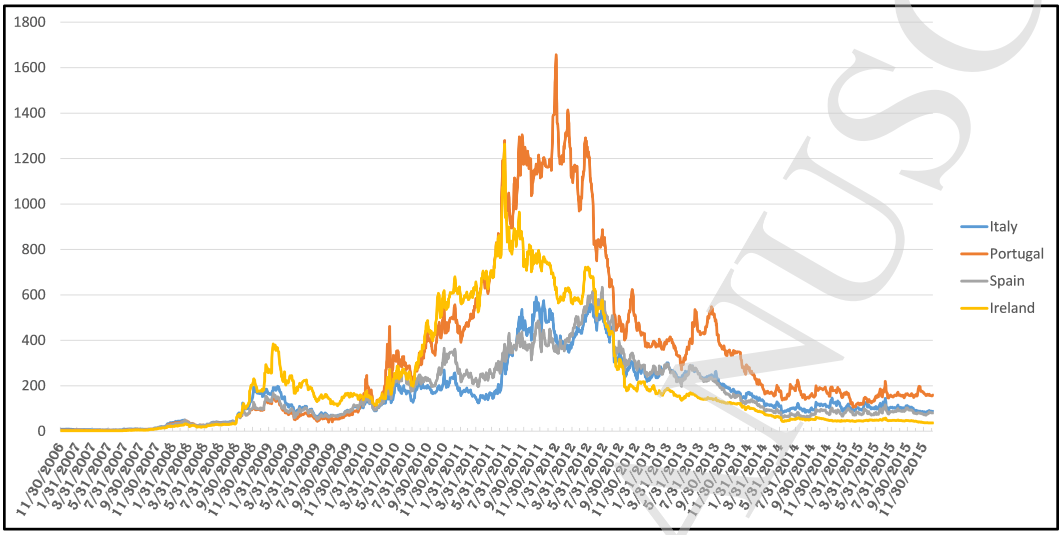


Figure 1: Historical time-series of some European five-year sovereign CDSs traded in USD (Markit).

spread, the FX rate and FX volatility. Given the increased correlation between FX moves and credit spreads, interest in this product grew significantly with years. For instance, Augustin (2014) reports that in 2012, single-name sovereign CDS accounted for approximately 11% of the overall market, which was then valued at 27 trillion USD in gross notional amounts outstanding. To compare, the corporate CDS market accounted for about 89% of the market, with single-name and multi-name contracts amounting to 16 trillion and 11 trillion USD, respectively. Based on Augustin et al. (2017), while the CDS market has somewhat shrunk in recent years, statistics from the Bank for International Settlements suggest that sovereign CDS represented with 1.715 trillion USD about 18% of the entire market in 2016.

A quanto CDS is quoted as a spread between the standard CDS and that of a different currency, and they are available for different maturities. For instance, one can observe that CDS on European sovereigns are usually traded in US dollars. That is because in case of default a euro-denominated credit protection would significantly drop down reflecting the default of the corresponding economy. So, the term structure of quanto CDS tells us how financial markets view the likelihood of a foreign default and associated currency devaluations at different horizons, see e.g., discussion in Augustin et al. (2017) and references therein.

In Fig. 1 historical time series of some European five-year sovereign CDSs traded in USD are presented for the period from 2006 to 2015. It can be seen that these spreads reach their maximum around 2011, and then drop down by factors 2-5 to their current level. However, since high levels of the spreads have been recorded, later in this paper when choosing test parameters of our numerical experiments we will look at the cases corresponding just to the period of raised spreads around 2011.

As far as the value of the quanto CDS spread is concerned, there are various data in the

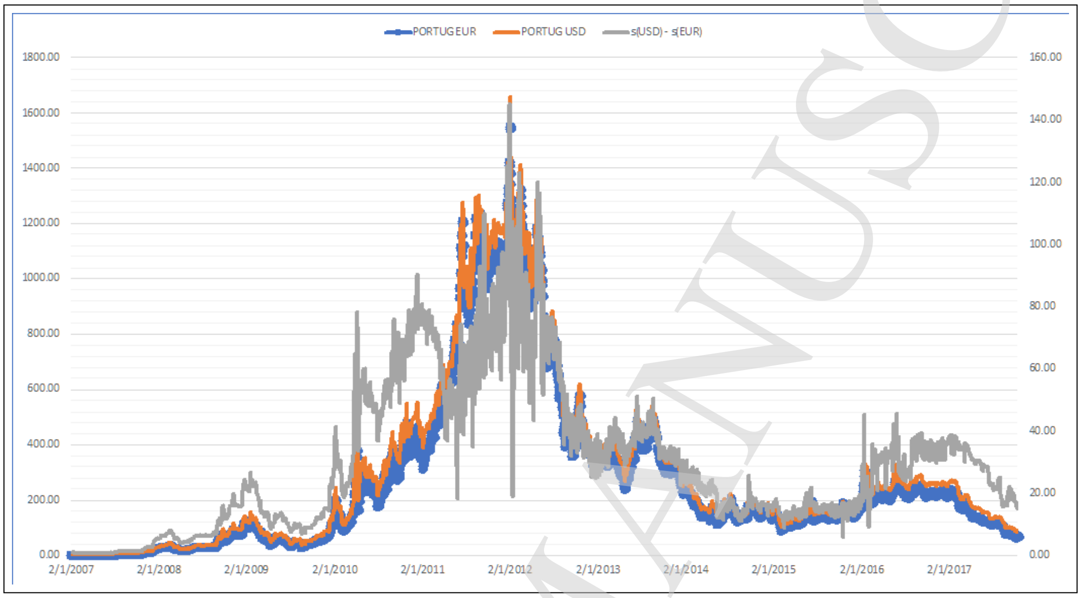


Figure 2: Historical time-series of five-year Portugal sovereign CDSs spread traded in USD and EUR and their basis spread (the difference) (Markit).

literature. For instance, in Augustin et al. (2017) the term structure of spreads, defined as the difference between the USD and EUR denominated CDS spreads, is presented for six Eurozone countries: Germany , Belgium, France, Ireland, Italy, Portugal and for maturities three, five, seven, ten and fifteen years relative to the one-year quanto spread. This difference could reach thirty bps at the time horizon fifteen years (France, Ireland). In Simon (2015) the five-year quanto CDS spreads are presented for Germany, Italy and France over the period from 2004 to 2013, which, e.g., for Italy could reach 500 bps in 2012. The results presented in Brigo et al. (2015) indicate a significant basis across domestic and foreign CDS quotes. For instance, for Italy, a USD CDS spread quote of 440 bps can translate into a EUR quote of 350 bps in the middle of the Euro-debt crisis in the first week of May 2012. More recently, from June 2013, the basis spreads between the EUR quotes and the USD quotes are in the range of around 40 bps.

Fig 2 shows historical time series of five-year sovereign Portugal CDSs traded in USD and EUR for the period from 2006 to 2017 as well as the basis spread which is a difference of those two. Again, around 2012, the basis spread reached about 160 bps while its current level is about 15 bps. Therefore, yet quanto CDS effect plays an important role, and, hence, requires a suitable theory to explain the observed behavior.

Indeed, quanto effects drew a lot of attention on a modelling side. Various aspects of the problem were under investigation including the relationship between sovereign credit and currency risks, the pricing of sovereign CDS, the impact of contagion on credit risk, see survey in Augustin et al. (2017) and references therein. But in this paper our particular attention will be directed to pricing quanto CDS, or, more rigorously, to determining and testing an appropriate framework that provides a reasonable explanation of these effects from a mathematical finance point of view. Our approach is close to that in Brigo et al. (2015) where a model of quanto CDS is built based on the reduced form model for credit risk. Within this setting the default time is modeled as a Cox process with explicit diffusion dynamics for default intensity/hazard rate and exponential jump to default, similar to the approach of Ehlers and Schönbucher (2006), El-Mohammadi (2009). But what is more important, Brigo et al. (2015) introduce an explicit jump-at-default in the FX dynamics. Then they show that this provides a more effective way to model credit/FX dependency as the results of simulation are able to explain the observed basis spreads during the Euro-debt crisis. In contrast, taking into account the instantaneous correlation between the driving Brownian motions of the default intensity and the FX rate alone is not sufficient for doing so.

However, in Brigo et al. (2015) only deterministic domestic and foreign interest rates (IR) were considered. On the one hand, it could be important to extend this approach by relaxing this assumption and letting the rates be stochastic. But it would be more important to account not just for the jump-at-default in the FX rate, but also for a simultaneous jump-at-default in the interest rate of the defaulted country. Relevant data on the subject could be found, e.g., in Catao and Mano (2015). This investigation shows that the interest rate premium on past default has been underestimated. This is partly due to narrower credit history indicators and, crucially, to the narrower data coverage of previous studies. Once this correction is made for these problems, a sizeable and persistent default premium emerges, and one which rises on the duration of the default. This means that the longer a country stays in default the higher premium it will pay once it resumes borrowing from private capital markets.

Another example is given in Katselas (2010). He provides a plot of the overnight inter-bank cash rate as quoted by the Reserve Bank of Australia for the period starting on 4 January 2000 and finishing on 31 December 2009. This rate serves as an approximation to the risk-free short rate applicable to borrowing/lending in Australia, and the plot indicates that not only are jumps evident in the short rate, but that a pure jump process may act as a suitable model for short rates. This observation prompted, e.g., Borovkov et al. (2003), to consider using a marked Poisson point process to model the short rate as a pure jump process.

Therefore, in this paper we extend the framework of Brigo et al. (2015) by introducing stochastic interest rates and account for jump-at-default in both FX and foreign (defaulted) interest rates. Our goal is to compare contribution of both jumps into the value of quanto CDS spread. It might seem, at the first glance, that if the model of Brigo et al. (2015) has been already able to qualitatively explain the observed quanto CDS by the order of magnitude, introducing an additional jump increases a risk of over fitting. However, our model enables the desks to introduce a richer behavior when they feel the market exhibits a jump at default in the foreign interest rate. For some cases it is s an expected behavior. Also, as follows from our results presented below in this paper, two jumps at default have opposite effects: the jump in the FX rate decreases the value of the foreign CDS, while the jump in the IR increases the value of the foreign CDS. Hence, taking the latter into account should improve the fitting (at least, potentially), and not vice versa.

As this problem has four stochastic drivers, plus time, we show that the corresponding CDS price solves a four-dimensional partial differential equation. It is well-known that this dimensionality is such that finite-difference method already immensely suffer from the curse of dimensionality, while using Monte Carlo methods is too computationally expensive. Therefore, here we used another method, namely, a radial basis function (RBF) method, which has already demonstrated its efficiency when solving various problems of intermediate ($10 > d > 3$) dimensionality including those in mathematical finance, see, e.g., Hon and Mao (1999), Fasshauer et al. (2004), Pettersson et al. (2008), thanks to its high order convergence. The latter allows for obtaining a high resolution scheme using just a few discretization nodes. In particular, in this paper a localized version of the RBF method is used. It is based on the partition of unity method (or RBF-PUM). The partition of unity was originally introduced by Babuška and Melenk (1997) for finite element methods, and later adapted for the RBF methods by several authors, Safdari-Vaighani et al. (2015), Shcherbakov and Larsson (2016). This approach enables a significant reduction in the number of non-zero elements that remain in the coefficient matrix, hence, lowering the computational intensity required for solving the system.

The rest of the paper is organized as follows. In Section 2 we describe our model, and derive the main partial differential equation (PDE) for the risky bond price under this model. In Section 2.2 we extend this framework by adding jumps-at-default into the dynamics of the FX and foreign (defaulted) interest rates. Again, the main PDE is derived for the risky bond (the detailed derivation is given in Appendix). The connection of this price with the prices of the quanto CDS is established in Section 4. In Section 5 the RBF-PUM method is described in detail. In Section 6 we present numerical results of our experiments with this model and discussion of the observed effects. Finally, Section 7 concludes the paper.

2. Model

We begin describing our model by giving some useful definitions which are heavily utilized throughout the rest of the paper.

By the *domestic currency* or the *liquid currency* we denote the most liquidly traded currency among all contractual currencies. In what follows this is the US dollar (USD) [a].

The other contractual currency we denote as *contractual* or *foreign currency*. In this paper it can be both USD and EUR. The premium and protection leg payments are settled in this currency.

Since in this paper we focus on pricing credit default swap (CDS) contracts, it is assumed that their market quotes are available in both domestic and foreign currencies. Let us denote these prices as CDS_d and CDS_f respectively. If so, every price CDS_f expressed in the foreign currency can be translated into the corresponding price in the domestic currency if the exchange rate Z_t for two currencies is provided by the market. In other words, the

[a] Perhaps, this could be a bit confusing as in our framework there is no distinction between domestic and liquid currencies. However, the latter could be a helpful notion as there is no domestic liquid CDS on the market.

theoretical price of the CDS contract in the foreign currency would be $Z_t\mathrm{CDS}_d$. However, it is known that the market demonstrates a spread $\mathrm{CDS}_f - Z_t\mathrm{CDS}_d$ which could reach hundreds of bps, Brigo et al. (2015). Hence, the availability of the market quotes on CDS contracts in both currencies together with the corresponding exchange rates allows one to capture these spreads.

It is worth mentioning that market quotes of the CDS contracts are usually expressed in terms of par spreads, and not as prices. Presumably, a quanto CDS spread would be the difference between two par spreads and not between two prices. However, any credit model is calibrated to the CDS prices. And CDS par rates indicate that swaps with the given premiums are priced to zero. Hence, when we calibrate the model prices of those CDS, the parameters are adjusted such that the CDS prices vanish. Also in the latest convention (post 2008) the CDS curves are quoted in terms of upfront prices (where the CDS is kept constant at 6%, and the the price is quoted). Anyway, regardless of that, once we are able to replicate the CDS price, we then can calibrate to any convention.

We continue our description by considering a framework where all underlying stochastic processes do not experience a jump-at-default except the default process itself. So, this is similar to what is presented in Brigo et al. (2015) with an exception that the interest rates in our model are stochastic. This will then be generalized with the allowance for jumps-at-default in other processes in Section 2.2.

2.1. *Simple jump-at-default framework*

Below we chose the risk neutral probability measure $\mathbb{Q}$ corresponding to the domestic (liquid) currency money market. Also, by $\mathbb{E}_t[\,\cdot\,]$ we denote the expectation conditioned on the information received by time t, i.e. $\mathbb{E}[\,\cdot\,|\mathcal{F}_t]$.

Consider two money markets: B_t associated with the domestic currency (USD), and $\hat{B}_t$ associated with the foreign currency (EUR), where $t \geq 0$ is the calendar time. We assume that the dynamics of the two money market accounts are given by

$$\begin{aligned} dB_t &= R_t B_t dt, \quad B_0 = 1, \\ d\hat{B}_t &= \hat{R}_t \hat{B}_t dt, \quad \hat{B}_0 = 1, \end{aligned} \tag{2.1}$$

where the stochastic interest rates $R_t, \hat{R}_t$ follow the Cox-Ingersoll-Ross (CIR) process, Cox et al. (1985)

$$\begin{aligned} dR_t &= a(b - R_t)dt + \sigma_r \sqrt{R_t} dW_t^{(1)}, \quad R_0 = r, \\ d\hat{R}_t &= \hat{a}(\hat{b} - \hat{R}_t)dt + \sigma_{\hat{r}} \sqrt{\hat{R}_t} dW_t^{(2)}, \quad \hat{R}_0 = \hat{r}. \end{aligned} \tag{2.2}$$

Here $a, \hat{a}$ are the mean-reversion rates, $b, \hat{b}$ are the mean-reversion levels, $\sigma_r, \sigma_{\hat{r}}$ are the volatilities, and $W_t^{(1)}, W_t^{(2)}$ are the Brownian motions. Without loss of generality, further we assume $a, \hat{a}, b, \hat{b}, \sigma_r, \sigma_{\hat{r}}$ to be constant. This assumption can be easily relaxed.

We assume that the exchange rate Z_t of two currencies is also stochastic, and its dynamics is driven by the following stochastic differential equation (SDE)

$$dZ_t = \mu_z Z_t dt + \sigma_z Z_t dW_t^{(3)}, \quad Z_0 = z, \tag{2.3}$$

where μ_z, σ_z are the corresponding drift and volatility, and $W_t^{(3)}$ is another Brownian motion. From the financial point of view Z_t denotes the amount of domestic currency one has to pay to buy one unit of foreign currency. Loosely speaking, this means that 1 euro could be exchanged for Z_t US dollars.

As the underlying security of a CDS contract is a risky bond, we need a model of a credit risk implied by the bond. For modeling the credit risk we use a reduced form model approach, see e.g., Jarrow and Turnbull (1995), Duffie and Singleton (1999), Bielecki and Rutkowski (2004), Jarrow et al. (2003) and references therein. We define the hazard rate λ_t to be a stochastic process given by

$$\lambda_t = e^{Y_t}, \quad t \geq 0, \tag{2.4}$$

where Y_t follows the Ornstein-Uhlenbeck process defined by the SDE

$$dY_t = \kappa(\theta - Y_t)dt + \sigma_y dW_t^{(4)}, \quad Y_0 = y, \tag{2.5}$$

with κ, θ, σ_y to be the corresponding mean-reversion rate, mean-reversion level and volatility, and $W_t^{(4)}$ to be another Brownian motion. Both Z_t and λ_t are defined and calibrated in the domestic measure.

Actually, for practical purposes the exponential model is better, than say, the CIR model, as lognormal volatilities are dimensionless and more intuitive to the traders. Perhaps, the main reason why the CIR model is also widely used in practice is because then the solution is easy to represent in the form of Laplace/Fourier transform. Since we do not have this requirement, this is not an advantage for our approach. We underline that there is no mandate to use the CIR model for credit. As long as the hazard rate stays positive, we are fine

We assume all Brownian motions $W_t^{(i)},\ i \in [1,4]$ to be dependent, and this dependence can be specified through the instantaneous correlation ρ between each pair of the Brownian motions, i.e., $< dW_t^{(i)}, dW_t^{(j)} >= \rho_{ij} dt$. Hence, the whole correlation matrix in our model is

$$\mathcal{P} = \begin{bmatrix} 1 & \rho_{r\hat{r}} & \rho_{rz} & \rho_{ry} \\ \rho_{\hat{r}r} & 1 & \rho_{\hat{r}z} & \rho_{\hat{r}y} \\ \rho_{zr} & \rho_{z\hat{r}} & 1 & \rho_{zy} \\ \rho_{yr} & \rho_{y\hat{r}} & \rho_{yz} & 1 \end{bmatrix}, \tag{2.6}$$

where all correlations $|\rho_{ij}| \leq 1,\ i,j \in [r, \hat{r}, z, y]$ are assumed to be constant.

Finally, we define the default process $(D_t,\ t \geq 0)$ as

$$D_t = \mathbf{1}_{\tau \leq t}, \tag{2.7}$$

where τ is the default time of the reference entity. In order to exclude trivial cases, we assume that $\mathbb{Q}(\tau > 0) = 1$, and $\mathbb{Q}(\tau \leq T) > 0$.

2.2. *Jumps-at-default in FX and foreign IR*

In this section we extend the above described framework by assuming the value of the foreign currency as well as the foreign interest rate to experience a jump at the default time.

As shown in Brigo et al. (2015) and mentioned in the introduction, including jump-at-default into the FX rate provides a more effective way of modeling the credit/FX dependency than the instantaneous correlations imposed among the driving Brownian motions of default intensity and FX rates. Moreover, the authors claim that it is not possible to explain the observed basis spreads during the Euro-debt crisis by using the latter mechanism alone.

However, looking at historical time-series, an existence of jump-at-default in the foreign interest rate could also be justified, especially in case when sovereign obligations are in question. For example, after the default of Russia in 1998, the Russian ruble lost about 75% of its value within 1.5 months, which in turn resulted in a jump of the corresponding FX rates. On the other hand, the jump in the interest rate can be even more pronounced since the default also lowers the creditability and dramatically increases the cost of borrowing. For the above mentioned example of the Russian crisis of 1998, the short interest rate grew from 20% in April 1998 to 120% in August 1998. Therefore, it would be interesting to see a relative contribution of each jump into the value of the quanto CDS spread.

It is worth noting that since we introduce a single jump in the foreign IR, the long term rates in the yield curve would be subject to the same jump. Despite this is not fully realistic, usually, as PCA analysis provided on most curves shows, a significant percentage of variance can be explained by parallel moves. However this is not the topic of this research. Certainly, someone could further elaborate on a more realistic behavior of the yield curve.

To add jumps to the dynamics of the FX rate in Eq.(2.3), we follow Brigo et al. (2015), Bielecki et al. (2005) who assume that at the time of default the FX rate experiences a single jump which is proportional to the current rate level, i.e.

$$dZ_t = \gamma_z Z_{t^-} dM_t, \tag{2.8}$$

where $\gamma_z \in [-1, \infty)$ [b] is a devaluation/revaluation parameter.

The hazard process Γ_t of a random time τ with respect to a reference filtration is defined through the equality $e^{-\Gamma_t} = 1 - \mathbb{Q}\{\tau \le t | \mathcal{F}_t\}$. It is well known that if the hazard process Γ_t of τ is absolutely continuous, so

$$\Gamma_t = \int_0^t (1 - D_s)\lambda_s ds, \tag{2.9}$$

and increasing, then the process $M_t = D_t - \Gamma_t$ is a martingale (which is called as the compensated martingale of the default process D_t) under the full filtration $\mathcal{F}_t \vee \mathcal{H}_t$ with $\mathcal{H}_t$ being the filtration generated by the default process. So, M_t is a martingale under $\mathbb{Q}$, Bielecki et al. (2005).

It can be shown that under the risk-neutral measure associated with the domestic currency, the drift μ_z is, (Brigo et al. (2015))

$$\mu_z = R_t - \hat{R}_t. \tag{2.10}$$

Therefore, with the allowance for Eq.(2.3), Eq.(2.8) we obtain

$$dZ_t = (R_t - \hat{R}_t) Z_t dt + \sigma_z Z_t dW_t^{(3)} + \gamma_z Z_t dM_t. \tag{2.11}$$

[b]This is to prevent Z_t to be negative, Bielecki et al. (2005).

Thus, Z_t is a martingale under the $\mathbb{Q}$-measure with respect to $\mathcal{F}_t \vee \mathcal{H}_t$ as it should be, since it is a tradable asset.

Certainly, we are more interested in the negative values of γ_z because a default of the reference entity has to negatively impact the value of its local currency. For instance, we expect the value of EUR expressed in USD to fall if some European country defaults.

Similarly, we add jump-at-default to the stochastic process for the foreign interest rate $\hat{R}_t$ as

$$d\hat{R}_t = \gamma_{\hat{r}} \hat{R}_{t^-} dD_t,$$

so Eq.(2.2) transforms to

$$d\hat{R}_t = \hat{a}(\hat{b} - \hat{R}_t)dt + \sigma_{\hat{r}} \sqrt{\hat{R}_t} dW_t^{(2)} + \gamma_{\hat{r}} \hat{R}_t dD_t. \tag{2.12}$$

Here $\gamma_{\hat{r}} \in [-1, \infty)$ is the parameter that determines the post-default cost of borrowing. We are interested in positive values of $\gamma_{\hat{r}}$ as the interest rate most likely will grow after a default has occurred. Note that $\hat{R}_t$ is not tradable, and so is not a martingale under the $\mathbb{Q}$-measure.

3. Pricing zero-coupon bonds

To price contingent claims where the contractual currency differs from the pricing currency, e.g., quanto CDS, we first need to determine the price of the underlying defaultable zero-coupon bond settled in foreign currency. The bond price under the foreign money market martingale measure $\hat{\mathbb{Q}}$ reads

$$\hat{U}_t(T) = \hat{\mathbb{E}}_t \left[\frac{\hat{B}_t}{\hat{B}_T} \hat{\Phi}(T) \right], \tag{3.1}$$

where $\hat{B}_t/\hat{B}_T = \hat{B}(t,T)$ is the stochastic discount factor from time T to time t in the foreign economy, and $\Phi(T)$ is the payoff function. However, we are going to find this price under the domestic money market measure $\mathbb{Q}$. Hence, converting the payoff to the domestic currency and discounting by the domestic money market account yields

$$U_t(T) = \mathbb{E}_t \left[B(t,T) Z_t \hat{\Phi}(T) \right], \tag{3.2}$$

where without loss of generality it is assumed that the notional amount of the contract is equal to one unit of the foreign currency. This implies the payoff function to be

$$\hat{\Phi}(T) = \mathbb{1}_{\tau > T}. \tag{3.3}$$

Further, we assume that if this bond defaults, the recovery rate $\mathcal{R}$ is paid at the time of default. Therefore, the price of a defaultable zero-coupon bond, which pays out one unit of the foreign currency in the domestic economy reads

$$\begin{aligned} U_t(T) &= \mathbb{E}_t \left[B(t,T) Z_T \mathbb{1}_{\tau > T} + \mathcal{R} B(t,\tau) Z_\tau \mathbb{1}_{\tau \le T} \right] \\ &= \mathbb{E}_t \left[B(t,T) Z_T \mathbb{1}_{\tau > T} \right] + \mathcal{R} \int_t^T \mathbb{E}_t \left[B(t,\nu) Z_\nu \mathbb{1}_{\tau \in (\nu - d\nu, nu]} \right] = w_t(T) + \mathcal{R} \int_t^T g_t(\nu) d\nu, \\ w_t(T) &:= \mathbb{E}_t \left[Z_T B(t,T) \mathbb{1}_{\tau > T} \right], \qquad g_t(\nu) := \mathbb{E}_t \left[B(t,\nu) Z_\nu \frac{\mathbb{1}_{\tau \in (\nu - d\nu, nu]}}{d\nu} \right]. \end{aligned} \tag{3.4}$$

As the whole dynamics of our underlying processes is Markovian, Bielecki et al. (2005), to find the price of such an instrument we use a PDE approach, so that the defaultable bond price just solves it. This is more efficient from the computationally point of view as compared, e.g., with the Monte Carlo method, despite the resulting PDE becomes four-dimensional. Indeed, in order to properly obtain the net present value and Greeks by the Monte Carlo method hundreds of thousands of paths is required (or special variance reduction tricks). The presence of jumps makes the Monte Carlo approach even less feasible, as the jump probability is relatively low, and thus even more paths is required. Also to calibrate the model (this is a subject that we don't discuss in this paper) we need to perform calibration of drifts, which could be done as usual by using the forward PDE approach, but is very cumbersome when using Monte Carlo.

Further, conditioning on $R_t = r, \hat{R}_t = \hat{r}, Z_t = z, Y_t = y, D_t = d$, and using the approach of Bielecki et al. (2005) (see Appendix A), we obtain that under the risk-neutral measure $\mathbb{Q}$ the price $U_t(T)$ is

$$U_t(T, r, \hat{r}, y, z) = \mathbb{1}_{\tau>t} f(t, T, r, \hat{r}, y, z, 0) + \mathbb{1}_{\tau\le t} f(t, T, r, \hat{r}, y, z, 1). \tag{3.5}$$

Here the function $f(t, T, r, \hat{r}, y, z, 1) \equiv u(t, T, X),\ X = \{r, \hat{r}, y, z\}$ solves the PDE

$$\frac{\partial u(t, T, X)}{\partial t} + \mathcal{L} u(t, T, X) - r u(t, T, X) = 0, \tag{3.6}$$

where the diffusion operator $\mathcal{L}$ reads

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\sigma_r^2 r \frac{\partial^2}{\partial r^2} + \frac{1}{2}\sigma_{\hat{r}}^2 \hat{r} \frac{\partial^2 u}{\partial \hat{r}^2} + \frac{1}{2}\sigma_z^2 z^2 \frac{\partial^2}{\partial z^2} + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} + \rho_{r\hat{r}}\sigma_r\sigma_{\hat{r}}\sqrt{r\hat{r}} \frac{\partial^2}{\partial r \partial \hat{r}} \\
&+ \rho_{rz}\sigma_r\sigma_z z\sqrt{r} \frac{\partial^2}{\partial r \partial z} + \rho_{\hat{r}z}\sigma_{\hat{r}}\sigma_z z\sqrt{\hat{r}} \frac{\partial^2}{\partial z \partial \hat{r}} + \rho_{ry}\sigma_r\sigma_y\sqrt{r} \frac{\partial^2}{\partial r \partial y} + \rho_{\hat{r}y}\sigma_{\hat{r}}\sigma_y\sqrt{\hat{r}} \frac{\partial^2}{\partial y \partial \hat{r}} \\
&+ \rho_{yz}\sigma_y\sigma_z z \frac{\partial^2}{\partial y \partial z} + a(b - r)\frac{\partial}{\partial r} + \hat{a}(\hat{b} - \hat{r})\frac{\partial}{\partial \hat{r}} + (r - \hat{r}) z \frac{\partial}{\partial z} + \kappa(\theta - y)\partial y.
\end{aligned} \tag{3.7}$$

The second function $f(t, T, r, \hat{r}, y, z, 0) \equiv v(t, T, X)$ solves the PDE

$$\begin{aligned}
&\frac{\partial v(t, T, X)}{\partial t} + \mathcal{L} v(t, T, X) - r v(t, T, X) - \lambda\gamma_z z \frac{\partial v(t, T, X)}{\partial z} \\
&\qquad + \lambda\left[u(t, T, X^+) - v(t, T, X)\right] = 0, \qquad X^+ = \{r, \hat{r}(1 + \gamma_{\hat{r}}), y, z(1 + \gamma_z)\}.
\end{aligned} \tag{3.8}$$

where according to Eq.(2.4), $\lambda = e^y$.

The boundary conditions for this problem should be set at the boundaries of the unbounded domain $(r, \hat{r}, y, z) \in [0, \infty] \times [0, \infty] \times [-\infty, 0] \times [0, \infty]$. However, this can be done in many different ways. As the value of the bond price is usually not known at the boundary, similarly to Brigo et al. (2015) we assume the second derivatives to vanish towards the boundaries

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \nu^2}\Big|_{\nu\uparrow 0} = \frac{\partial^2 u}{\partial \nu^2}\Big|_{\nu\uparrow\infty} = 0, \quad \nu \in [r, \hat{r}], \\
&\frac{\partial^2 u}{\partial y^2}\Big|_{y\uparrow 0} = \frac{\partial^2 u}{\partial y^2}\Big|_{y\uparrow -\infty} = 0, \qquad \frac{\partial^2 u}{\partial z^2}\Big|_{z\uparrow 0} = \frac{\partial^2 u}{\partial z^2}\Big|_{y\uparrow\infty} = 0.
\end{aligned} \tag{3.9}$$

We assume that the default has not yet occured at the validation time t, therefore, Eq.(3.5) reduces to

$$U_t(T, r, \hat{r}, y, z) = v(t, T, X). \tag{3.10}$$

Therefore, it could be found by solving Eq.(3.6), Eq.(3.8) as follows. Since the payoff in Eq.(3.4) is a sum of two terms, and our PDE is linear, it can be solved independently for each term. Then the solution is just a sum of the two.

3.1. *Solving the PDE for $w_t(T)$*

The function $w_t(T)$ solves exactly the same set of PDEs as in Eq.(3.6), Eq.(3.8) [c]. Therefore, it can be found in two steps.

Step 1 We begin by solving the PDE in Eq.(3.6) for function u. Since this function corresponds to $d = 1$, it describes the evolution of the bond price *at or after* default. Accordingly, the terminal condition for u becomes $u(T, T, X) = 0$. Indeed, this payoff does not assume any recovery paid at default, therefore, the bond expires worthless. Then, a simple analysis shows that the function $u(t, T, X) \equiv 0$ is the solution at $d = 1$ as it solves the equation itself and obeys the terminal and boundary conditions. Therefore, at this step the solution can be found analytically.

Step 2 As the solution of the first step vanishes, it implies that $u(t, T, X^+) \equiv 0$ in Eq.(3.8).

By the definition before Eq.(3.8), the function v corresponds to the states with no default. Accordingly, from Eq.(3.4) the payoff function (which is the terminal condition for Eq.(3.8) at $t = T$) reads

$$v(T, T, X) = z. \tag{3.11}$$

The boundary conditions again are set as in Eq.(3.9).

The PDE Eq.(3.8) for $v(t, T, X)$ now takes the form

$$\frac{\partial v(t, T, X)}{\partial t} + \mathcal{L}v(t, T, X) - (r + \lambda)v(t, T, X) - \lambda\gamma_z z \frac{\partial v(t, T, X)}{\partial z} = 0,$$

subject to the terminal condition $v(T, T, X) = z$. Then, obviously $w_t(T) = v(t, T, X)$.

It can be seen, that in case of no recovery, the defaultable bond price does depend on jump in the FX rate, but does not depend on the jump in the foreign interest rate.

3.2. *Solving the PDE for $g_t(\nu)$*

As far as the second part of the payoff in Eq.(3.4) is concerned, it could be noticed that the integral in Eq.(3.4) is a Riemann–Stieltjes integral in ν. Therefore, it can be approximated

[c] The PDEs remain unchanged since the model is same, and only the continent claim $G(t, T, r, \hat{r}, y, z, d)$, which is a function of the same underlying processes, changes.

by a Riemann–Stieltjes sum where the continuous time interval $[t,T]$ could be replaced by a discrete uniform grid with sufficiently small step $\Delta\nu = h$. So

$$\int_t^T g_t(\nu)d\nu \approx h\sum_{i=1}^N g_t(t_i), \tag{3.12}$$

where $t_i = t + ih,\ i \in [0,N],\ N = (T-t)/h$. Accordingly, each term in this sum can be computed independently by solving the corresponding pricing problem in Eq.(3.6), Eq.(3.8) with the maturity t_i.

Note, that since the pricing problem in Eq.(3.6), Eq.(3.8) is formulated via backward PDEs, computation of $g_t(t_i)$ for every maturity $t_i,\ i \in [1,m]$ requires an independent solution of such a problem. This could be significantly improved if instead of the backward PDE we would work with the forward one for the corresponding density function. In that case all $U_t(t_i),\ i \in [1,m]$ can be computed in one run (by a marching method). However, we leave this improvement to discuss in detail elsewhere. Do not confuse m and N since m is the total number of coupon payments, while N is the number of discretisation steps in the integral Eq.(3.12).

Again, it can be observed that the function $g_t(T)$ solves exactly the same set of PDEs as in Eq.(3.6), Eq.(3.8), and, thus, again it can be found in two steps.

Step 1 The problem for u should be solved subject to the terminal condition

$$g_T(T) = ze^y(1+\gamma_z). \tag{3.13}$$

Indeed, by the definition of $g_t(T)$, we can set $t = T$ and condition on $R_t = r, \hat{R}_t = \hat{r}, Z_t = z, Y_t = y, d = 1$. Then

$$g_T(T)dT = \mathbb{E}_t\left[B(t,T)Z_T\mathbb{1}_{\tau\in(T-dT,T]}\Big|t=T\right] = z\mathbb{E}_t\left[\lambda_t dt|t=T\right] = ze^y dT, \tag{3.14}$$

see Schonbucher (2003), Section 3.2. However, the dynamics of Z_t in Eq.(2.11) implies that when the default occurs, the value of Z_{τ^-} jumps proportionally to the value $Z_\tau = Z_{\tau^-}(1+\gamma_z)$. Thus, we arrive at Eq.(3.13).

Step 2 Having an explicit representation of the function $u(t,T,X)$ obtained as the solution of the previous step, one can find $u(t,T,X^+)$ as the values of parameters $\gamma_z, \gamma_{\hat{r}}$ are known, and the values of λ are also given (for instance, at some grid which is used to numerically solve the PDE problem in Step 1). Then, Eq.(3.8) can be solved with respect to $v(t,T,X)$.

By the definition before Eq.(3.8), the function v corresponds to states with no defaults. Accordingly, the recovery is not paid, and the terminal condition for this step is $v(T,T,X) = 0$. This, however, does not mean that $v=0$ solves the problem. That is because Eq.(3.8) contains the term $\lambda u(t,T,X^+) \neq 0$ (since the terminal condition at the previous step is not zero), and so $v \neq 0$ if $\lambda \neq 0$.

It can be seen that according to this structure in case of non-zero recovery the defaultable bond price does depend on jumps in both FX and foreign IR rates.

4. From bond prices to CDS prices

As this paper is mostly dedicated to modeling quanto CDS contracts, we use the setting developed in the previous sections for risky bonds and apply it to CDS contracts. Let us remind that a CDS is a contract in which the protection buyer agrees to pay a periodic coupon to a protection seller in exchange for a potential cashflow in the event of default of the CDS reference name before the maturity of the contract T.

We assume that a CDS contract is settled at time t and assures protection to the CDS buyer until time T. We consider CDS coupons to be paid periodically with the payment time interval Δt, and there will be totally m payments over the life of the contract, i.e., $m\Delta t = T - t$. Assuming unit notional, this implies the following expression for the CDS coupon leg L_c, Lipton and Savescu (2014), Brigo and Morini (2005)

$$L_c = \mathbb{E}_t \left[\sum_{i=1}^{m} cB(t,t_i)\Delta t \mathbb{1}_{\tau > t_i} \right], \tag{4.1}$$

where c is the CDS coupon, t_i is the payment date of the i-th coupon, and $B(t,t_i) = B_t/B_{t_i}$ is the stochastic discount factor.

However, if the default occurs in between of the predefined coupon payment dates, there must be an accrued amount from the nearest past payment date till the time of the default event τ. The expected discounted accrued amount L_a reads

$$L_a = \mathbb{E}_t \left[cB(t,\tau)(\tau - t_{\beta(\tau)}) \mathbb{1}_{t < t_{\beta}(\tau) \le \tau < T} \right], \tag{4.2}$$

where $t_{\beta(\tau)}$ is the payment date preceding the default event. In other words, $\beta(\tau)$ is a piecewise constant function of the form

$$\beta(\tau) = i, \quad \forall \tau: \ t_i < \tau < t_{i+1}.$$

These cashflows are paid by the contract buyer and received by the contract issuer. The opposite expected protection cashflow L_p is

$$L_p = \mathbb{E}_t \left[(1 - \mathcal{R}) B(t,\tau) \mathbb{1}_{t < \tau \le T} \right], \tag{4.3}$$

where the recovery rate $\mathcal{R}$ is unknown beforehand, and is determined at or right after the default, e.g., in court. In modern mathematical finance theory it is customary to consider the recovery rate to be stochastic, see e.g., Cohen and Costanzino (2017)) and references therein, however, throughout this paper we assume the recovery rate being constant and known in advance.

Further, we define the so-called *premium* $\mathcal{L}_{pm} = L_c + L_a$ and *protection* $\mathcal{L}_{pr} = L_p$ legs, and, as usual, define the CDS par spread s as the coupon which equalizes these two legs and makes the CDS contract fair at time t. Similar to Section 3, if we price all instruments under the domestic money market measure $\mathbb{Q}$ we need to convert the payoffs to the domestic currency and discount by the domestic money market account. Then s solves the equation

$$\begin{aligned} \sum_{i=1}^{m} \mathbb{E}_t \left[sZ_T B(t,t_i)\Delta t \mathbb{1}_{\tau > t_i} \right] + \mathbb{E}_t \left[sZ_T B(t,\tau)(\tau - t_{\beta(\tau)}) \mathbb{1}_{t < \tau < T} \right] \\ = \mathbb{E}_t \left[(1 - \mathcal{R}) Z_\tau B(t,\tau) \mathbb{1}_{t < \tau \le T} \right]. \end{aligned} \tag{4.4}$$

In the spirit of Ehlers and Schönbucher (2006) and Brigo (2011), we develop a numerical procedure for finding the par spread s from the bond prices. Consider each term in Eq.(4.4) separately.

Coupons For the coupon payment one has

$$L_c = \mathbb{E}_t \left[\sum_{i=1}^{m} s Z_{t_i} B(t,t_i) \Delta t \mathbb{1}_{\tau \geq t_i} \right] = s \Delta t \sum_{i=1}^{m} \mathbb{E}_t \left[Z_{t_i} B(t,t_i) \mathbb{1}_{\tau \geq t_i} \right] = s \Delta t \sum_{i=1}^{m} w_t(t_i). \quad (4.5)$$

where $t_m = T$. Computation of $w_t(T)$ is described in Section 3.1.

Note, that as follows from the analysis of the previous section, $w_t(T)$ (and, respectively, the coupon payments) does depend on the jump in the FX rate, but does not depend on the jumps in the foreign interest rate which is financially reasonable.

Protection leg A similar approach is provided for the protection leg

$$\begin{aligned} L_p &= \mathbb{E}_t \left[(1-\mathcal{R}) Z_\tau B(t,\tau) \mathbb{1}_{t<\tau\leq T} \right] = (1-\mathcal{R}) \int_t^T \mathbb{E}_t \left[Z_\nu B(t,\nu) \mathbb{1}_{\tau \in (\nu - d\nu, \nu]} \right] d\nu \qquad (4.6) \\ &= (1-\mathcal{R}) \int_t^T g_t(\nu) d\nu, \end{aligned}$$

where computation of $g_t(T)$ is described in Section 3.2.

Accrued payments For the accrued payment one has

$$\begin{aligned} L_a &= \mathbb{E}_t \left[s Z_\tau B(t,\tau)(\tau - t_{\beta(\tau)}) \frac{\mathbb{1}_{t<\tau<T}}{d\nu} \right] = s \int_t^T \mathbb{E}_t \left[Z_\nu B(t,\nu)(\nu - t_{\beta(\nu)}) \frac{\mathbb{1}_{\tau \in (\nu - d\nu, \nu]}}{d\nu} \right] d\nu \qquad (4.7) \\ &= s \sum_{i=0}^{m-1} \left\{ \int_{t_i}^{t_{i+1}} (\nu - t_i) \mathbb{E}_t \left[Z_\nu B(t,\nu) \frac{\mathbb{1}_{\tau \in (\nu - d\nu, \nu]}}{d\nu} \right] d\nu \right\} = s \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (\nu - t_i) g_t(\nu) d\nu, \end{aligned}$$

where $t_0 \equiv t$, and $t_m \equiv T$.

As was mentioned in Section 3.2, both final integrals in Eq.(4.6), Eq.(4.7) are Riemann–Stieltjes integrals in ν. Therefore, each one can be approximated by a Riemann–Stieltjes sum where the continuous time interval $[t, T]$ could be replaced by a discrete uniform grid with a sufficiently small step $\Delta\nu = h$.

Now we have all necessary componets to compute the CDS spread. Introducing new notation

$$
\begin{aligned}
A_i &= \int_{t_i}^{t_{i+1}} w_t(\nu) d\nu \approx h \sum_{k=1}^{N} w_t(\nu_k), \\
B_i &= \int_{t_i}^{t_{i+1}} g_t(\nu) d\nu \approx h \sum_{k=1}^{N} g_t(\nu_k) \\
C_i &= \int_{t_i}^{t_{i+1}} \nu g_t(\nu) d\nu \approx h \sum_{k=1}^{N} \nu_k g_t(\nu_k) \\
\nu_k &= t_i + kh, \quad k = 1, \dots, N, \quad h = (t_{i+1} - t_i)/N,
\end{aligned}
\tag{4.8}
$$

we re-write Eq.(4.5), Eq.(4.6) and Eq.(4.7) in the form

$$
L_p = (1 - \mathcal{R}) \sum_{i=1}^{m} B_i, \qquad L_c = s \Delta t \sum_{i=1}^{m} A_i, \qquad L_a = s \sum_{i=1}^{m} \left[C_i - t_i B_i \right]. \tag{4.9}
$$

Finally, combining together Eq.(4.4) and Eq.(4.9) we obtain

$$
s = (1 - \mathcal{R}) \frac{\sum_{i=1}^{m} B_i}{\sum_{i=1}^{m} \left[\Delta t A_i + C_i - t_i B_i \right]}. \tag{4.10}
$$

5. Radial Basis Function Partition of Unity Method

In order to numerically solve Eq.(3.6), Eq.(3.8) subject to the corresponding terminal and boundary conditions we use a radial basis function method. Radial basis function methods become increasingly popular for applications in computational finance, e.g., Hon and Mao (1999), Fasshauer et al. (2004), Pettersson et al. (2008), thanks to their high order convergence that allows for obtaining a high resolution scheme using just a few discretization nodes. This is a crucial property when solving various multi-dimensional problems, e.g., pricing derivatives written on several assets (basket options), or those for models whose settings use several stochastic factors. Indeed, all these models suffer immensely from the curse of dimensionality, in particular, an increasing storage (memory) becomes the dominant limiting factor. This, however, can be successfully overcome by using the RBF methods. For instance, in Shcherbakov and Larsson (2016) it is shown that standard finite difference methods require about three times as many computational nodes per dimension as RBF methods to obtain the same accuracy, thus, significantly reducing the memory consumption.

Nevertheless, it should be emphasized, that the original global RBF method is computationally very expensive and rather unstable due to dense and ill-conditioned coefficient matrices[d]. This is a consequence of the global connections between the basis functions.

[d]More details could be found, e.g., in Fasshauer (2007).

Therefore, here we eliminate from the global RBF method in favour of its localised version based on the idea of partition of unity. The partition of unity method was originally introduced by Babuška and Melenk (1997) for finite element methods, and later adapted for the RBF methods by several authors, Safdari-Vaighani et al. (2015), Shcherbakov and Larsson (2016). This approach (which further on is referred as RBF-PUM) enables a significant reduction in the number of non-zero elements that remain in the coefficient matrix, hence, lowering the computational intensity required for solving the system. In addition, this concept is supported, say in Matlab, by making use of sparse operations. Typically, as applied to our problem of pricing quanto CDS, only about one percent of all elements remain to have non-zero values.

In order to construct an RBF-PUM approximation we start by defining an open cover $\{\Omega_j\}_{j=1}^P$ of the computational domain $\Omega \subset \mathbb{R}^d$ such that

$$\Omega \subseteq \bigcup_{j=1}^{P} \Omega_j. \tag{5.1}$$

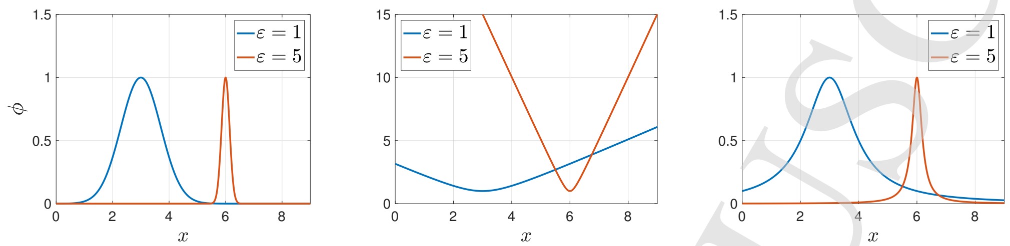
We select the patches Ω_j to be of a spherical form. Inside each patch a local RBF approximation of the solution u is defined as

$$\tilde{u}_j(x) = \sum_{i=1}^{n_j} \lambda_i^j \phi(\varepsilon, ||x - x_i^j||), \tag{5.2}$$

where n_j is the number of computational nodes belonging to the patch Ω_j, $\phi(\varepsilon, ||x - x_i^j||)$ is the i-th basis function centred at x_i^j, which is the i-th local node in the j-th patch Ω_j, ε is the shape parameter that determines the widths of basis functions, and λ_i^j are the unknown coefficients. Some popular choices of the basis functions are listed in Table 1, while their behavior as a function of the parameter ε is presented in Fig. 3.

RBF	$\phi(\varepsilon, r)$
Gaussian (GA)	$\exp\left(-\varepsilon^2 r^2\right)$
Multiquadric (MQ)	$\sqrt{1+\varepsilon^2 r^2}$
Inverse Multiquadric (IMQ)	$1/\sqrt{1+\varepsilon^2 r^2}$
Inverse Quadratic (IQ)	$1/(1+\varepsilon^2 r^2)$

Table 1: Commonly used radial basis functions.



(a) Gaussian (b) Multiquadric (c) Inverse quadratic

Figure 3: Commonly used basis functions with respect to the value of the shape parameter ε.

In addition to the patches, we also construct partition of unity weight functions $w_j(x)$, $j = 1, \ldots, P$, subordinated to the open cover, such that

$$\sum_{j=1}^{P} w_j(x) = 1, \quad \forall x \in \Omega. \tag{5.3}$$

Functions $w_j(x)$ can be obtained, e.g., by Shepard's method, Shepard (1968), from compactly supported generating functions $\varphi_j(x)$

$$w_j(x) = \frac{\varphi_j(x)}{\sum_{i=1}^{P} \varphi_i(x)}, \quad j = 1, \ldots, P, \quad \forall x \in \Omega. \tag{5.4}$$

The generation functions $\varphi_j(x)$ must fulfil some smoothness requirements. For instance, for the problem considered in this paper they should be at least $C^2(\Omega)$. To proceed, as a suitable candidate for $\varphi_j(x)$ we choose Wendland functions, Wendland (1995)

$$\varphi(r) = (5r + 1)(1 - r)_+^5, \quad r \in \mathbb{R}, \tag{5.5}$$

with the support $\varphi(r) \in \mathbb{B}^d(0, 1)$, where $\mathbb{B}^d(0, 1)$ is a unit d-dimensional ball centred at the origin. In order to map the generating function to the patch Ω_j with the centre c_j and radius ρ_j, it is shifted and scaled as

$$\varphi_j(x) = \varphi_j\left(\frac{||x - c_j||}{\rho_j}\right), \quad \forall x \in \Omega. \tag{5.6}$$

Further we blend the local RBF approximations with the partitions of unity weight and obtain a combined RBF-PUM solution $\tilde{u}(x)$ as

$$\tilde{u}(x) = \sum_{j=1}^{P} w_j(x)\tilde{u}_j(x). \tag{5.7}$$

The RBF-PUM approximation in the given form allows to maintain accuracy similar to that of the global method while significantly reducing the computational effort (see e.g., Shcherbakov and Larsson (2016), Ahlkrona and Shcherbakov (2017)). Moreover, it was shown in von Sydow et al. (2015) that RBF-PUM is the most efficient numerical method for higher-dimensional problems among deterministic methods that rely on a node discretization.

The error estimates for RBF-PUM have been studied by several authors, e.g., Safdari-Vaighani et al. (2015), Larsson et al. (2017). These studies demonstrate that the PDE approximation error is strongly dominated by the RBF interpolation error $\mathcal{E}_{\mathcal{L}}$, which dictates the overall convergence order of the numerical method. The interpolation error under action of the differential operator $\mathcal{L}$ is defined as

$$\mathcal{E}_{\mathcal{L}} = \mathcal{L}(\mathcal{I}(u) - u), \tag{5.8}$$

where $\mathcal{I}(u)$ is the RBF-PUM interpolant

$$\mathcal{I}(u) = \sum_{j=1}^{P} w_j \mathcal{I}(u_j), \tag{5.9}$$

and $\mathcal{I}(u_j)$ is the local RBF interpolant defined as in Eq.(5.2) satisfying the interpolation condition

$$\mathcal{I}(u_j)(x^j) = u(x^j), \tag{5.10}$$

where x^j are the n_j local nodes that belong to the patch Ω_j.

Based on Rieger and Zwicknagl (2010), Safdari-Vaighani et al. (2015) and Larsson et al. (2017) demonstrate that either an exponential or an algebraic convergence of the numerical solution can be achieved depending on the refinement mode, when the solution is approximated by Gaussian functions. Keeping the number of patches fixed and refining the node set (i.e., gradually increasing the number of nodes per patch) yields the following bound on the interpolation error

$$||\mathcal{E}_{\mathcal{L}}||_{L_\infty(\Omega)} \le K C^E \max_{1 \le j \le P} e^{\gamma \log(h_j)/\sqrt{h_j}} ||u||_{\mathcal{N}(\tilde{\Omega}_j)}, \tag{5.11}$$

which indicates an exponential convergence with respect to the internodal distance h_j for smooth problems. Here $\tilde{\Omega}_k = \Omega \cap \Omega_k$, the norm $||\cdot||_{\mathcal{N}(\tilde{\Omega}_k)}$ denotes the native space norm, see, e.g., Fasshauer (2007), associated with the chosen RBF, K is some constant, the constant C^E and the rate γ both depend on the problem dimension d and the order of differential operator α, and C^E additionally depends on the chosen weight function.

Keeping the number of nodes per patch fixed while refining the node set (i.e., simultaneously increasing the number of patches and the number of nodes in the node set) yields that the interpolation error is bounded as

$$||\mathcal{E}_{\mathcal{L}}||_{L_\infty(\Omega)} \le K \max_{1 \le j \le P} C_j^A H_j^{q(n_j)+1-\frac{d}{2}-|\alpha|} ||u||_{\mathcal{N}(\tilde{\Omega}_j)}, \tag{5.12}$$

where $q(n_j)$ corresponds to the polynomial degree q supported by the local number of points n_j, K is some constant, and the constant C^A depends on the dimension d, the chosen weight function, the number of local points n_j, and the order of the differential operator α.

6. Numerical Experiments

In this section we perform numerical experiments to find the quanto-adjusted CDS par spread value s and its sensitivity to market conditions. The par spread is computed as in Eq.(4.10) while the bond price is obtained from Eq.(3.10) by approximating the PDEs in Eq.(3.6), Eq.(3.8) using radial basis function partition of unity method with 1296 patches. We select Gaussian functions to construct a finite RBF basis on 28561 nodes. As $[r, \hat{r}, z] \in [0, \infty)$ and $y \in (-\infty, \infty)$, we truncate each semi-infinite ot infinite domain of definition sufficiently far away from the evaluation point, so an error brought by this truncation is relatively small. In particular, we use $r_{\min} = \hat{r}_{\min} = z_{\min} = 0$, $y_{\min} = -6$, $r_{\max} = \hat{r}_{\min} = z_{\min} = 4$, $y_{\max} = -2$. Accordingly, we move the boundary conditions, defined in Eq.(3.9), to the boundaries of this truncated domain.

Note, that in our numerical method (see Section 5), we substitute Eq.(3.9) into the pricing PDEs Eq.(3.6), Eq.(3.8) and then derive a corresponding reduced form discrete (boundary) operator. As this explicitly incorporates the boundary conditions into the pricing scheme, the latter can be implemented uniformly with no extra check that the boundary conditions are satisfied[e].

For integrating in time we use the backward differentiation formula of second order (BDF-2), Endre and David (2003). This scheme is A-stable and provides a second order convergence, i.e., halving the time step leads to a drop in the time discretisation error by factor 4. In order to compute the accrued amount L_a as in Eq.(4.9) we use the time discretisation with two-weeks intervals. The method is implemented in Matlab 2017a, and the experiments were run on a MacBook Pro with a Core i7 processor with 16 GB RAM.

To investigate quanto effects and their impact on the price of a CDS contract, we consider two similar CDS contracts. The first one is traded in the foreign economy, e.g., in Italy, but is priced under the domestic risk-neutral $\mathbb{Q}$-measure, hence is denominated in the domestic currency (US dollars). To find the price of this contract our approach described in the previous sections is utilized. The second CDS is the same contract which is traded in the domestic economy and is also priced in the domestic currency. As such, its price can be obtained by solving the same problem as for the first CDS, but where the equations for the foreign interest rate $\hat{R}_t$ and the FX rate Z_t are excluded from consideration. Accordingly, all related correlations which include index z and $\hat{r}$ vanish, and the no-jumps framework is used. However, the terminal conditions remain the same as in Section 4 as they are already expressed in the domestic currency[f].

Below we denote the CDS spread found by using the first contract as s, and the second one as s_d. So the impact of quanto effects could be determined as the difference between these two spreads

$$\Delta s = s - s_d, \tag{6.1}$$

which below is quoted as “basis” spread.

[e]Our experience shows that this approach works better and provides a more stable RBF approximation.
[f]Alternatively, the whole four-dimensional framework could be used if one sets $z = 1, \hat{r} = r, \gamma_z = \hat{a} = \sigma_{\hat{r}} = \gamma_{\hat{r}} = 0$, and $\rho_{\cdot,z} = \rho_{\cdot,\hat{r}} = \rho_{z,\hat{r}} = 0$, where $\langle\cdot\rangle \in [r, z, y]$.

A default set of parameter values used in our numerical experiments is given in Table 2. It is also assumed that in this default set all correlations are zero. If not stated otherwise, we use these values and assume the absence of jumps in the FX and foreign interest rates. The reference five-year CDS par spread value s_d under these assumptions is $s_d = 365$ bps.

Interest rates							
r	a	b	σ_r	$\hat{r}$	$\hat{a}$	$\hat{b}$	$\sigma_{\hat{r}}$
0.02	0.08	0.1	0.01	0.03	0.08	0.1	0.08

Hazard and FX rates, Tenor, and Recovery							
y	a_y	b_y	σ_y	z	σ_z	T	$\mathcal{R}$
-4.089	0.0001	-210	0.4	1.15	0.1	5	0.45

Table 2: The default set of parameter values used in the experiments.

The impact of the jump amplitude on the basis spread is presented in Fig. 4 for jumps in the interest rate (left panel) and exchange rate (right panel). In the absence of jumps ($\gamma_{\hat{r}} = 0$ or $\gamma_z = 0$) the domestic and foreign spreads have a basis about 3 bps. This is close to the normal situation where no currency and interest rate depreciation occurs. In fact, this was the case until recently when quanto effects were not taken into account. For example, Greek CDS with payments in dollars and in euros were traded with a 1 bp difference in 2006, Thomson-Reuters (2011).

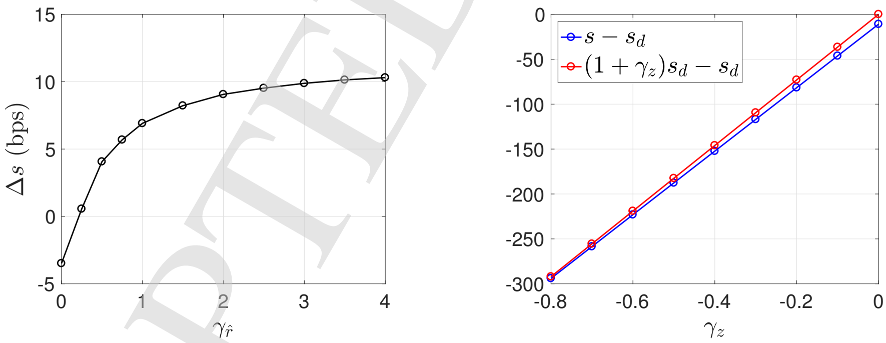


Figure 4: The influence of the jump-at-default amplitude on the five-year CDS par spread.

The results displayed in the left panel of Fig. 4 demonstrate that the impact of jump in $\hat{R}_t$ increases rapidly for $\gamma_{\hat{r}} \in [0, 2]$ and then saturates at some level. We explain this saturation by investor's indifference to whether the interest rate increases by 300% or 400% since the interest rate level does not directly affect the protection amount, rather it influences the investment climate in the foreign economy. In contrast, the FX rate has an immediate impact (right panel) on the protection since a depreciation of the foreign currency diminishes the amount being paid out when converted to the US dollars. Through the well-known approximation of the hazard rate via the spread and bond recovery rate[g]

$$\lambda \approx \frac{s}{1-\mathcal{R}},$$

and using the results in Brigo et al. (2015), we identify that

$$s \approx (1+\gamma_z)s_d.$$

That is, the CDS spread in the foreign currency is approximately proportional to the reference USD spread with the coefficient $(1+\gamma_z)$. Therefore, in the case of the foreign currency devaluation the coupon payments in the foreign currency should be lower. It can be observe that the results provided by our model perfectly align with this intuition.

We emphasize, that since the effect of the jump-at-default in the FX rate was thoroughly investigated in Brigo et al. (2015)[h], in this paper we mainly focus on examining the impact of the jump-at-default in the foreign interest rate. However, influence of the other model parameters is also investigated and reported.

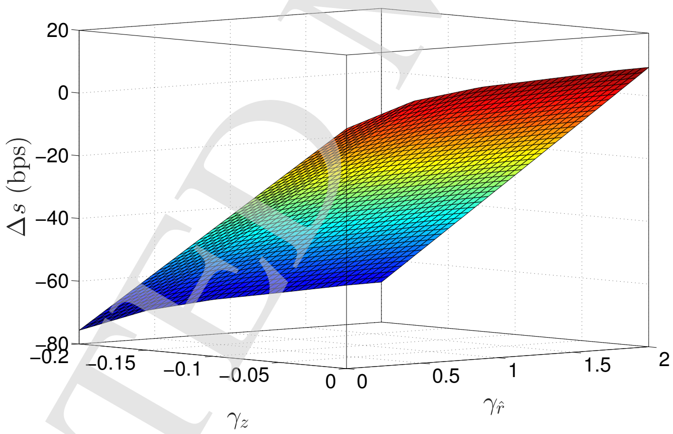


Figure 5: Basis spread as a function of the jump amplitude in the foreign exchange and interest rates.

In Fig. 5 the joint influence of jumps in the FX and foreign IR on the value of the basis spread is presented. It can be seen that the jump-at-default in $\hat{R}_t$, which occurs

[g]Which is correct if the hazard rate λ_t is constant.
[h]In Brigo et al. (2015), however, only constant foreign and domestic interest rates are considered, while in this paper they are stochastic even in the no-jumps framework.

simultaneously with the jump-at-default in Z_t, decreases the basis spread magnitude as compared with a similar case where $\hat{R}_t$ does not jump. This decrease slightly depends on the level of γ_z and for our set of parameters is about 10 bps. To better illustrate this point Fig. 6 represents some slices of the surface in Fig. 5. It can be seen that the smaller is γ_z the bigger is the impact of $\gamma_{\hat{r}}$, which, however, reaches some saturation at $\gamma_{\hat{r}} \approx 4$.

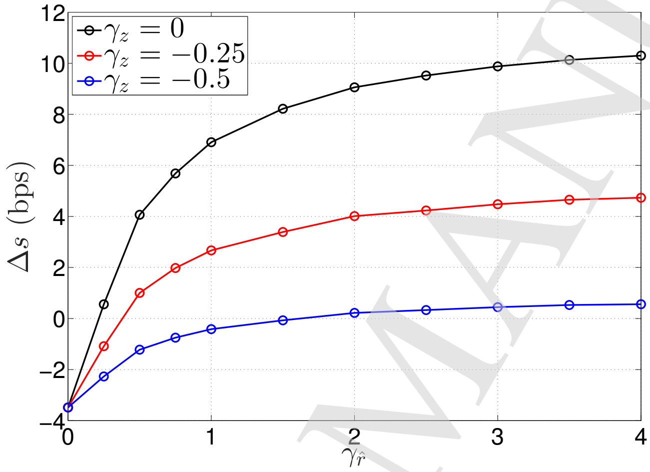


Figure 6: The influence of the jump amplitude $\gamma_{\hat{r}}$ at various values of the jump amplitude γ_z. Note, the lines are shifted to start from the same point.

In the next series of experiments we look at the influence of correlations among the stochastic factors on the quanto-adjusted CDS value. The results presented in Fig. 7 indicate that only the correlations between the hazard rate λ_t (or Y_t) and the stochastic factors that experience a jump-at-default, $\hat{R}_t, Z_t$, are relevant. The impact of the correlation between the hazard and FX rates ρ_{yz} can range in 45 bps, while the impact of the correlation $\rho_{y\hat{r}}$ between the hazard rate and the foreign interest rate does not exceed 3 bps.

Fig. 8 shows how the level of correlation between the foreign interest rate $\hat{R}_t$ and the other three stochastic factors affects the basis spread at various values of $\gamma_{\hat{r}}$ at $\gamma_z = 0$. In accordance with what was already mentioned, the results show that the correlations just slightly affect the basis spread value, except correlations with the hazard rate $\rho_{yz}, \rho_{y\hat{r}}$.

Fig. 9 shows the sensitivity of the foreign CDS to volatilities of the stochastic factors. We notice that the impact of the hazard rate volatility σ_y is the strongest, and under the jump-free setup can make the CDS quotes varying in range in 17 bps. The effect of the FX rate volatility σ_z is slightly weaker, while the effect of the interest rate volatilities $\sigma_r, \sigma_{\hat{r}}$ is almost negligible.

To analyze the impact of the two most influential volatilities under the presence of jumps in $\hat{R}_t$, we test how the volatility level affects the foreign CDS par spread with respect to the

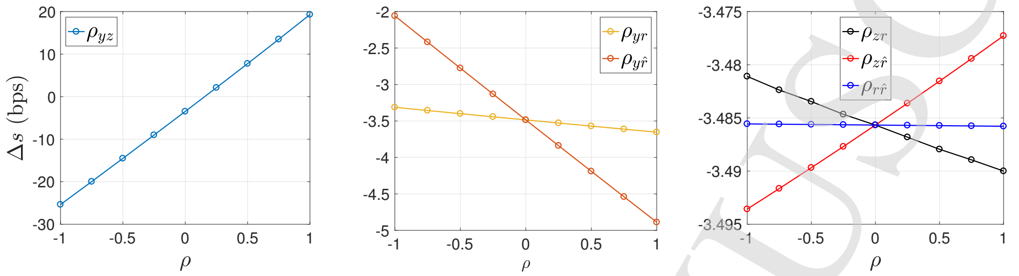


Figure 7: The influence of correlations on the five-year CDS par spread. No jumps-at-default are assumed.

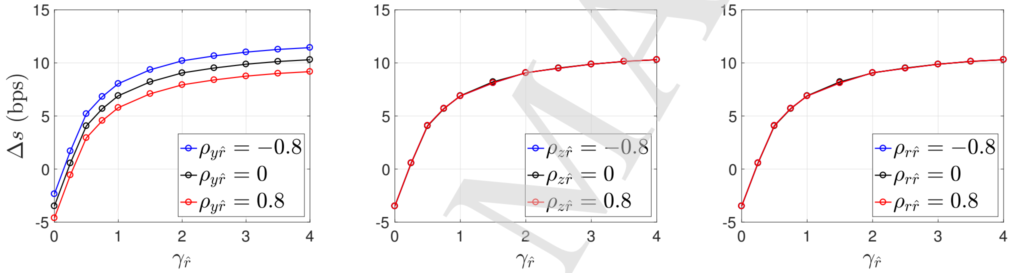


Figure 8: The influence of correlations on the basis spread for various $\gamma_{\hat{r}}$ at $\gamma_z = 0$.

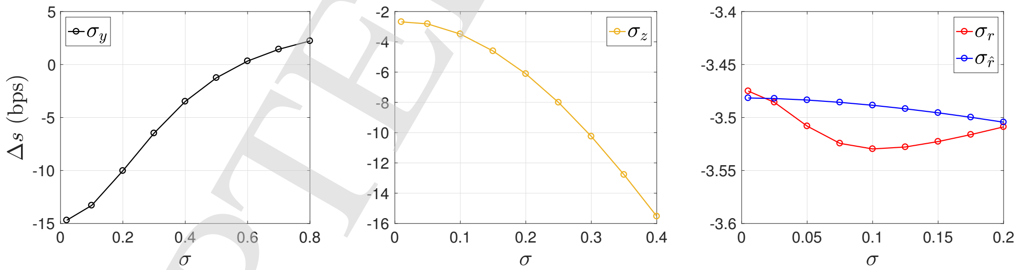


Figure 9: The influence of the volatilities on the five-year CDS par spread. No jumps-at-default are assumed.

jump amplitude. These results are presented in Fig. 10, 11. Increasing σ_y in combination with a 100% raise in $\hat{R}$ gives rise to the basis spread changing the sign from being negative to positive, while the absolute value of the growth in Δs is about 15 bps. However, the influence of σ_z is just opposite. Larger values of σ_z give rise to a negative basis spread, which though can be somewhat compensated by the increasing amplitude $\gamma_{\hat{r}}$ of the jump-at-default in the foreign interest rate $\hat{R}_t$.

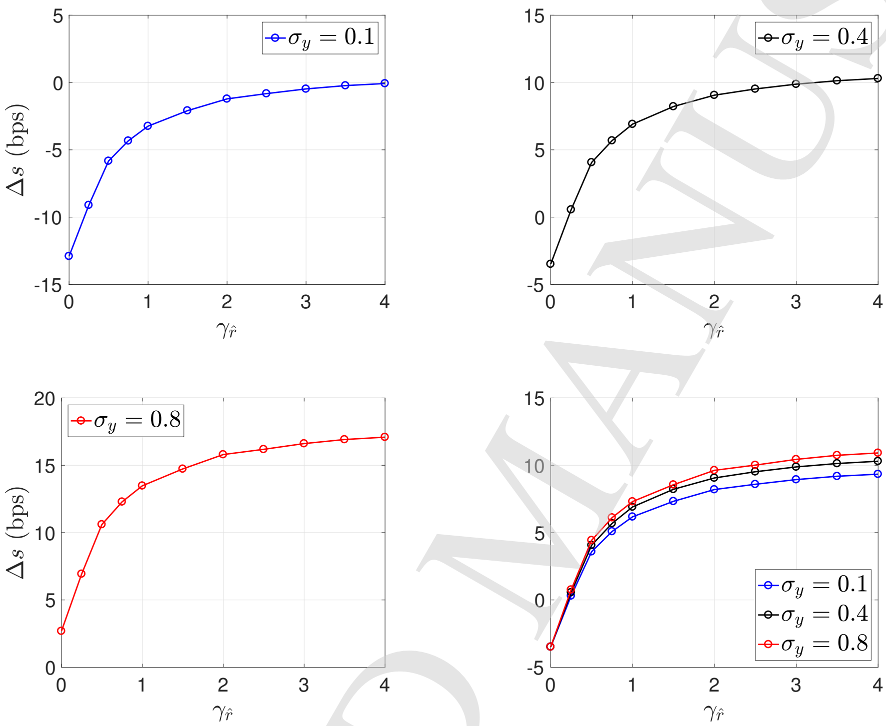


Figure 10: The influence of the hazard rate volatility σ_y on the basis spread as a function of $\gamma_{\hat{r}}$ at $\gamma_z = 0$. Note, in the lower right panel the lines are shifted to start from the same point.

Thus, we observe that the jump-at-default in the FX rate is the most prominent factor that explains the largest portion of the known discrepancies between quanto CDS quotes in US dollars and the foreign currency. Nevertheless, the potential jump in the foreign interest rate might be responsible for about 20 bps in the basis spread value. However, it is important to notice that the two jumps have opposite effects: the jump in the FX rate decreases the value of the foreign CDS, while the jump in the IR increases the value of the foreign CDS.

Despite calibration of the model is out of scope of this paper, nevertheless in the end we wish to shortly discuss main ideas of how this could be done in practice. It can be observed that overall our model contains 18 parameters which can be conventionally grouped as follows:

(a) 7 parameters of the interest rates and FX models $a, b, \sigma_r, \hat{a}, \hat{b}, \sigma_{\hat{r}}, \sigma_z$ and 3 corresponding correlations $\rho_{r\hat{r}}, \rho_{rz}, \rho_{\hat{r}z}$;

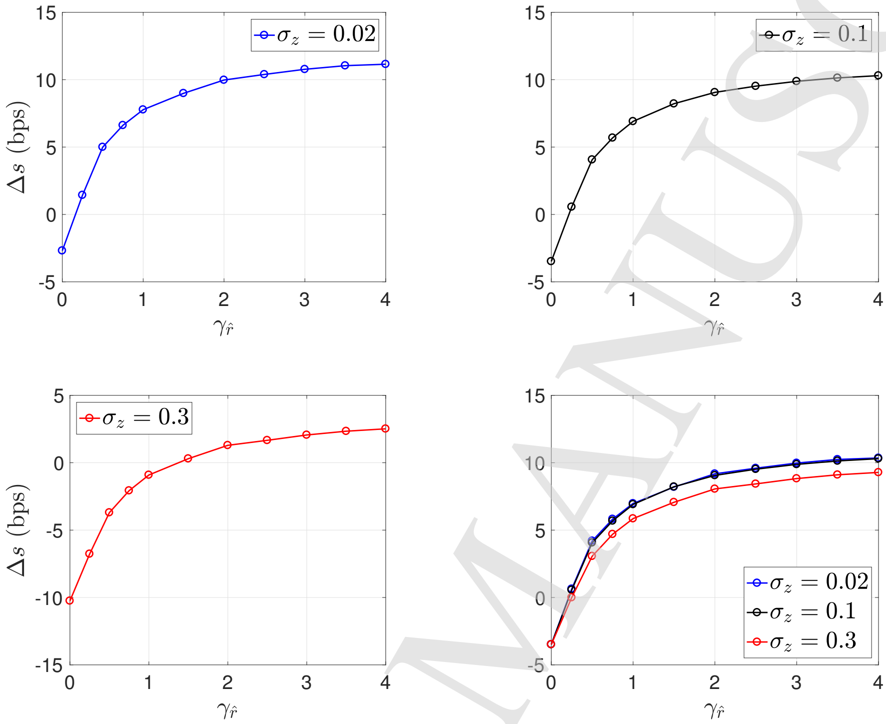


Figure 11: The influence of σ_z on the basis spread as a function of $\gamma_{\hat{r}}$ at $\gamma_z = 0$. Note, in the lower right panel the lines are shifted to start from the same point.

(b) 3 parameters of the hazard rate model θ, κ, σ_y;
(c) the remaining 3 correlations $\rho_{zy}, \rho_{y\hat{r}}, \rho_{yr}$;
(d) 2 jump intensities $\gamma_x, \gamma_{\hat{r}}$.

Obviously, calibrating all 18 parameters simultaneously is computationally expensive. On the other hand, as follows from the results of our numerical experiments, the model parameters which significantly affect the magnitude of a quanto CDS spread are: the jump intensities $\gamma_z, \gamma_{\hat{r}}$ and correlations between the hazard rate and the factors that incorporate jumps, i.e., ρ_{yz} and $\rho_{y\hat{r}}$ (in less degree this is also volatilities of the hazard process σ_y and the FX rate σ_z). Therefore, with this in mind, the whole calibration could be split into few steps.

At the first step, we can independently calibrate drifts and volatilities of the domestic r and foreign $\hat{r}$ interest rates to swaption implied volatilities, similar to how this is described in Gurrieri et al. (2009). However, it makes even more sense to calibrate the interest rates and FX together. Again, the interest rates could be calibrated to interest rate swaption volatilities, while parameters of the FX model are calibrated to FX option volatilities of different maturities. On this way one can also calibrate correlations $\rho_{r\hat{r}}, \rho_{rz}, \rho_{\hat{r}z}$, see Osajima

(2005), Sokol (2015) for more details. Thus, this determines parameters of the group (a).

Parameters of the hazard rate model (group (b)) in the first order approximation can be found by calibrating this model independently to the CDS term structures observed on the market. For instance, in Cariboni† and Schoutens (2009) the authors consider the 125 CDS constituting the iTraxx Europe Index. For each asset, weekly term structure data are available for a total of 58 market observations (i.e. covering a time period of bit more than 1 year). This allows not only to investigate in depth the calibration capabilities of the OU-models, but also to check the stability of the processes' parameters over time.

At the next step correlation ρ_{yr} could be calibrated by using historical market data on risky bonds. However, as this is discussed in Brigo and Alfonsi (2005) and also demonstrated by our numerical experiments, the impact of ρ_{yr} on the value of quanto CDS spread is small, so alternatively in the zero-order approximation it can be set to zero.

Finally, to calibrate the remaining jump parameters $\gamma_z, \gamma_{\hat{r}}$ and correlations ρ_{yz} and $\rho_{y\hat{r}}$ we can use market data on quanto CDS spreads of different maturities. When doing so we need to compute quanto CDS spreads according to the method described in this paper. However, at this step the number of yet unknown model parameters is 4, rather than 18, hence, this is computationally feasible.

7. Conclusion

This paper has introduced a new model which can be used, e.g., for pricing quanto CDS. The model operates with four stochastic factors, namely, the hazard rate, the foreign exchange rate, the domestic interest rate, and the foreign interest rate, and allows for jumps-at-default in the FX and foreign interest rates. Corresponding systems of PDEs for both the risky bond price and the CDS price have been derived similar to how this is done in Bielecki et al. (2005).

In order to solve these equations we have developed a localized radial basis function method that is based on the partition of unity approach. The advantage of the method is that in our four-dimensional case it maintains high accuracy while uses less resources than, for example, the corresponding finite difference or Monte Carlo methods. Potentially, the RBF method can be a subject of parallelization which would improve the computational efficiency.

The results of our numerical experiments have qualitatively explained the discrepancies observed in the marked values of CDS spreads traded in domestic and foreign economies and, accordingly, denominated in the domestic (USD) and foreign (euro, ruble, real, etc.) currencies. The quanto effect (the difference between the prices of the same CDS contract traded in different economies, but represented in the same currency) can, to a great extent, be explained by the devaluation of the foreign currency. This would yield a much lower protection payout if converted to the US dollars. These results are similar to those obtained in Brigo et al. (2015). We underline, however, that in Brigo et al. (2015) only constant foreign and domestic interest rates are considered, while in this paper they are stochastic even in the no-jumps framework.

In contrast to Brigo et al. (2015), in this paper we have also analyzed the impact of the jump-at-default in the foreign interest rate which could occur simultaneously with the default in the FX rate. We have found that this jump is a significant component of the process and is able to explain about twenty bps of the basis spread value. However, it is worth noticing that the jumps in the FX rate and IR have opposite effects. In other words, devaluation of the foreign currency will decrease the value of the foreign CDS, while the increase of the foreign interest rate will increase the foreign CDS value.

The other important parameters of the model are correlations between the hazard rate and the factors that incorporate jumps, i.e., ρ_{yz} and $\rho_{y\hat{r}}$, and (in less degree) volatilities of the hazard process σ_y and the FX rate σ_z. Therefore, they have to be properly calibrated. Varying the other correlations just slightly contributes to the basis spread value. Large values of the volatilities can in some cases explain up to fifteen bps of the basis spread value.

If one needs to hedge quanto CDSs, in our model this could be done as in any other model, namely by using delta hedging to the underlying market parameters. Jumps will obviously have to be determined historically.

We also have to mention that the pricing problem have been formulated via backward PDEs. Therefore, computation of the CDS spread requires to independently solve these PDEs for every discrete time point on a temporal grid lying below the contract maturity. Despite this can be done in parallel, e.g., using a Matlab's parallel toolbox, it could be significantly improved if instead of the backward PDE we would work with the forward one for the corresponding density function. We leave this improvement to be implemented elsewhere.

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Appendix A Derivation of main PDE

Below we give a sketch of derivation of the main PDE for the defaultable zero-coupon bond price which follows from our model introduced in Section 2.2, as the detailed derivation is rather long. Therefore, we utilize some results known in the literature, and explain only the main steps of the derivation.

According to our model setting which is presented in Section 2.2, all underlying stochastic processes $R_t, \hat{R}_t, Y_t, Z_t, D_t$ possess a strong Markovian property, see, e.g., Bielecki et al. (2005). Denote by $r, \hat{r}, y, z$, and d the initial values of these processes at time t, respectively. For Markovian underlyings it is well-known, e.g., Ethier and Kurtz (2009), that evolution of U_t represented as a function of variables $(t, r, \hat{r}, y, z, d)$ can be described by a corresponding

PDE (or PIDE if jumps are also taken into account). In this section we derived such a PIDE in the explicit form.

Let us remind that in the jump-at-default framework the dynamics of Z_t and $\hat{R}_t$ is given by Eq.(2.11), Eq.(2.12)

$$\begin{aligned} dZ_t &= (R_t - \hat{R}_t) Z_t dt + \sigma_z Z_t dW_t^{(3)} + \gamma_z Z_t dM_t, \\ d\hat{R}_t &= \hat{a}(\hat{b} - \hat{R}_t)dt + \sigma_{\hat{r}} \sqrt{\hat{R}_t} dW_t^{(2)} + \gamma_{\hat{r}} R_t dD_t. \end{aligned} \tag{A.1}$$

For the sake of convenience the second SDE could be re-written in the form of the first one

$$\begin{aligned} d\hat{R}_t &= \hat{a}\left(\hat{b} - \hat{R}_t\right) dt + d\Gamma_t + \sigma_{\hat{r}} \sqrt{\hat{R}_t} dW_t^{(2)} + \gamma_{\hat{r}} R_t dM_t \\ &= \hat{a}\left(\hat{b} - \hat{R}_t - \frac{\lambda_t}{\hat{a}}(1 - D_t)\right) dt + d\Gamma_t + \sigma_{\hat{r}} \sqrt{\hat{R}_t} dW_t^{(2)} + \gamma_{\hat{r}} R_t dM_t. \end{aligned} \tag{A.2}$$

So we replaced D_t with a compensated martingale M_t by subtracting a compensator of D_t, and, accordingly, added this compensator to the drift. When doing so, we take into account Eq.(2.9) to obtain $d\Gamma_t = (1 - D_t)\lambda_t dt$.

Below we need the following theorem from Jacod and Shiryaev (1987)[i], which provides a generalization of Itô's lemma to the class of semimartingales

Theorem A.1. *Let $X = (X_t)_{0 \le t \le T}$ be a Lévy process which is a real-valued semimartingale with the triplet (b, c, ν), and f be a function on $\mathbb{R}$, $f \in C^2$. Then, $f(X)$ is a semimartingale, and $\forall t \in [0, T]$ the following representation holds*

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_0^t f'(X_{s^-}) dX_s + \frac{1}{2}\int_0^t f''(X_{s^-}) d\langle X^c \rangle_s \\ &+ \sum_{0 \le s \le t} \left[f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s \right]. \end{aligned} \tag{A.3}$$

Here $X_{s^-} = \lim_{u \nearrow s}$ is the value just before a potential jump, $\Delta X_s = X_s - X_{s^-}$, X^c is the continuous martingale part of X_t, i.e. $X_t^c = \sqrt{c} W_t$, and $\langle \cdot \rangle$ determines a quadratic variation.

Alternatively, if the random measure of jumps $\mu^X(ds, dx)$ is used, we have

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_0^t f'(X_{s^-}) dX_{s^-} + \frac{1}{2}\int_0^t f''(X_{s^-}) d\langle X^c \rangle_{s^-} \\ &+ \int_0^t \int_{\mathbb{R}} \left[f(X_{s^-} + x) - f(X_{s^-}) - x f'(X_{s^-}) \right] \mu^X(ds, dx). \end{aligned} \tag{A.4}$$

Proof. See Theorem I.4.57 in Jacod and Shiryaev (1987). □

Further let us consider only jumps of a finite variation and finite activity, so

$$\sum_{0 \le s \le t} f(X_s) < \infty, \qquad \sum_{0 \le s \le t} f'(X_{s^-})\Delta X_s < \infty.$$

[i] See also Itkin (2017) and references therein.

Our model allows only for a single jump to occur at the default time τ. Therefore,

$$\mu^X(dsdx) = \delta(s-t)D_s\nu(dx)ds,$$

with $\nu(dx)$ being a Lévy measure of jumps in $\mathbb{R}$, and where $\delta(x)$ is the Dirac delta function.

Respectively, in the differential form and for a multidimensional case Eq.(A.4) reads

$$\begin{aligned} df(\mathbf{X}_s) &= \frac{\partial f(\mathbf{X}_{s^-})}{\partial \mathbf{X}_{s^-}} * d\mathbf{X}_s + \frac{1}{2}\frac{\partial^2 f(\mathbf{X}_{s^-})}{\partial {\mathbf{X}_{s^-}}^2} * d\langle \mathbf{X}^c\rangle_s \\ &+ \int_{\mathbb{R}} \left[f(\mathbf{X}_{s^-}+\mathbf{x}) - f(\mathbf{X}_{s^-}) - \mathbf{x} * \frac{\partial f(\mathbf{X}_{s^-})}{\partial \mathbf{X}_{s^-}} \right] \nu(d\mathbf{x}) dD_t, \end{aligned} \tag{A.5}$$

where $\mathbf{X}_s$ is a vector of independent variables, $\mathbf{x}$ is a vector of the corresponding jump values, and $\langle * \rangle$ is an inner product.

Also, according to Eq.(A.1) the size of the jump in both the foreign interest rate and the FX rate is proportional to the value of the corresponding process right before the jump occurs at time τ with a constant rate γ.

Combining Eq.(A.5) and Eq.(A.1) gives rise to the Lévy measure $\nu(d\mathbf{x})$ of this multi-dimensional jump process to be

$$\nu(d\mathbf{x}) = \delta(x_z - \gamma_z z)\delta(x_{\hat{r}} - \gamma_{\hat{r}}\hat{r})dx_z dx_{\hat{r}}, \tag{A.6}$$

compare, e.g., with Crosby (2013).

Therefore, the last line in Eq.(A.5) changes to

$$\begin{aligned} J &= \left[f(t,\mathbf{X}_s) - f(t,\mathbf{X}_{s^-}) - \Delta\mathbf{X}_{s^-} * \frac{\partial f(t,\mathbf{X}_{s^-})}{\partial \mathbf{X}_{s^-}} \right] dD_t, \\ \mathbf{X}_s &= \mathbf{X}_{s^-} + \Delta\mathbf{X}_{s^-} = f(t, r, \hat{r}(1+\gamma_{\hat{r}}), y, z(1+\gamma_z), d=1), \\ \mathbf{X}_{s^-} &= f(t, r, \hat{r}, y, z, d=0). \end{aligned} \tag{A.7}$$

Having all these results, the PDE for the discounted defaultable bond price can be derived by using a standard technique for jump-diffusion processes, see, e.g., Papapantoleon (2008). However, for the sake of brevity, we will utilize the approach of Bielecki et al. (2005), where a similar problem is considered, and, hence, making a reference to the corresponding theorems proved in that paper.

Note, that

$$\mathbb{E}_t[dD_t|D_t d, Y_t = y] = d\mathbb{E}_t[D_t|D_t = d, Y_t = y] \quad = \lambda_t \mathbb{1}_{t\le\tau}\Big|_{(D_t=d, Y_t=y)} dt = (1-d)e^y dt, \tag{A.8}$$

where the last by one equality follows from Lemma 7.4.1.3 in Jeanblanc et al. (2009).

Using Eq.(A.8), it can be seen that after the default occurs, $D_t = 1_{\tau\le t} = 1$, and thus the jump term J disappears. However, before the default at time $t < \tau$ the jump term is

$$J = f(t,\mathbf{X}_s) - f(t,\mathbf{X}_{s^-}) - \Delta\mathbf{X}_{s^-} * \frac{\partial f(t,\mathbf{X}_{s^-})}{\partial \mathbf{X}_{s^-}}. \tag{A.9}$$

So, conditional on the value of D_t the solution could be represented in the form

$$f(t,\mathbf{X}_s) = \mathbb{1}_{t<\tau} f(t,\mathbf{X}_{s^-}) + \mathbb{1}_{\tau\le t} f(t,\mathbf{X}_s). \tag{A.10}$$

Then the remaining derivation of the PDE could be done based on the following Proposition:

Proposition A.1 (Proposition 3.1 in Bielecki et al. (2005)). *Let the price processes* $Y^i,\ i = 1, 2, 3$ *satisfy*

$$dY_t^i = Y_{t^-}^i \left[\mu_i dt + \sigma_i dW_t^i + k_i dM_t\right]$$

with $k_i > -1$ *for* $i = 1, 2, 3$, μ, σ *being the corresponding drifts and volatilities. Then the arbitrage price of a contingent claim* Y *with the terminal payoff* $G(t, Y_T^1, Y_T^2, Y_T^3, D_T)$ *equals*

$$\pi_t(Y) = \mathbb{1}_{t \le \tau} C(t, Y_t^1, Y_t^2, Y_t^3, 0) + \mathbb{1}_{t \ge \tau} C(t, Y_t^1, Y_t^2, Y_t^3, 1)$$

for some function $C : [0, T] \times \mathbb{R}_+^3 \times 0, 1 \to \mathbb{R}$. *Assume that for* $d = 0$ *and* $d = 1$ *the auxiliary function* $C(\cdot, d) : [0, T] \times \mathbb{R}_+^3 \to \mathbb{R}$ *belongs to the class* $C^{1,2}([0, T] \times \mathbb{R}_+^3)$. *Then the functions* $C(\cdot, 0)$ *and* $C(\cdot, 1)$ *solve the following PDEs:*

$$\begin{aligned}
\partial_t C(\cdot, 0) &+ \sum_{i=1}^3 (\alpha - \lambda k_i) y_i \partial_i C(\cdot, 0) + \frac{1}{2} \sum_{i,j=1}^3 \rho_{ij} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot, 0) \\
&+ \lambda \left[C(t, y_1(1 + k_1), y_2(1 + k_2), y_3(1 + k_3), 1) - C(t, y_1, y_2, y_3, 0)\right] - \alpha C(\cdot, 0) = 0, \\
\partial_t C(\cdot, 1) &+ \sum_{i=1}^3 \alpha y_i \partial_i C(\cdot, 1) + \frac{1}{2} \sum_{i,j=1}^3 \rho_{ij} \sigma_i \sigma_j y_i y_j \partial_{ij} C(\cdot, 1) - \alpha C(\cdot, 1) = 0.
\end{aligned}$$

subject to the terminal conditions

$$\begin{aligned}
C(T, y_1, y_2, y_3, 0) &= G((T, y_1, y_2, y_3, 0), \\
C(T, y_1, y_2, y_3, 1) &= G((T, y_1, y_2, y_3, 1).
\end{aligned}$$

Proof. See Bielecki et al. (2005). □

Two important notes should be made in order to apply this proposition to our problem.

Tradable assets In Bielecki et al. (2005) all underlying assets are assumed to be tradable. Therefore, they have to be martingales under some unique martingale measure (a particular choice of Y^1 as is made to be a numeraire). To achieve this, additional conditions on the drifts, volatilities and the jump rates k_i should be imposed. In particular, this requires that the coefficient α in Proposition A.1 would be

$$\alpha = \mu_i + \sigma_i \frac{c}{a},$$

where the determinants c, a are the explicit functions of $\mu_i, \sigma_i, k_i,\ i = 1, 2, 3$ and given in Bielecki et al. (2005). Moreover, it is shown that the right-hand side of this formula does not depend on i.

However, for our problem among all the underlying processes the only tradable one is that for the FX rate. This allows one to fully eliminate these conditions on $\mu_i, \sigma_i, k_i,\ i = 1, 2, 3$.

As a consequence, e.g., the term

$$\sum_{i=1}^{3}(\alpha - \lambda k_i) y_i \partial_i C(\cdot, 0)$$

in the Proposition A.1 is now replaced with

$$\sum_{i=1}^{3}(\mu_i - \lambda k_i) y_i \partial_i C(\cdot, 0).$$

Risk-neutrality Proposition A.1 derives an arbitrage price (under real measure) of the contingent claim written on the given underlyings. To get this price under a risk-neutral measure $\mathbb{Q}$, one needs to construct a replication (*self-financing*) strategy of a generic claim. In particular, to hedge out the risk of $\hat{R}_t$ and R_t, corresponding non-defaultable zero-coupon bonds (perhaps, of a longer maturity) should be used as a hedge, Bielecki and Rutkowski (2004), Wilmott (1998).

This problem is solved by Proposition 3.3 of Bielecki et al. (2005). Accordingly, the previously derived PDEs remain the same, with the only change of the killing term where the coefficient α is replaced with the interest r corresponding to measure $\mathbb{Q}$ (as expected based on a general theory of asset pricing).

We proceed by combining these results together and applying them to our model. First, we revert the notation back to that used in this paper. Then, taking into account an explicit form of the stochastic differential equations describing the dynamics of our underlying processes, and conditioning on $R_t = r, \hat{R}_t = \hat{r}, Z_t = z, Y_t = y, D_t = d$, we obtain that under the risk-neutral measure $\mathbb{Q}$ the price $U_t(T)$ is

$$U_t(T, r, \hat{r}, y, z) = \mathbb{1}_{t<\tau} f(t, T, r, \hat{r}, y, z, 0) + \mathbb{1}_{t \geq \tau} f(t, T, r, \hat{r}, y, z, 1). \tag{A.11}$$

Here the function $f(t, T, r, \hat{r}, y, z, 1) \equiv u(t, T, X),\ X = \{t, r, \hat{r}, y, z\}$ solves the PDE

$$\frac{\partial u(t, T, X)}{\partial t} + \mathcal{L}u(t, T, X) - ru(t, T, X) = 0, \tag{A.12}$$

where the diffusion operator $\mathcal{L}$ reads

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}{\sigma_r}^2 r \frac{\partial^2}{\partial r^2} + \frac{1}{2}\sigma_{\hat{r}}^2 \hat{r} \frac{\partial^2 u}{\partial \hat{r}^2} + \frac{1}{2}\sigma_z^2 z^2 \frac{\partial^2}{\partial z^2} + \frac{1}{2}\sigma_y^2 \frac{\partial^2}{\partial y^2} + \rho_{r\hat{r}}\sigma_r\sigma_{\hat{r}}\sqrt{r\hat{r}}\frac{\partial^2}{\partial r \partial \hat{r}} \\
&+ \rho_{rz}\sigma_r\sigma_z z\sqrt{r}\frac{\partial^2}{\partial r \partial z} + \rho_{\hat{r}z}\sigma_{\hat{r}}\sigma_z z\sqrt{\hat{r}}\frac{\partial^2}{\partial z \partial \hat{r}} + \rho_{ry}\sigma_r\sigma_y\sqrt{r}\frac{\partial^2}{\partial r \partial y} + \rho_{\hat{r}y}\sigma_{\hat{r}}\sigma_y\sqrt{\hat{r}}\frac{\partial^2}{\partial y \partial \hat{r}} \\
&+ \rho_{yz}\sigma_y\sigma_z z\frac{\partial^2}{\partial y \partial z} + a(b - r)\frac{\partial}{\partial r} + \hat{a}(\hat{b} - \hat{r})\frac{\partial}{\partial \hat{r}} + (r - \hat{r})z\frac{\partial}{\partial z} + \kappa(\theta - y)\partial y.
\end{aligned} \tag{A.13}$$

The second function $f(t, T, r, \hat{r}, y, z, 0) \equiv v(t, T, X)$ solves the PDE

$$\begin{aligned}
&\frac{\partial v(t, T, X)}{\partial t} + \mathcal{L}v(t, T, X) - rv(t, T, X) - \lambda\gamma_z z\frac{\partial v(t, T, X)}{\partial z} \\
&\qquad + \lambda\left[u(t, T, X^+) - v(t, T, X)\right] = 0, \qquad X^+ = \{r, \hat{r}(1 + \gamma_{\hat{r}}), y, z(1 + \gamma_z)\},
\end{aligned} \tag{A.14}$$

where according to Eq.(2.4), $\lambda = e^y$. Note, that the term $\lambda\gamma_{\hat{r}}\hat{r}v_{\hat{r}}(t,T,X)$ in the drift of Eq.(A.2) cancels out with the corresponding compensator in Eq.(A.9) as it should be as the process $\hat{R}_t$ is not a martingale.