Low Latency Interest Rate Markets

Theory, Pricing & Practice

Nicholas Burgess

PART ONE: Theory

IR Markets, Products & Models

- Introduction to IR Markets
- Interest Rate Swaps
- IR Products & CDS
- Yield Curves
- IR Risk
- Credit Models

PART TWO: Pricing & Practice

Case Studies

- IRS Pricing Formulae
- IRS Pricing Case Study
- Asset Swap Structuring
- Asset Swap Pricing Case Study
- Pricing Tricks & Rules of Thumb

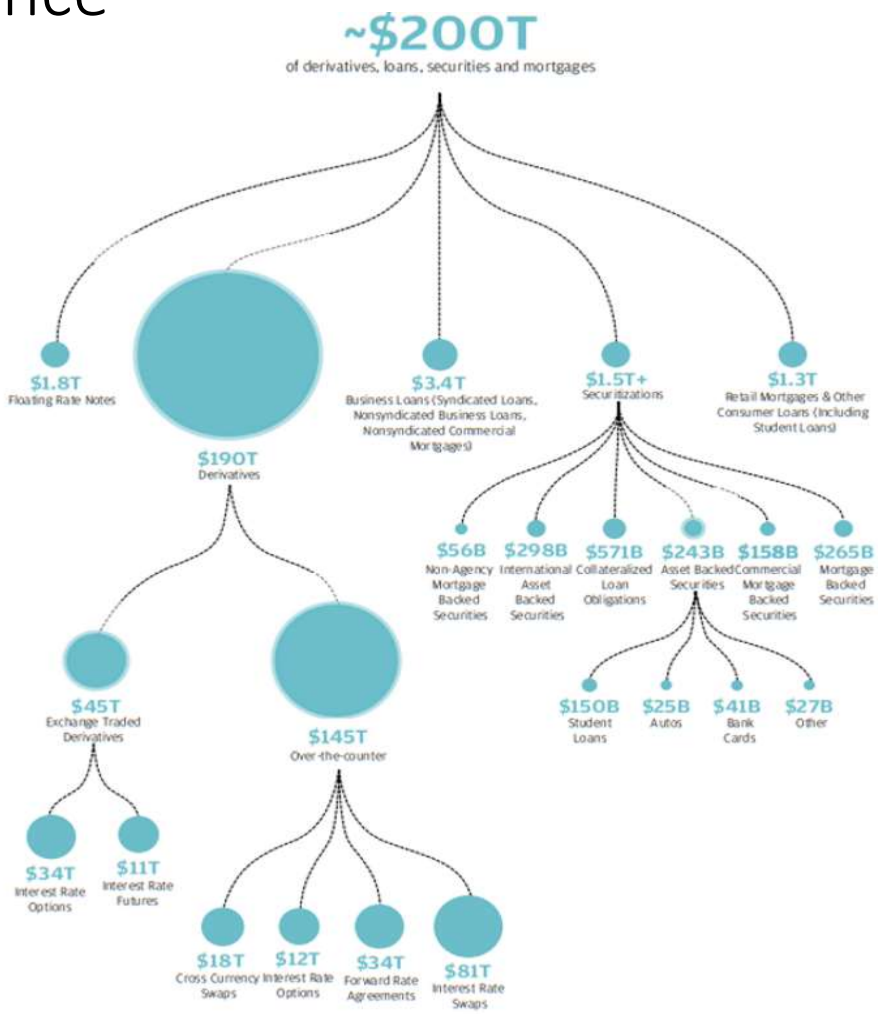
Quant Research Papers
https://ssrn.com/author=1728976

Support Materials: Quant Research, C++ and Excel Examples
https://github.com/nburgessx/SwapsBook

PART ONE - THEORY

IR Markets, Products & Models

Interest Rate Markets - Project Finance



Purpose

- To Facilitate Government, Corporate & Project Finance
- Mortgages, Corporate Loans, Gov Projects & Infrastructure
- e.g. Hospitals, Transport (HS2), Energy & Defence Projects

Market Size

- Market Size by Notional: $200T (US) + $150T (EU)
- Derivatives, Loans & Securities
- All Referencing LIBOR, until Recently

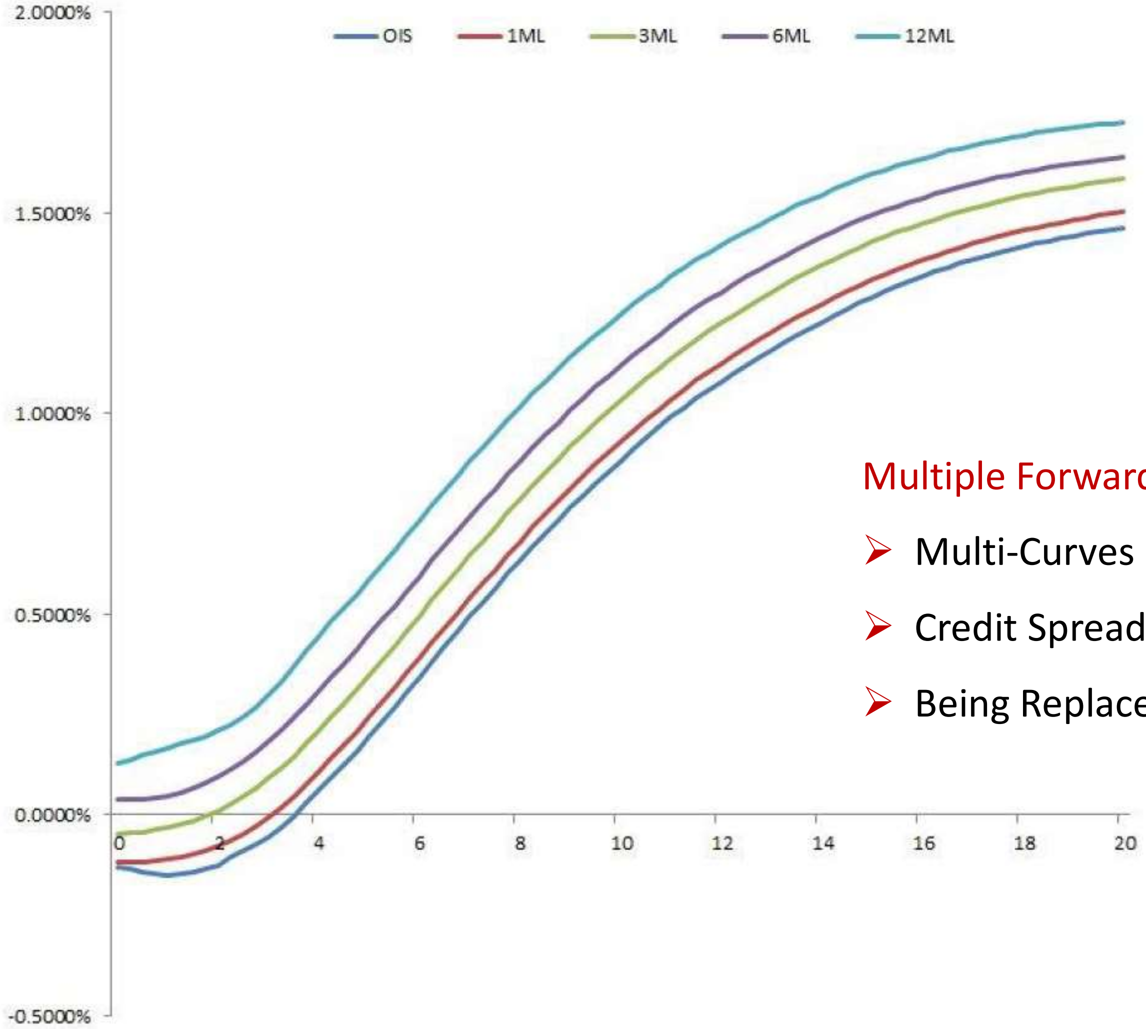
Interest Rate Markets – Why the need for Speed?

- Cleared Electronic Trading & Auto-Hedging
- Real-Time, Highly Liquid & High Precision (Bid-Offer 1/10th bps i.e. USD 10 per MM)
- Trading Horizon: High Frequency Trading (HFT) vs Long-Term Fund Performance

USD Semi vs 3M Libor			
31) 1 Year	0.750 / 0.754	+0.014	≣
32) 2 Year	1.045 / 1.049	+0.017	≣
33) 3 Year	1.284 / 1.287	+0.018	≣
34) 4 Year	1.467 / 1.471	+0.015	≣
35) 5 Year	1.617 / 1.621	+0.014	≣
36) 6 Year	1.750 / 1.754	+0.012	≣
37) 7 Year	1.866 / 1.870	+0.011	≣
38) 8 Year	1.966 / 1.970	+0.011	≣
39) 9 Year	2.052 / 2.056	+0.011	≣
40) 10 Year	2.126 / 2.129	+0.011	≣
41) 12 Year	2.250 / 2.254	+0.007	≣
42) 15 Year	2.376 / 2.380	+0.006	≣
43) 20 Year	2.497 / 2.501	+0.002	≣
44) 25 Year	2.558 / 2.563	+0.003	≣
45) 30 Year	2.592 / 2.597	+0.000	≣
46) 40 Year	2.612 / 2.621	+0.003	≣
47) 50 Year	2.598 / 2.604	+0.004	≣

USD Spreads vs Treasuries			
71) 1 Year	4.282 / 5.295	+0.687	
72) 2 Year	10.248 / 10.806	-0.073	≣
73) 3 Year	3.337 / 3.895	-0.029	≣
74) 4 Year	1.350 / 1.900	+0.161	
75) 5 Year	-4.020 / -3.454	+0.138	≣
76) 6 Year	-8.100 / -7.550	+0.157	
77) 7 Year	-13.577 / -13.036	+0.382	≣
78) 8 Year	-11.100 / -10.550	+0.335	
79) 9 Year	-9.888 / -9.088	+0.492	
80) 10 Year	-9.775 / -9.275	+0.537	≣
81) 12 Year	2.520 / 3.320	+0.204	
82) 15 Year	-3.599 / -2.799	+0.110	
83) 20 Year	-10.100 / -9.600	+0.150	
84) 25 Year	-22.800 / -22.250	+0.150	
85) 30 Year	-38.058 / -37.491	+0.351	≣

Interest Rate Markets – Yield Curve Models



Required to Forecast Future Interest Rates

- Use Liquid Market Instruments
- To Imply Forward Rates & Disc. Factors

Multiple Forward Curves

- Multi-Curves Have In-Built Credit Spread (Tenor Homogenous)
- Credit Spread Determined by Loan Repayment Frequency
- Being Replaced by Single RFR Curves (Similar to OIS Curve)

Interest Rate Markets – The LIBOR Problem

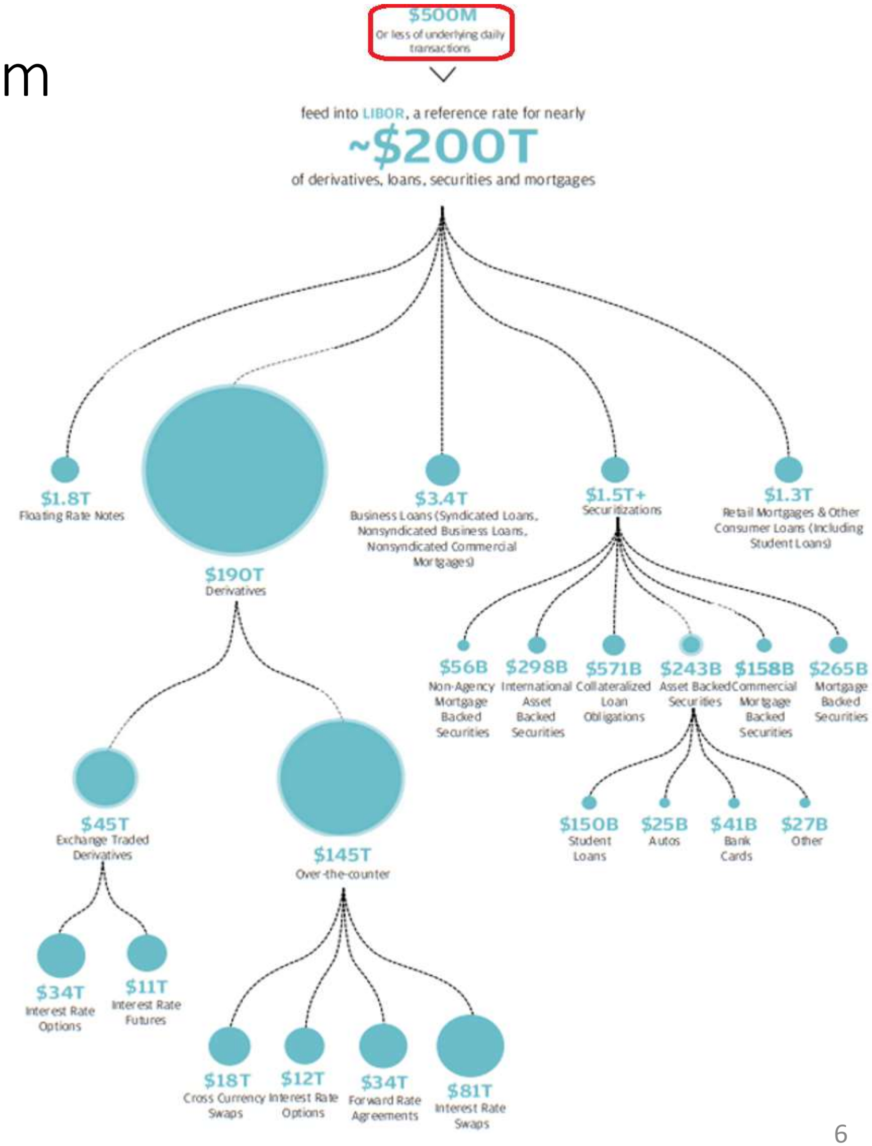
The Problem with LIBOR

- LIBOR Market Transactions < $500M
- Rates Do Not Reflect Actual Borrowing Levels
- LIBOR Levels Increasingly Set by Panel/Expert Judgement

Market Size

- Market Size by Notional: $200T (US) + $150T (EU)

Large Market Driven by Small Number of LIBOR Transactions!!!



Interest Rate Markets – LIBOR Benchmark Replacement

LIBOR Rates

- Low Transaction Volume / Panel Based
- Forward Looking Term Rate, known In-Advance
- In Built Credit Risk Component

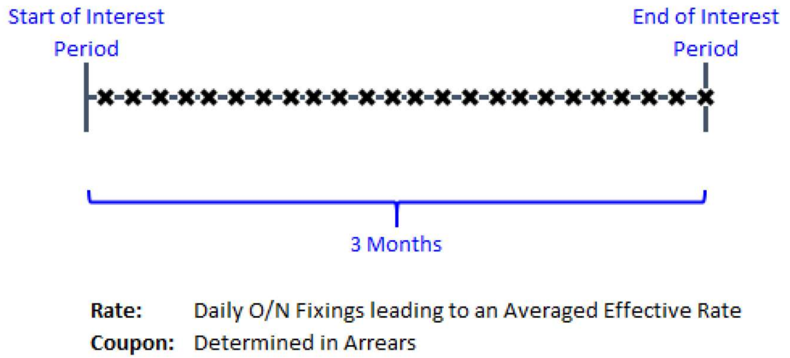
Risk-Free Rates (RFRs)

- Transaction Based
- Backward Looking Rate, Known In-Arrears
- No Credit Component i.e. Risk-Free

Market Changes

- Legacy LIBOR Contracts, Fall-Back Rates
- New RFR Products & Yield Curve Model Changes

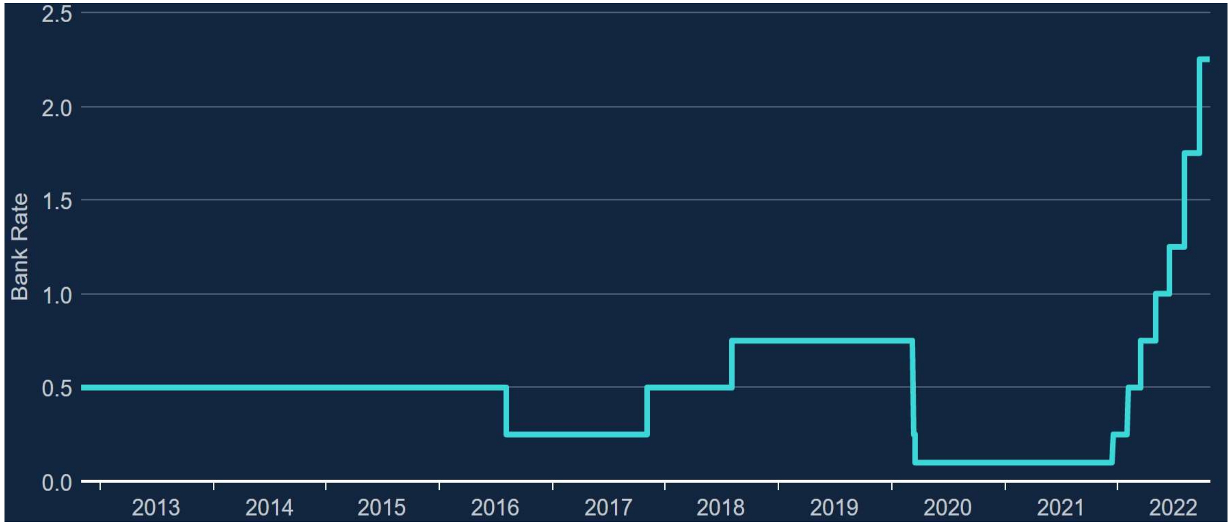
3 Month Risk-Free Rate



Interest Rate Markets – Project Finance Risks & Solutions

1. Interest Rate Risk

- Finance linked to variable interest rates
- Use IRS to Fix Borrowing Costs



2. Foreign Exchange / Currency Risk

- International Finance
- Use Cross Currency Swaps to Fix FX Rates

3. Credit Default Risk

- Bonds, Bi-Lateral and Non-Cleared Transactions
- Risk of Counterpart Default
- Credit Default Swaps, Collateral & CSA Agreements

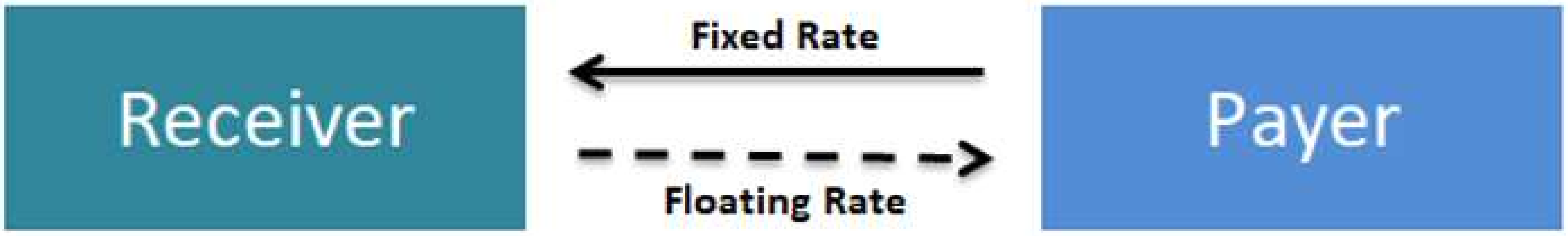
4. No money to invest?

- Use Asset Swaps to Borrow Funds to Invest in Bonds
- Pay LIBOR + Spread (Finance) to Receive Bond Coupons
- Floating Spread includes Funding + Credit Costs

Interest Rate Swaps – Fixed or Variable Borrowing Costs?

Project Finance

- Project Finance Naturally Incurs Variable Interest Costs (LIBOR + Spread)
- Exposed to Interest Rate Risk (Market may Move Against Us)



Hedging Interest Rate Risk

- Use IRS to Exchange Floating for Fixed Interest (or Vice Versa)
- We Can Choose to Fix Borrowing Costs
- We Also Trade IRS for Speculative Purposes

Interest Rate Swaps –Market Quotes & Pricing

USD Semi vs 3M Libor

31)	1 Year	0.750 / 0.754	+0.014	≣
32)	2 Year	1.045 / 1.049	+0.017	≣
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- Standard Tenors: Spread Over US Treasury Yields
- New Swaps: Par Rate (%), since PV=0
- Existing Swaps: Present Value (USD)

Interest Rate Swaps – Present Value

Present Value is the Sum of Discounted Cash Flows

$$Swap\ PV = \underbrace{\sum_{i=1}^{n} N\, r\, \tau_i\, P(t_0, t_i)}_{Fixed\ Cash\ Flows} - \underbrace{\sum_{j=1}^{m} N\left(l_{j-1} + s\right) \tau_j\, P\left(t_0, t_j\right)}_{Floating\ Cash\ Flows}$$

Interest Rate Swaps – Par Rate

- New Swaps Trade at Par i.e. PV = 0
- Consequently such Swaps Quote as a Par Rate
- This is the fixed rate that makes both trade legs equal

$$Swap\ PV = r \underbrace{\sum_{i=1}^{n} N\,\tau_i\,P(t_0, t_i)}_{Fixed\ Cash\ Flows} - \underbrace{\sum_{j=1}^{m} N(\,l_{j-1} + s)\,\tau_j\,P(t_0, t_j)}_{Floating\ Cash\ Flows} = 0$$

Rearrange for the Fixed Rate r and call this the Par Rate, p

$$Par\ Rate, p = \frac{PV(Float\ Leg)}{\sum_{i=1}^{n} N\,\tau_i\,P(t_0, t_i)} = \frac{PV(Float\ Leg)}{Annuity(Fixed\ Leg)^{1}}$$

[1] Par Rates calculated in terms of Annuity or PV01

Interest Rate Swaps - Specification

- Majority of Swap Booking Schedule Related
- Trading Templates, Generators & Static Data

Swap Generator Template: **USD_SWAP_3M**

Section	LEG TYPE	LEG1:FIXED	LEG2:FLOAT
Dynamic Trade Info	PAY / RECEIVE	PAY	RECEIVE
	NOTIONAL	1,000,000	1,000,000
	FIXED RATE (%)	1.00%	-
	FLOAT SPREAD (BPS)	-	0.00
	EFFECTIVE DATE / LAG	2D	2D
	MATURITY DATE / TENOR	2Y	2Y
Static Data + Schedule Info	LEG CURRENCY	USD	USD
	NOTIONAL EXCHANGE	NONE	NONE
	LEVERAGE	1.00	1.00
	FRONT STUB INDEX	-	NATURAL
	BACK STUB INDEX	-	NATURAL
	VALUATION CURRENCY	USD	USD
	FORECAST INDEX	-	USD3M
	DISCOUNT INDEX	USDOIS	USDOIS
	INDEX COMPOUND METHOD	-	NONE
	SPREAD COMPOUND METHOD	-	NONE
	ROLL DAY	END	END
	STUB TYPE	SHORT START	SHORT START
	FIXING BUS DAY ADJUSTMENT	-	MODIFIED_FOLLOWING
	FIXING CALENDAR	-	NY+LDN
	FIXING LAG	-	2D
	FIXING IN-ADVANCE / IN-ARREARS	-	IN-ADVANCE
	ACCRUAL FREQUENCY	SEMI-ANNUAL	QUARTERLY
	ACCRUAL BUS DAY ADJUSTMENT	MODIFIED_FOLLOWING	MODIFIED_FOLLOWING
	ACCRUAL CALENDAR	NY	NY
	ACCRUAL DAYCOUNT	30/360	ACT/360
	PAYMENT FREQUENCY	SEMI-ANNUAL	QUARTERLY
	PAYMENT BUS DAY ADJUSTMENT	MODIFIED_FOLLOWING	MODIFIED_FOLLOWING
	PAYMENT CALENDAR	NY	NY
	PAYMENT LAG	2D	2D

	TRADE PARAMETERS	LEG1	LEG2
TRADE ECONOMICS	LegType	FLOAT	FLOAT
	Currency	EUR	USD
	Notional	8,769,622	10,000,000
	NotionalExchange	ALL	ALL
	PayReceive	PAY	RECEIVE
	EffectiveDate	Fri, 26-Oct-18	Fri, 26-Oct-18
	MaturityDateOrTenor	1Y	1Y
	FixedRate (%)	-	-
	FloatSpread (Bps)	0.00	0.00
	IndexCompoundMethod	-	NONE
	SpreadCompoundMethod	-	NONE
	Leverage	1.00	1.00
	ForecastCurve	EUR3M	USD3M
	DiscountCurve	EURDF_USDCSA	USDDF
MTM SWAPS	isMTMResetLeg	FALSE	TRUE
	ResetBaseFX	1.00000	1.14030
	ValuationCurrency	USD	USD
COUPON & STUB CONVENTIONS	CouponRollDay	NATURAL	NATURAL
	isEndOfMonth	TRUE	TRUE
	StubType	SHORT_START	SHORT_START
	FrontStubCurveIndex	NATURAL	NATURAL
	BackStubCurveIndex	NATURAL	NATURAL
	FrontStubDate	-	-
	BackStubDate	-	-
SCHEDULE INFORMATION	AccrualFrequency	QUARTERLY	QUARTERLY
	AccrualCalendar	TGT+NY+LON	TGT+NY+LON
	AccrualBusDayConv	MOD_FOLLOWING	MOD_FOLLOWING
	AccrualDaycount	ACT/360	ACT/360
	IRFixingBusDayConv	MOD_FOLLOWING	MOD_FOLLOWING
	IRFixingCalendar	TGT+NY+LON	TGT+NY+LON
	IRFixingLag	2D	2D
	IRFirstFixingLag	-	-
	PaymentFrequency	QUARTERLY	QUARTERLY
	PaymentBusDayConv	MOD_FOLLOWING	MOD_FOLLOWING
	PaymentCalendar	TGT+NY+LON	TGT+NY+LON
	PaymentLag	2D	2D
NON-DELIVERABLES	IsNonDeliverable	FALSE	FALSE
	SettlementCurrency	-	-
	FXFixingLag	-	-
	FXFixingBusDayConv	-	-
	FXFixingCalendar	-	-

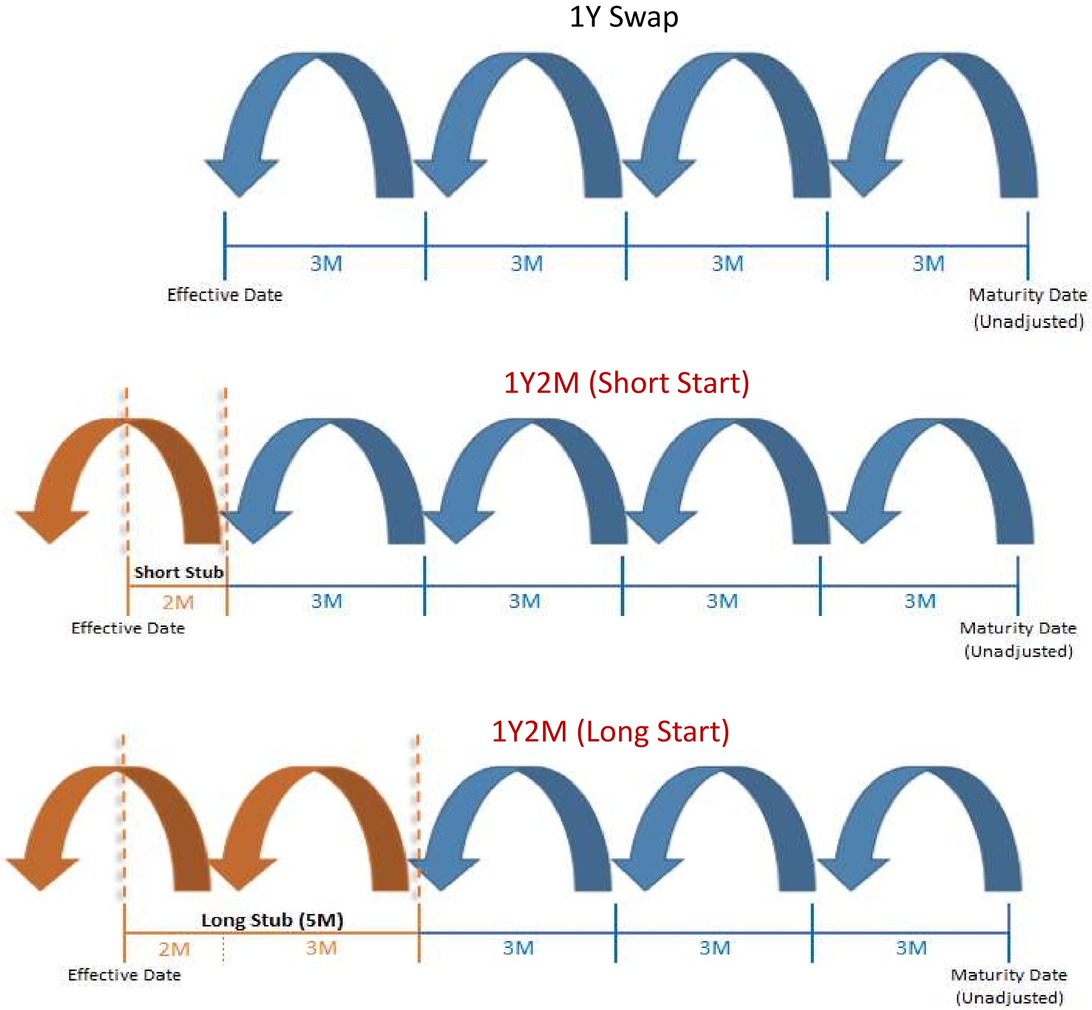
Interest Rate Swaps - Schedules & Stubs

Swap Schedules

- Backwards vs Forward Rolling Schedules
- Unadjusted to Preserve Roll Day
- Holiday Adjustments Ex-Ante
- Accrual Day Count Conventions

Broken-Dated Swaps

- Stubs & Stub Rates (Linear Interp)
- Short Start/End, Long Start/End
- Market Default: Short Start



IR Products – Tenor & Xccy Basis Swaps

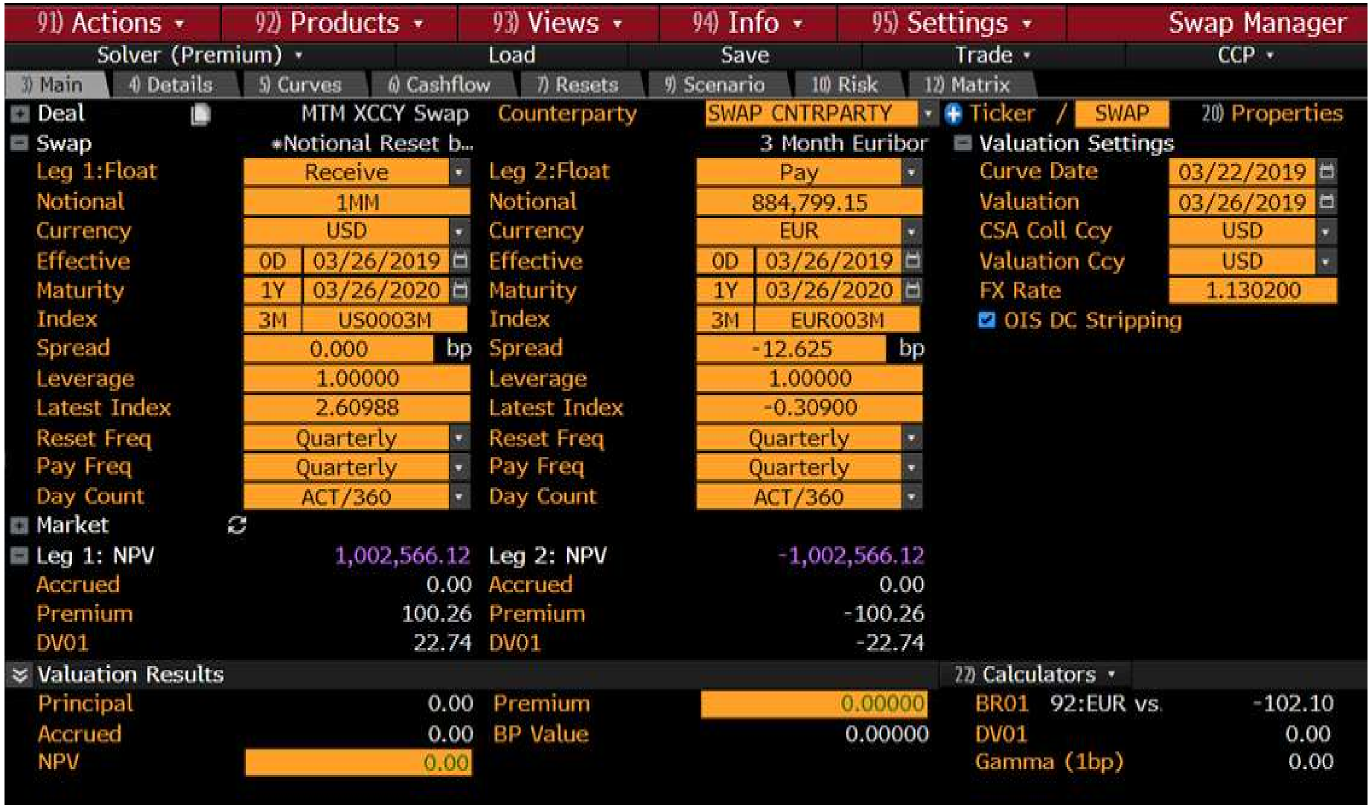
Tenor Basis Swaps

- Float vs Float (Same Currency)
- Exchange USD3M for USD6M say
- Match Project Cash Flow Frequency

Tenor Basis Swap Formulae (December 30, 2015).
Available at SSRN: https://ssrn.com/abstract=2959605

Xccy Basis Swaps

- Float vs Float (Different Currencies)
- Exchange USD3M for EUR3M say
- Marked-to-Market / FX Notional Resets
- Reduces XVA Costs

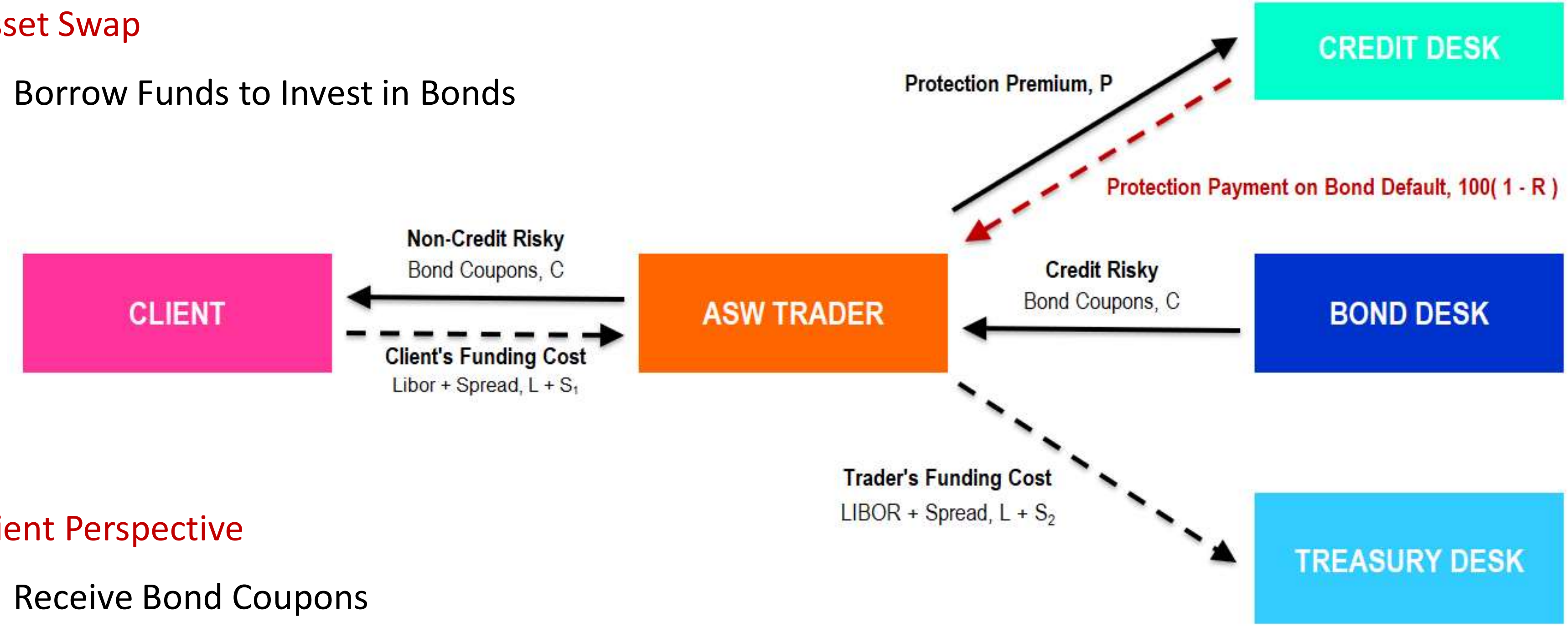


An Illustrated Step-by-Step Guide of How to Price Cross Currency Swaps (November 11, 2018).
Available at SSRN: https://ssrn.com/abstract=3278907

IR Products – Asset Swaps

Asset Swap

- Borrow Funds to Invest in Bonds



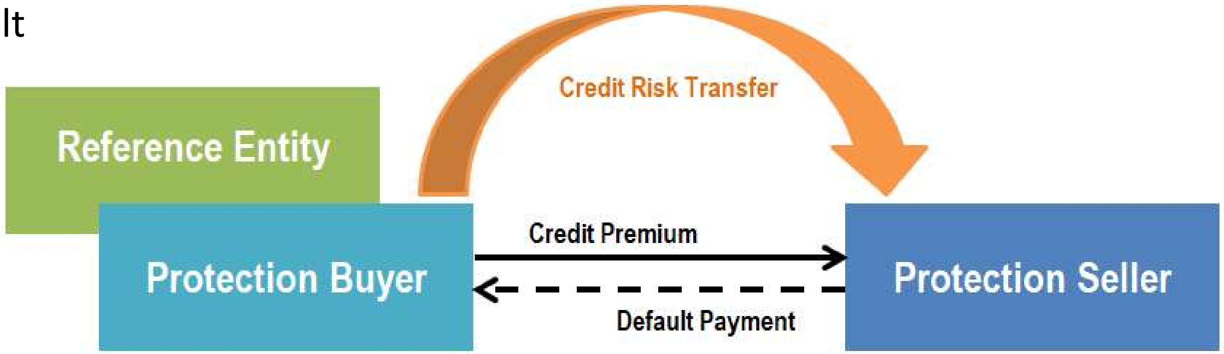
Client Perspective

- Receive Bond Coupons
- Pay LIBOR + Spread
- Spread Includes Finance + Credit Costs

IR Products – Credit Default Swaps (CDS)

Insurance Against Counterparty Default

- Insuring Bond Notional Invested
- Pay Fixed Insurance Premium
- Receive Protection Payment on Default



Credit Crisis & ISDA Big Bang (2008)

- Standardized & Cleared Contracts (IMM Dates[1])
- Increased Liquidity
- Accrued Interest, Clean & Dirty Prices

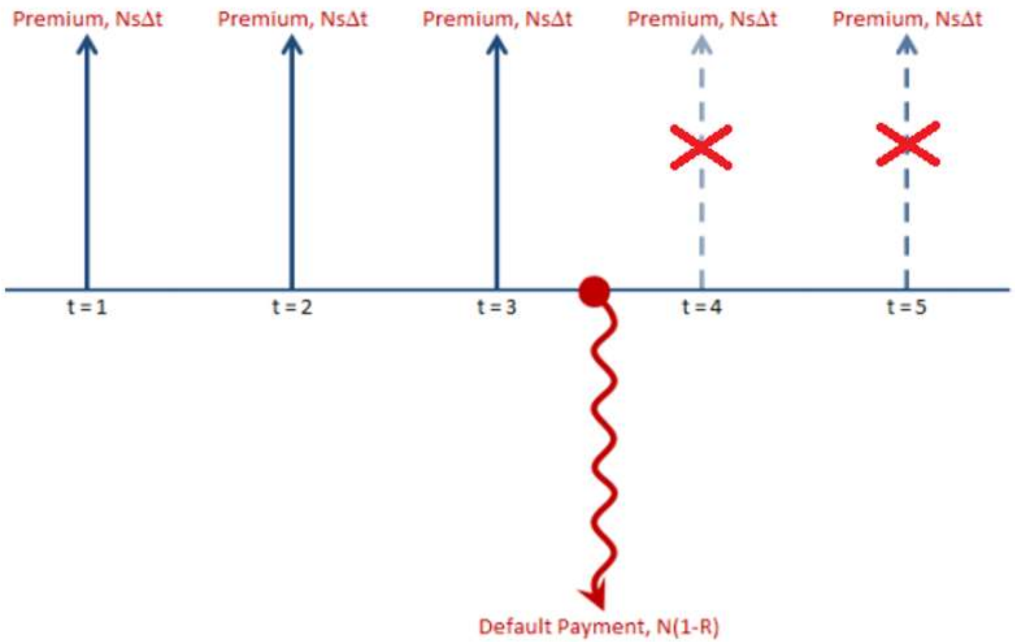
[1] Third Wednesday of Mar, June, Sep and Dec

IR Products – CDS Pricing

(Reference)

Pricing

- Similar to Interest Rate Swap Pricing
- With Additional Survival Probability Term, Q(t,T)
- $Q(t,T) = exp\left(-\int_t^T \lambda(t,u)du\right)$
- λ is the 'Hazard Rate' (instantaneous prob of default)



Buying Credit Protection

$$PV = PV(Protection\ Leg) - PV(Premium\ Leg)$$

$$PV(Premium\ Leg) = \sum_{i=1}^{n} \underbrace{N\ s\ \Delta(t_{i-1}, t_i)}_{Coupon}\ \underbrace{Q(t_i)}_{P(Survive)}\ \underbrace{P(t_0, t_i)}_{\substack{Discount\\Factor}}$$

$$PV(Protection\ Leg) = \sum_{i=1}^{n} \underbrace{N(1-R)}_{\substack{Loss\ Given\\Default}}\ \underbrace{[Q(t_{i-1}) - Q(t_i)]}_{\substack{Default\ within\\Premium\ Period}}\ \underbrace{P(t_0, t_i)}_{\substack{Discount\\Factor}}$$

IR Risk

What are the main IR risks?

- Discount Risk (DF01)
- Forward Risk (PV01)
- Discount + Forward Risk (DV01)

Risk Calculation Methods

- Analytical
- Numerical Risk (Benchmark)
- Using Yield Curve Jacobian
- Automatic Adjoint Differentiation (AAD)

USD SOFR YIELD CURVE - CALIBRATION INSTRUMENTS

Instrument	Term	Rate
USD SOFR Swap	ON	2.37000%
USD SOFR Swap	1W	2.36510%
USD SOFR Swap	2W	2.34960%
USD SOFR Swap	3W	2.35200%
USD SOFR Swap	1M	2.34550%
USD SOFR Swap	2M	2.30320%
USD SOFR Swap	3M	2.25590%
USD SOFR Swap	4M	2.19610%
USD SOFR Swap	5M	2.14750%
USD SOFR Swap	6M	2.10350%
USD SOFR Swap	1Y	1.89350%
USD SOFR Swap	2Y	1.68360%
USD SOFR Swap	3Y	1.62600%
USD SOFR Swap	4Y	1.61700%
USD SOFR Swap	5Y	1.64200%
USD SOFR Swap	6Y	1.67900%
USD SOFR Swap	7Y	1.71600%
USD SOFR Swap	8Y	1.75700%
USD SOFR Swap	9Y	1.79800%
USD SOFR Swap	10Y	1.83200%
USD SOFR Swap	15Y	1.96800%
USD SOFR Swap	20Y	2.03300%
USD SOFR Swap	25Y	2.04100%
USD SOFR Swap	30Y	2.04900%

Bucketed DV01, USD

Instrument	Tenor	DV01
USD SOFR Swap	ON	8
USD SOFR Swap	1W	0
USD SOFR Swap	2W	0
USD SOFR Swap	3W	0
USD SOFR Swap	1M	0
USD SOFR Swap	2M	0
USD SOFR Swap	3M	0
USD SOFR Swap	4M	0
USD SOFR Swap	5M	-1
USD SOFR Swap	6M	1
USD SOFR Swap	1Y	92
USD SOFR Swap	2Y	213
USD SOFR Swap	3Y	294
USD SOFR Swap	4Y	409
USD SOFR Swap	5Y	453
USD SOFR Swap	6Y	541
USD SOFR Swap	7Y	723
USD SOFR Swap	8Y	736
USD SOFR Swap	9Y	852
USD SOFR Swap	10Y	892
USD SOFR Swap	15Y	1,320
USD SOFR Swap	20Y	1,662
USD SOFR Swap	25Y	1,979
USD SOFR Swap	30Y	2,252
Total Risk		**12,428**

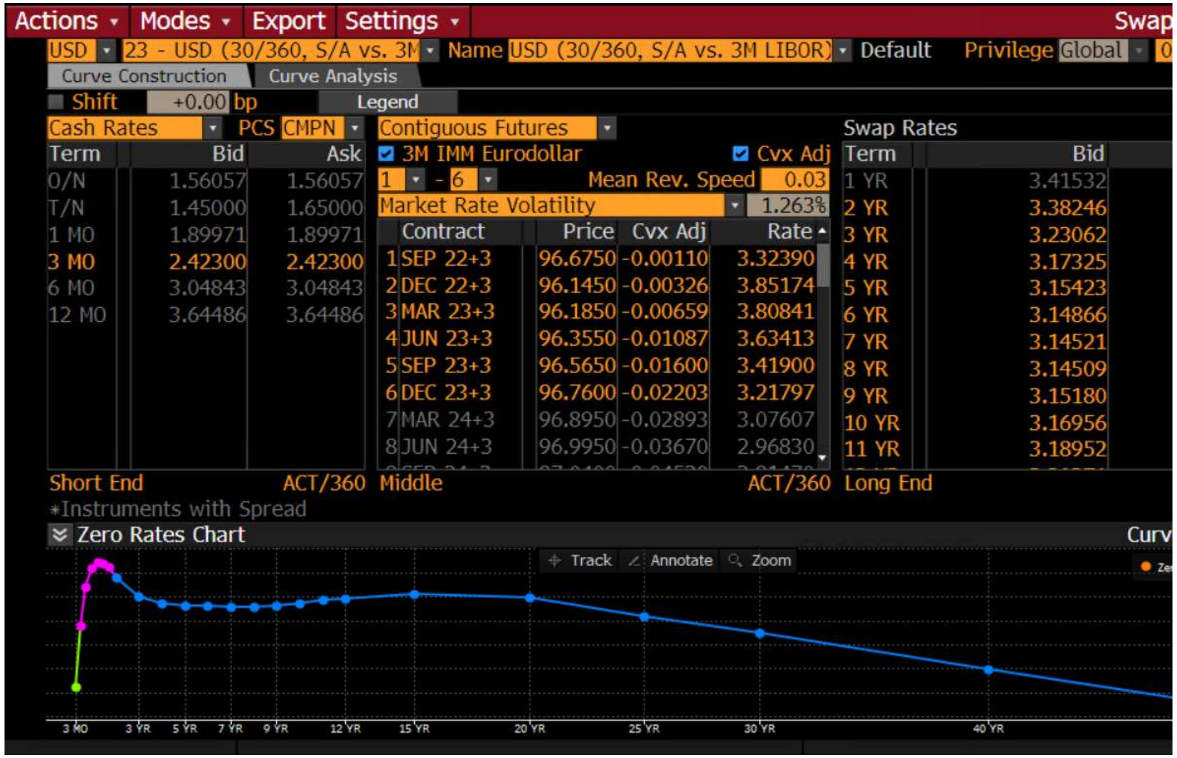
Yield Curves - Calibration

Model Inputs & Outputs

- Liquid Market Instrument Quotes [IN]
- Forward Rates [OUT]
- Discount Factors [OUT]

Calibration Process

- Choose State Variable[1]
- Choose Interpolator (Functional Form)
- Solve and Imply Forwards & Disc Factors[2]



[1] Popular choices: forward rate, disc factor, logDF, zero rate etc.

[2] May need to differentiate and/or integrate state variable, $P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$

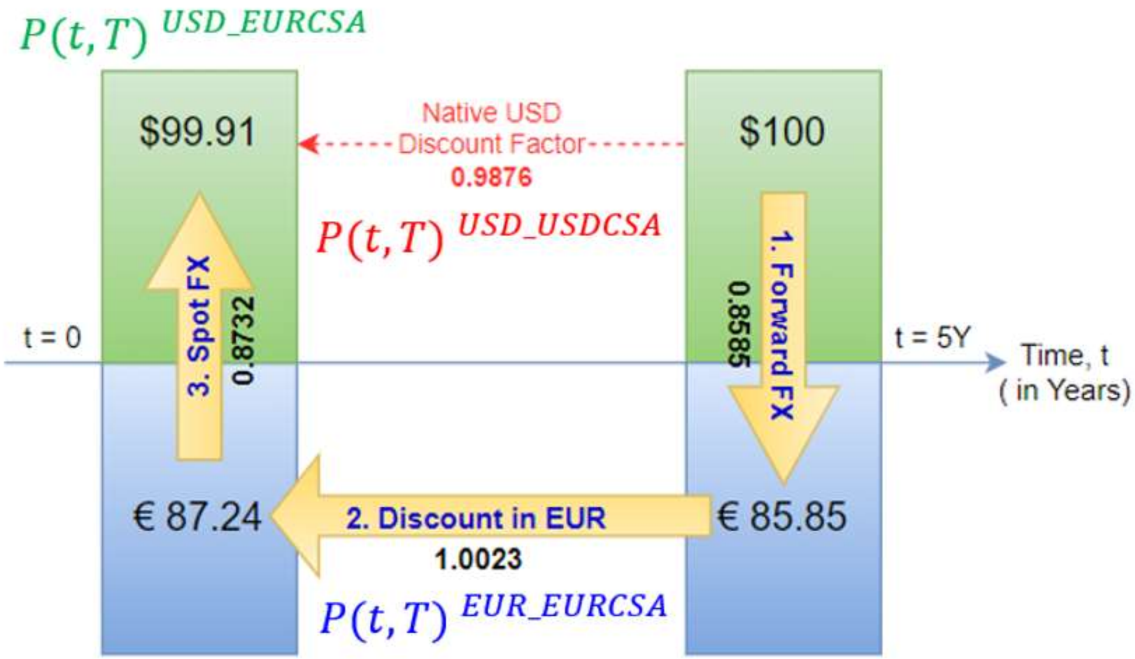
Yield Curves – Collateral & CSA Curves

Collateral & CSA Curves

- Calibrate to FX Forwards & Xccy Swaps
- FX Forward Invariance (FX Carry Trade)
- Impacts Discount Factors Only
- No Impact on Forward Rates

Advanced CSA Topics

- Cheapest to Deliver (Multiple CSAs)
- Collateral Switch Options

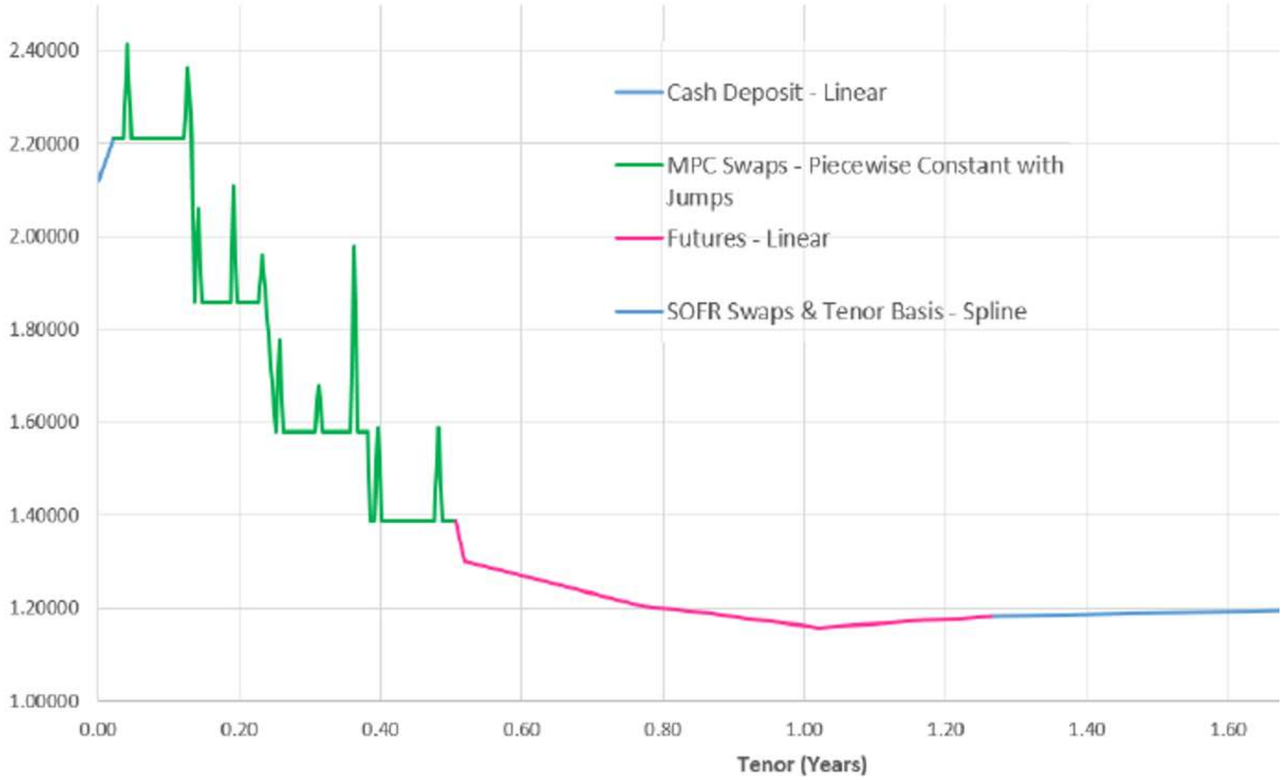


$$f(t,T)^{USD/EUR} = s(t)^{USD/EUR} \underbrace{\left(\frac{P(t,T)^{EUR\_USDCSA}}{P(t,T)^{USD\_USDCSA}}\right)}_{USD\ CSA} = s(t)^{USD/EUR} \underbrace{\left(\frac{P(t,T)^{EUR\_EURCSA}}{P(t,T)^{USD\_EURCSA}}\right)}_{EUR\ CSA}$$

Yield Curves - Features

Curve Features & Considerations

- Underlying Instrument Behaviour
- Mixed Interpolation Schemes
- Turn-of-Year Effects (ToYs)



Advanced Features for Electronic Markets

- Curve Jacobian
- Ultra-Fast Curves & Analytical Risk
- Automatic Adjoint Differentiation (AAD)

Yield Curves – Curve Jacobian

Electronic HFT Usage

- Ultra-Fast Rebuilds
- Real-Time Risk
- Auto-Hedging

Inverse Curve Jacobian, dL/dP

Forward Pillars	Curve Calibration Instruments									
	dP_{1Y}^{OIS}	dP_{2Y}^{OIS}	dP_{3Y}^{OIS}	dP_{4Y}^{OIS}	dP_{5Y}^{OIS}	dP_{1Y}^{IRS}	dP_{2Y}^{IRS}	dP_{3Y}^{IRS}	dP_{4Y}^{IRS}	dP_{5Y}^{IRS}
dO_{1Y}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{2Y}	-1.01	2.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{3Y}	0.00	-2.04	3.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{4Y}	0.00	0.00	-3.08	4.08	0.00	0.00	0.00	0.00	0.00	0.00
dO_{5Y}	0.00	0.00	0.00	-4.13	5.13	0.00	0.00	0.00	0.00	0.00
dL_{1Y}	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
dL_{2Y}	0.00	0.00	0.00	0.00	0.00	-1.01	2.01	0.00	0.00	0.00
dL_{3Y}	0.00	0.00	0.00	0.00	0.00	0.00	-2.04	3.04	0.00	0.00
dL_{4Y}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-3.08	4.08	0.00
dL_{5Y}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-4.13	5.13

By-Product of Calibration Process

- Measures Changes in Market Instrument Quotes (P) on Forward Rates (L)
- First Order Derivative Matrix, dP/dL (Inverse Required)
- Controls Hedge and Risk Buckets (Same as Numerical Bumping)
- Use Implicit Function Theorem (IFT) to modify Risk Buckets (see Appendix)

Yield Curves – Ultra-Fast Rebuilds

New Forwards

$$L_{New} = L_{Old} + dL$$

$$= L_{Old} + \left({}^{dL}/_{dP}\right).\, dP$$

New Forwards

	L_{NEW}
L_{1Y}^{OIS}	1.44591%
L_{2Y}^{OIS}	1.24323%
L_{3Y}^{OIS}	1.26107%
L_{4Y}^{OIS}	1.30130%
L_{5Y}^{OIS}	1.40782%
L_{1Y}^{IRS}	1.71896%
L_{2Y}^{IRS}	1.48359%
L_{3Y}^{IRS}	1.50531%
L_{4Y}^{IRS}	1.56934%
L_{5Y}^{IRS}	1.63999%

=

Original Forwards

	L_{OLD}
L_{1Y}^{OIS}	1.43591%
L_{2Y}^{OIS}	1.23323%
L_{3Y}^{OIS}	1.25107%
L_{4Y}^{OIS}	1.29130%
L_{5Y}^{OIS}	1.39782%
L_{1Y}^{IRS}	1.70896%
L_{2Y}^{IRS}	1.47359%
L_{3Y}^{IRS}	1.49531%
L_{4Y}^{IRS}	1.55934%
L_{5Y}^{IRS}	1.62999%

+

Inverse Jacobian, dL/dP

	OIS_{1Y}	OIS_{2Y}	OIS_{3Y}	OIS_{4Y}	OIS_{5Y}	IRS_{1Y}	IRS_{2Y}	IRS_{3Y}	IRS_{4Y}	IRS_{5Y}
L_{1Y}^{OIS}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
L_{2Y}^{OIS}	-1.01	2.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
L_{3Y}^{OIS}	0.00	-2.04	3.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
L_{4Y}^{OIS}	0.00	0.00	-3.08	4.08	0.00	0.00	0.00	0.00	0.00	0.00
L_{5Y}^{OIS}	0.00	0.00	0.00	-4.13	5.13	0.00	0.00	0.00	0.00	0.00
L_{1Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
L_{2Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	-1.01	2.01	0.00	0.00	0.00
L_{3Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	0.00	-2.04	3.04	0.00	0.00
L_{4Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-3.08	4.08	0.00
L_{5Y}^{IRS}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-4.13	5.13

×

Change in Mkt Data

	dP
L_{1Y}^{OIS}	0.01%
L_{2Y}^{OIS}	0.01%
L_{3Y}^{OIS}	0.01%
L_{4Y}^{OIS}	0.01%
L_{5Y}^{OIS}	0.01%
L_{1Y}^{IRS}	0.01%
L_{2Y}^{IRS}	0.01%
L_{3Y}^{IRS}	0.01%
L_{4Y}^{IRS}	0.01%
L_{5Y}^{IRS}	0.01%

Implementation

- Slow Curve (Full-Rebuild) Ticks in Background (ca. 10ms)
- Fast Curve (Jacobian Method) Used Between Refreshes (Real-Time)

Yield Curves – Real-Time Bucketed Risk

Requirements

- Curve Jacobian
- Trade or Portfolio Jacobian

$$DV01(Analytical) = 1bps \times \underbrace{\frac{dPV}{dL}}_{\substack{Pricing\\Jacobian}} \times \underbrace{\frac{dL}{dP}}_{\substack{Curve\\Jacobian}}$$

Risk as a Matrix Operation

- Can be Parallelized / Vectorized
- Matrix Dimensions Must Agree
- Interpolation & Forward Mapping
- Barycentric Weights, $w_j(t)$

$$p(t) = \sum_{j=0}^{n} w_j(t) f(t_j), \qquad w_j(t) = \frac{\prod_{k=0,\, k\neq j}^{n}(t - t_k)}{\prod_{k=0,\, k\neq j}^{n}(t_j - t_k)}$$

Inverse Curve Jacobian, dL/dP

Curve Calibration Instruments

Forward Pillars	dP_{1Y}^{OIS}	dP_{2Y}^{OIS}	dP_{3Y}^{OIS}	dP_{4Y}^{OIS}	dP_{5Y}^{OIS}	dP_{1Y}^{IRS}	dP_{2Y}^{IRS}	dP_{3Y}^{IRS}	dP_{4Y}^{IRS}	dP_{5Y}^{IRS}
dO_{1Y}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{2Y}	-1.01	2.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{3Y}	0.00	-2.04	3.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
dO_{4Y}	0.00	0.00	-3.08	4.08	0.00	0.00	0.00	0.00	0.00	0.00
dO_{5Y}	0.00	0.00	0.00	-4.13	5.13	0.00	0.00	0.00	0.00	0.00
dL_{1Y}	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
dL_{2Y}	0.00	0.00	0.00	0.00	0.00	-1.01	2.01	0.00	0.00	0.00
dL_{3Y}	0.00	0.00	0.00	0.00	0.00	0.00	-2.04	3.04	0.00	0.00
dL_{4Y}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-3.08	4.08	0.00
dL_{5Y}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-4.13	5.13

×

Trade Jacobian, dPV/dL

3Y Par Swap

Forward Pillars

Trade	OIS Curve (Discount Risk)					Swap Curve (Forward Risk)				
	dO_{1Y}	dO_{2Y}	dO_{3Y}	dO_{4Y}	dO_{5Y}	dL_{1Y}	dL_{2Y}	dL_{3Y}	dL_{4Y}	dL_{5Y}
dS_{3Y}^{IRS}	0	0	0	0	0	98	97	96	0	0

=

Risk Bucket	Trade: IRS 3Y
OIS 1Y	0
OIS 2Y	0
OIS 3Y	0
OIS 4Y	-1
OIS 5Y	1
IRS 1Y	0
IRS 2Y	0
IRS 3Y	291
IRS 4Y	0
IRS 5Y	0

Total Trade DV01

IRS 3Y
291

Yield Curves – Automatic Adjoint Differentiation (AAD)

Trade Jacobian

- AAD Can Compute Instrument Price & Risk Simultaneously
- Direct Differentiation of Code + Implicit Function Theorem (IFT)
- Exact & Fast (X4 Pricing Time)

Tangent & Adjoint Modes

- Tangent Mode (dot) : Forward Mode - One Risk at a Time
- Adjoint Mode (bar) : Backward Mode - All Risks Simultaneously
- Activation Inputs Control Risk Outputs

Implementation Methods

- By Hand (See Appendix for Swap DV01 Risk Example)
- Derivative Code by Overloading, DCO/C++
- Professional Tools: Adept, NAG

Pricing Calculations

$$x \rightarrow f(x) \rightarrow g(f) \rightarrow h(g) \rightarrow y$$

Chain Rule: Forwards

$$\frac{\mathrm{d}f}{\mathrm{d}x} \cdot \frac{\mathrm{d}g}{\mathrm{d}f} \cdot \frac{\mathrm{d}h}{\mathrm{d}g} \cdot \frac{\mathrm{d}y}{\mathrm{d}h} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Chain Rule: Backwards

$$\frac{\mathrm{d}y}{\mathrm{d}h} \cdot \frac{\mathrm{d}h}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}f} \cdot \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Yield Curves – AD Tangent Mode Example

Tangent Mode

- Differentiate Forwards using ‘Dot’ Notation
- One Risk at a Time, Controlled by Dot Input Activation Variables 1 or 0
- For $\frac{df}{dx_1}$ and $\frac{df}{dx_2}$ must call tangent method twice

```
tangent(2.0, 3.0, 1.0, 0.0);    // Input: x1 = 2, x2 = 3, x1_d = 1, x2_d = 0    Output: 8
tangent(2.0, 3.0, 0.0, 1.0);    // Input: x1 = 2, x2 = 3, x1_d = 0, x2_d = 1    Output: 3
```

Function Derivatives using Tangent Mode

```
double tangent( double x1, double x2, double x1_dot, double x2_dot )
{
    double a = x1*x1;               // Step 1:    $a = x_1^2$
    double a_dot = 2*x1*x1_dot;     // Tangent:   $\dot{a} = 2x_1 \cdot \dot{x}_1$    $\dot{a} = 2x_1$
    double b = 2*a;                 // Step 2:    $b = a$
    double b_dot = 2*a_dot;         // Tangent:   $\dot{b} = 2 \cdot \dot{a}$    $\dot{b} = 4x_1$
    double c = x2;                  // Step 3:    $c = x_2$
    double c_dot = x2_dot;          // Tangent:   $\dot{c} = \dot{x}_2$    $\dot{c} = 1$
    double d = 3*c;                 // Step 4:    $d = 3c$
    double d_dot = 3*c_dot;         // Tangent:   $\dot{d} = 3 \cdot \dot{c}$    $\dot{d} = 3$
    double f = b + d;               // Step 5:    $f = 2x_1^2 + 3x_2$
    double f_dot = b_dot + d_dot;   // Tangent:   $\dot{f} = \dot{b} + \dot{d}$
    return f_dot;                   // Result:    $\dot{f} = 4x_1 + 3$
}
```

Simple Function $f(x_1, x_x) = 2x_1^2 + 3x_2$ with Tangent Derivatives

```
double function( double x1, double x2 )
{
    double a = x1*x1;     // Step 1:    $a = x_1^2$
    double b = 2*a;       // Step 2:    $b = 2x_1^2$
    double c = x2;        // Step 3:    $c = x_2$
    double d = 3*c;       // Step 4:    $d = 3x_2$
    double f = b + d;     // Step 5:    $f = 2x_1^2 + 3x_2$
    return f;
}
```

Simple Function: $f(x_1, x_x) = 2x_1^2 + 3x_2$

Source Code: https://onlinegdb.com/kKqaS6hJT

Yield Curves – AD Adjoint Mode Example

Adjoint Mode (Reverse Mode)

- Backwards Differentiation with ‘Bar’ Notation
- Forward Sweep then Back Propagate Risk
- Computes All Risks at Same Time
- Risk Controlled By Bar Input Activation Variable 1 or 0
- Adjoint Method Calculates Both $\frac{df}{dx_1}$ and $\frac{df}{dx_2}$

```
double function( double x1, double x2 )
{
    double a = x1*x1;        // Step 1:    a = x1^2
    double b = 2*a;          // Step 2:    b = 2x1^2
    double c = x2;           // Step 3:    c = x2
    double d = 3*c;          // Step 4:    d = 3x2
    double f = b + d;        // Step 5:    f = 2x1^2 + 3x2
    return f;
}
```

Simple Function: $f(x_1, x_x) = 2x_1^2 + 3x_2$

```
adjoint(2.0, 3.0, 1.0);  // Input: x1 = 3, x2 = 2, f_bar   Output: df/dx1 = 8 and df/dx2 = 3
```

Function Derivatives using Adjoint Mode

```
void adjoint( double x1, double x2, double f_bar )
{
    // Forward Sweep
    double a = x1*x1;                 // Step 1:    a = x1^2
    double b = 2*a;                   // Step 2:    b = 2x1^2
    double c = x2;                    // Step 3:    c = x2
    double d = 3*c;                   // Step 4:    d = 3x2
    double f = b + d;                 // Step 5:    f = 2x1^2 + 3x2

    // Back Propagation
    double b_bar = f_bar;             // Step 5:    b_bar  = 1      from input variable
    double d_bar = f_bar;             // Step 5:    d_bar  = 1      from input variable
    double c_bar = 3*d_bar;           // Step 4:    c_bar  = 3
    double x2_bar = c_bar;            // Step 3:    x2_bar = 3      df/dx2 = 3
    double a_bar = 2*b_bar;           // Step 2:    a_bar  = 2
    double x1_bar = 2*x1*a_bar;       // Step 1:    x1_bar = 4x1    df/dx1 = 4x1

    // Display Results
    std::cout << "df/dx1: " << x1_bar << std::endl;      // x̄1 = df/dx1 = 4x1
    std::cout << "df/dx2: " << x2_bar << std::endl;      // x̄2 = df/dx2 = 3
}
```

Simple Function $f(x_1, x_x) = 2x_1^2 + 3x_2$ **with Adjoint Derivatives**

Source Code: https://onlinegdb.com/kKqaS6hJT

Credit Models – Hazard Rates & Survival Probabilities

Calibration Summary

- Yield Curve is an Input
- Calibrate to Bonds or CDS
- Imply Hazard Rates, λ
- Used for Survival Prob, Q(t,T)

Common Assumptions

- Piecewise Constant[1]
- Deterministic Hazard Rates

Rule of Thumb

$$\lambda = \frac{S}{(1-R)}$$

$$Q(t,T) = exp\left(-\int_t^T \lambda(t,u)du\right)$$

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

[1] As often there is only a single calibration instrument

PART TWO – PRICING & PRACTICE

Interest Rate Swap – Annuity is the Key Pricing & Risk Factor

It's All About Annuity

- Pricing & Risk Expressed in Terms of Annuity
- Similarly Float Legs Expressed in Annuity Terms
- Can Be Used to Convert a Float Leg to Fixed Leg
- Useful for Low Latency Pricing

$$A_N = N\sum_{i=1}^{n}\tau_i P(t_0, t_i)$$

Low Latency Interest Rate Swap Pricing

Electronic Rates Markets & Low Latency Interest Rate Swap Calculations (May 31, 2022).
Available at SSRN: https://ssrn.com/abstract=4125565

Key Formulae:

- PV = (r – p) Annuity(Fixed)
- Par Rate = PV(Float) / Annuity(Fixed)
- PV01 = Annuity(Fixed) x 0.01%
- DV01 = PV01 + DF01 = PV01 for Par Swaps

$$Swap\ PV = PV^{Fixed\ Leg} - PV^{Float\ Leg}$$

$$= r\sum_{i=1}^{n} N_i\tau_i P(t_0, t_i) - \sum_{j=1}^{m} N_j l_{j-1}\tau_j P(t_0, t_j)$$

$$= (r-p)A_{Fixed}$$

Interest Rate Swap – Pricing & Risk Example

Compute Annuity A_N

= USD 4,863,971.74

$$A_N = N \sum_{i=1}^{n} \tau_i P(t_0, t_i)$$

PV = (r – p) A_N

= (5.00% - 1.59%) A_N

= USD 167,892.11

Par Rate = PV(Float) / A_N

= 75,306 / A_N

= 1.5482%

PV01

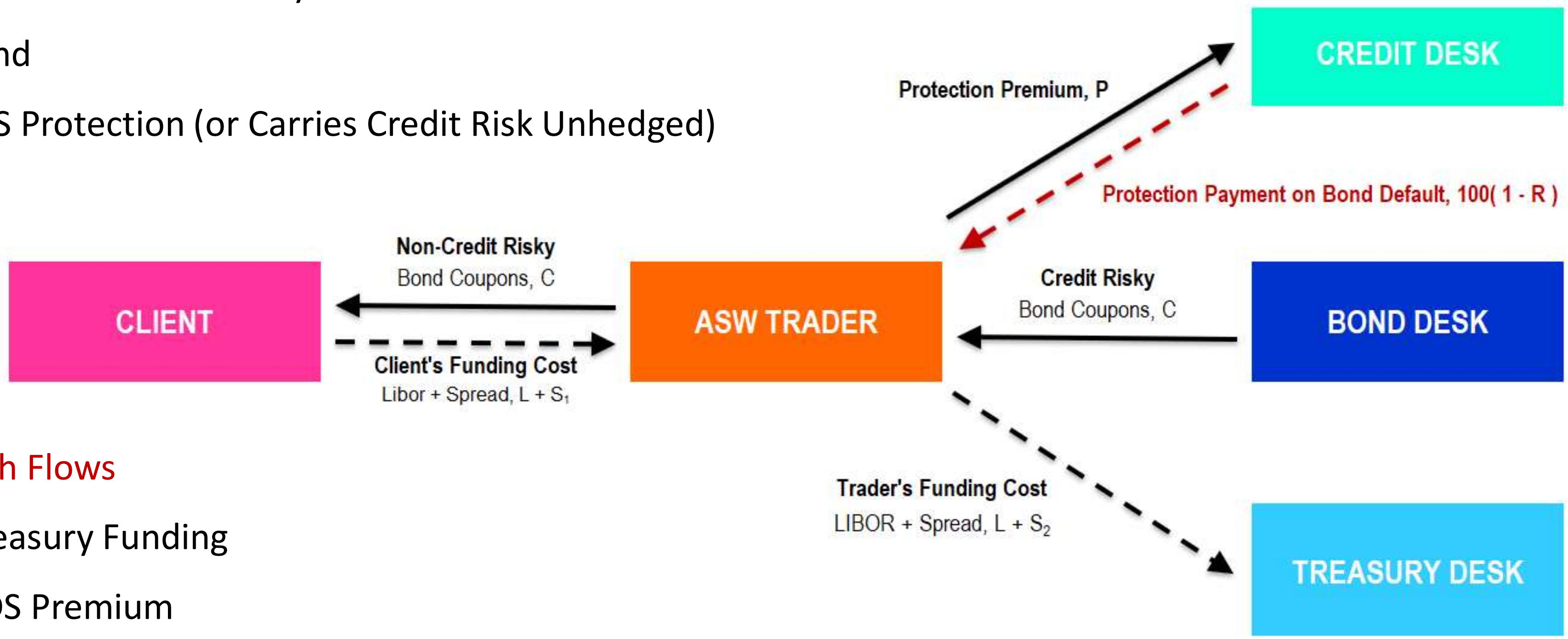
= $A_N \times 0.01\%$

= USD 486.40

Asset Swap – Structuring the Asset Swap Spread

Trader Creates Synthetic Asset Swap

- Borrow Cash from Treasury to Purchase Bond
- Buy Bond
- Buy CDS Protection (or Carries Credit Risk Unhedged)



Trader Cash Flows

- Pays Treasury Funding
- Pays CDS Premium
- Receives Bond Coupons and Passes on to Client
- Client Pays All Costs + Commission as a Spread over LIBOR (or RFR)

Asset Swap – Pricing as a Spread Over LIBOR (or RFR)

- ASW Spread - Par-Par Spread
- MMS Spread - Yield-Yield Spread[1]

[1] Y/Y Spread Between Swap Rate and Benchmark Gov't Bond Yield

Asset Swap – Pricing using Par-Par Method

Pricing as a PV

- Valuation Method for Existing Swaps, Unwinds and Novations (trade transfers)
- Again Present Value is Simply the Sum of Incoming and Outgoing Cash Flows
- An Upfront Par-Adjustment is Made if the Underlying Bond not Trading at Par, i.e., 100

$$PV^{Asset\ Swap} = \underbrace{\phi \mathrm{r}^{Fixed} \sum_{i=1}^{n} N_i \tau_i P(t_0, t_i)}_{Fixed\ Leg} - \underbrace{\phi \sum_{j=1}^{m} N_j \left(l_{j-1} + s\right) \tau_j P\left(t_0, t_j\right)}_{Float\ Letg} + \underbrace{\phi \mathrm{N}_1 \left(\frac{100 - B}{100}\right)}_{Par\ Adjustment}$$

Pricing as a Par Spread

- New Asset Swaps Price to Par i.e., zero
- Instead Quote as a Par Spread s
- Rearrangement of PV formula with PV=0

$$s = \left(\frac{\left(\mathrm{r}^{Fixed} - p^{Market}\right) A^{Fixed} + \left(\frac{100 - B}{100}\right)}{A^{Float}}\right)$$

Pricing Tricks & Rules of Thumb

Annuity Assumption

- Need to know market par rates for standard swap maturities
- Assume Annual Coupons and Discount Factors = 1.0
- This means Annuity = Time to Maturity
- Used to Gain Intuition when Pricing IR Products & CDS

Interest Rate Swap PV

- $PV = N \, (r - p) \, A(Fixed)$

Approximate PV

- $PV = N \, \Delta r \, T$

This gives PV as USD 100 per Million per Year per Δr in bps

IRS Rule of Thumb – Multiples of a Base Case

- $PV = 100 \times \Delta N \times \Delta r \times \Delta T$
- $DV01 = PV01 = 100 \times \Delta N \times \Delta T$

Pricing Tricks – Interest Rate Swap

PV as USD 100 per Million per Year per Δr in bps

IRS – Rule of Thumb

- PV = 100 x ΔN x Δr in bps x ΔT
- DV01 = PV01 = 100 x ΔN x ΔT

Market Par Rate

- 5Y Par Rate = 150 bps
- Δr = (r-p) = (500-150) = 350 bps

Present Value

- Here ΔN = 1, Δr = 350, ΔT = 5
- PV = USD 175,000
- DV01 = PV01 = USD 500

PV = 100 x 1 x 350 x 5 = USD 170K

DV01 = 100 x 1 x 5 = USD 500

Pricing Tricks – Asset Swaps

We Can Make the Same Annuity Assumption to Price Asset Swaps

Par-Par Spread

$$s = \left(\frac{\left(\mathrm{r}^{Fixed} - p^{Market}\right)A^{Fixed} + \left(\frac{100 - B}{100}\right)}{A^{Float}}\right)$$

IRS Rule of Thumb

s = (r – p) + (100-B/100) / T

= Δr – (ΔB / T)

where Δr = (r – p) in bps

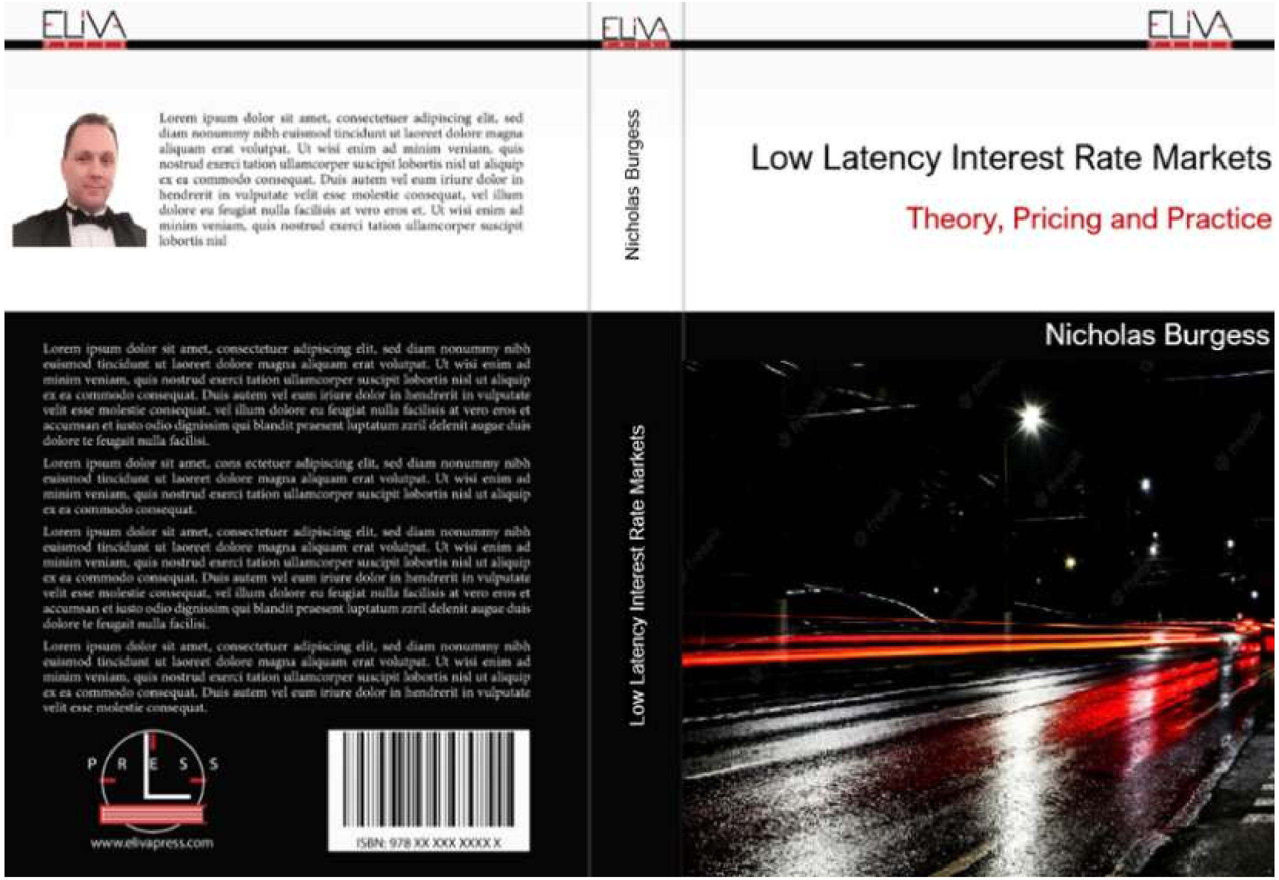
and ΔB = (B% - 100%) in bps

Par-Par Spread

Here Δr = 0.50% – 0.44% = 6 bps, ΔB = 458 bps and T = 10

S = 6 – (458/10) ≈ 6 - 46 = -40 bps

References



Quant Research Papers
https://ssrn.com/author=1728976

Support Materials, C++ & Excel Examples
https://github.com/nburgessx/SwapsBook

Appendix – Implicit Function Theorem (IFT)

IFT Theorem

To gain some intuition consider the following function $f(x,y) = 0$ for which we have a solution (a,b). Near the solution we can express y as function of x namely $f(x, y(x)) = 0$. Using this expression, we can compute the derivative in terms of x only by differentiating with respect to x as follows,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0$$

which gives,

$$\frac{\partial y}{\partial x} = -\left(\frac{\partial f/_{\partial x}}{\partial f/_{\partial y}}\right)$$

We have a solution under the condition, ${}^{\partial f}/_{\partial y} \neq 0$, since we cannot divide by zero.

Yield Curve Application

In the context of a yield curve calibration, we solve for the solution of a helper target function, $H(L,P) = 0$, where L is the LIBOR forward rate state variable (model output) and P the yield curve par rate (model input). The helper target function computes the difference between model par rates as a function of the forward state variable L and a market instrument par rate quote,

$$H(L,P) = Model\ Par\ Rate(L) - Market\ Par\ Rate$$

How does this Help with Sensitivity Calculations?

The IFT theorem says that having found a solution to the continuously differentiable function $H(L,P) = 0$ in two variables we can express the solution solely in terms of the model output L namely $H(L, P(L)) = 0$ and that the Jacobian derivative can be computed independent of model inputs i.e., the yield curve instruments and par rates as,

$$\frac{\partial P}{\partial L} = -\left(\frac{\partial H/_{\partial L}}{\partial H/_{\partial P}}\right)$$

Now, from the definition of the function $H(L,P)$ we can easily determine $dH/dP = -1$ which leads to,

$$\frac{\partial P}{\partial L} = -\left(\frac{\partial H/_{\partial L}}{\partial H/_{\partial P}}\right) = \frac{\partial H}{\partial L}$$

$$= \frac{d}{dL}(Model\ Par\ Rate)$$

For an Interest Rate Swap

$$Par\ Rate, p = \frac{PV(Float\ Leg)}{\sum_{i=1}^{n} N\ \tau_i\ P(t_0, t_i)} = \frac{\sum_{j=1}^{m} N\left(\ l_{j-1} + s\right)\tau_j\ P\left(t_0, t_j\right)}{Annuity(Fixed)}$$

- The derivative with respect to L is trivial to calculate
- We can calculate for any set of calibration instruments
- This allows us to modify and select any risk & hedge buckets

Appendix – Swap DV01 Risk Example using AAD (Part I)

IRS Present Value Code

- Swap Price Implementation
- Simplified for Demo Purposes
- For Full Example See

  https://bit.ly/SwapCodeAAD

```
// Swap Inputs
//   phi      Pay or Receive Fixed: Pay = 1,  Receive = -1
//   n        Swap Notional
//   r        Fixed rate
//   tau      Accrual year fraction
//   t        Coupon Payment Time
//   f        Floating Forward Rate
//   s        Floating Spread
//   z        Discounting Zero Rate for Discount Factor, where df = exp(-z*t)

double swap_pv(double phi, double n, double r, double tau, double t, double f, double s,
double z)
{
      double df           = exp(-z*t);               // Step 1.  Discount Factor using zero rate, z
      double pv_fixed = phi*n*r*tau*df;          // Step 2.  Fixed PV = φ N r τ1 P(0, t1)
      double pv_float  = -phi*n*(f+s)*tau*df;  // Step 3.  Float PV = φ N(l1 + s)τ1 P(0, t1)
      double pv_swap = pv_fixed+pv_float;      // Step 4.  Swap PV = Fixed PV + Float PV
      return pv_swap;
}
```

Swap Price

Appendix – Swap DV01 Risk Example using AAD (Part II)

Analytical DV01 Risk

- Using Adjoint Mode (AAD)
- Forward Sweep for Price
- Back Propagation for Risk
- Simultaneous Forward and Discount Risk

Source Code: https://www.onlinegdb.com/edit/al8aNASJnQ

```
double adjoint(double phi, double n, double r, double tau, double t, double f, double s,  double z,
double pv_bar)
{
    // Forward Sweep
    double df              = exp(-z*t);             // Step 1.  Discount Factor using zero rate, z
    double pv_fixed        = phi*n*r*tau*df;        // Step 2.  Fixed PV = ϕ N r τ_1 P(0,t_1 )
    double pv_float        = -phi*n*(f+s)*tau*df;   // Step 3.  Float PV = ϕ N(l_1+s) τ_1 P(0,t_1 )
    double pv_swap         = pv_fixed+pv_float;     // Step 4.  Swap PV = Fixed PV + Float PV

    // Backward Propagation
    double pv_fixed_bar    = pv_bar;                                        // Step 4.
    double pv_float_bar    = pv_bar;                                        // Step 4.
    double f_bar           = -phi*n*tau*df*pv_float_bar*shift_size_f;       // Step 3. *
    double df_bar          = -phi*n*f*tau*pv_float_bar*shift_size_df;       // Step 3. *
    df_bar                 += phi*n*r*tau*pv_fixed_bar*shift_size_df;       // Step 2. *
    double z_bar           = -t*exp(-z*t)*df_bar;                           // Step 1.

    // DV01 Result
    return f_bar + df_bar;  // Sensitivity to 1 bps change in forwards and discount factors
}
```

Swap DV01 using AD in Adjoint Mode

```
// inputs( phi, n, r, tau, t, f, s, z, pv_bar )
adjoint( 1, 1000000, 0.02, 1, 1, 0.01, 0, 0.02, 1 );  // Output DV01 Risk
```

Swap DV01 Risk using Adjoint Mode

Have questions or want further info?

Contact

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