LEAST SQUARES MONTE CARLO

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Outline

1. Review
   - Last time...
   - Today's lecture

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2. MONTE CARLO METHODS FOR AMERICAN OPTIONS
   - The least squares method
   - Longstaff and Schwartz (2001)

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3. Longstaff and Schwartz - general procedure
   - The Algorithm
   - Performance

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Monte Carlo Extensions

- We have looked through a variety of extensions to the standard Monte Carlo in an effort to reduce the variance of the error or to improve the convergence.
- Most of the improvements are simple to apply such as antithetic variables and moment matching, others are more complex such as low discrepancy sequences.

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- We have looked at using Monte Carlo methods for most types of options, but they struggle with early exercise
- This is the subject of lots of research at the moment and we have seen the basic idea of some of the methods.
- What we present here is one of the easiest methods to understand and implement.
- There are still issues with how it performs in practise which we will also deal with here.

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THE KEY ISSUE:

How to find what is the expected value for continuation?

- The continuation value is the discounted expected option value at the next instance in time.

Outline

- This is a technique for fitting a set of functions to (given) data.
- Here we describe the procedure for an mth degree polynomial - it is straightforward to extend the idea to a general class of polynomial
- When using an mth degree polynomial

$$y = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m$$

to approximate the given set of data, (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), where $n \geq 3$, the best fitting curve $f(x)$ has the least square error, i.e.,

$$\Pi = \min(\sum_{i=1}^{n} [y_i - f(x_i)]^2) = \min(\sum_{i=1}^{n} [y_i - (a_0 + ... + a_m x_i^m)]^2)$$

- Note that a_0, a_1, a_2, ..., and a_m are unknown coefficients while all x_i and y_i are given. To obtain the least square error, the unknown coefficients a_0, a_1, a_2, ..., and a_m must yield zero first derivatives.

$$\frac{\partial \Pi}{\partial a_0} = 2\sum_{i=1}^{n}[y_i - (a_0 + a_1 x_i + a_2 x_i^2 + ... + a_m x_i^m)] = 0$$

$$\frac{\partial \Pi}{\partial a_1} = 2\sum_{i=1}^{n} x_i[y_i - (a_0 + a_1 x_i + a_2 x_i^2 + ... + a_m x_i^m)] = 0$$

$$\frac{\partial \Pi}{\partial a_2} = 2\sum_{i=1}^{n} x_i^2[y_i - (a_0 + a_1 x_i + a_2 x_i^2 + ... + a_m x_i^m)] = 0$$

....

$$\frac{\partial \Pi}{\partial a_m} = 2\sum_{i=1}^{n} x_i^m[y_i - (a_0 + a_1 x_i + a_2 x_i^2 + ... + a_m x_i^m)] = 0$$

- Expanding the above equations, we have

$$\sum_{i=1}^{n} y_i = a_0 \sum_{i=1}^{n} 1 + a_1 \sum_{i=1}^{n} x_i + \alpha_2 \sum_{i=1}^{n} x_i^2 + \ldots + a_m \sum_{i=1}^{n} x_i^m$$

$$\sum_{i=1}^{n} x_i y_i = a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + \alpha_2 \sum_{i=1}^{n} x_i^3 + \ldots + a_m \sum_{i=1}^{n} x_i^{m+1}$$

$$\sum_{i=1}^{n} x_i^2 y_i = a_0 \sum_{i=1}^{n} x_i^2 + a_1 \sum_{i=1}^{n} x_i^3 + \alpha_2 \sum_{i=1}^{n} x_i^4 + \ldots + a_m \sum_{i=1}^{n} x_i^{m+2}$$

....

$$\sum_{i=1}^{n} x_i^m y_i = a_0 \sum_{i=1}^{n} x_i^m + a_1 \sum_{i=1}^{n} x_i^{m+1} + \alpha_2 \sum_{i=1}^{n} x_i^{m+2} + \ldots + a_m \sum_{i=1}^{n} x_i^{2m}$$

- The unknown coefficients a_0, a_1, a_2, ..., and a_m can hence be obtained by solving the above linear equations.
- There are also many library routines available to do the job.

OUTLINE

Longstaff and Schwartz (2001)

- The Longstaff and Schwartz method estimates the conditional expected option value by:
  - simulating lots of paths
  - carrying out a regression analysis on the resulting option values
- This gives an approximation for the continuation value that can then be compared to the early exercise value and then we know the option value at each point in time on each path.

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- This gives an approximation for the continuation value that can then be compared to the early exercise value and then we know the option value at each point in time on each path.
- In terms of Monte Carlo pricing, all we actually need to know is the rule for early exercising, so we know when we receive the cash flows and the value of the option is the average of the discounted payoffs for each path.
- We will explain the method via an example and then describe the general method.

Example

- We will attempt to value a Bermudan put option where exercise is possible now and at three future dates. $S_0 = 1$, $X = 1.1$, $r = 0.06$.
- The first step is to simulate some paths, the table below denotes the results:

Stock price paths

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

- We need to use this information to determine the continuation value at each point in time for each path. To do this we will construct a "Cash Flow Matrix" at each point in time.

CONTINUATION VALUE AT $t = 2$

- The table below denotes the cash flows at $t = 3$ assuming that we held the option that far:

Cash-flow matrix at $t = 3$

Path	$t = 1$	$t = 2$	$t = 3$
1	–	–	.00
2	–	–	.00
3	–	–	.07
4	–	–	.18
5	–	–	.00
6	–	–	.20
7	–	–	.09
8	–	–	.00

- The next step is to attempt to find a function that describes the continuation value at time 2 as a function of the value of S at time 2.

REGRESS...

- To do this we use a regression technique, that takes the values at time 2 as the "x" values and the discounted payoff at time 3 as the "y" values.

Continuation value at $t = 2$

Path	y	x
1	$.00 \times .94176$	1.08
2	–	–
3	$.07 \times .94176$	1.07
4	$.18 \times .94176$	.97
5	–	–
6	$.20 \times .94176$	.77
7	$.09 \times .94176$	.84
8	–	–

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4	$.18 \times .94176$	.97
5	–	–
6	$.20 \times .94176$	.77
7	$.09 \times .94176$	.84
8	–	–

- Note that the regression is only carried out on paths that are in the money at time 2.
- The regression here is simple where y is regressed upon x and x^2.
- In this particular example (using least squares): $y = -1.070 + 2.938x - 1.813x^2$

Option value at $t = 2$

Optimal early exercise decision at time 2

Path	Exercise	Continuation
1	.02	.0369
2	–	–
3	.03	.0461
4	.13	.1176
5	–	–
6	.33	.1520
7	.26	.1565
8	–	–

- This then allows you to decide at which points in time you would exercise and thus determine the cash flows at $t = 2$ (below). Notice that for each path, if you exercise at $t = 2$ then you do not also exercise at $t = 3$

Cash-flow matrix at time 2

Path	$t = 1$	$t = 2$	$t = 3$
1	–	.00	.00
2	–	.00	.00
3	–	.00	.07
4	–	.13	.00
5	–	.00	.00
6	–	.33	.00
7	–	.26	.00
8	–	.00	.00

CONTINUATION VALUE AT $t = 1$

- We can apply the same process to $t = 1$, for each of the paths that are in the money we regress the discounted future cash flows (y) on the current value of the underlying asset (x), where x and y are as given below:

Regression at time 1

Path	y	x
1	$.00 \times .94176$	1.09
2	–	–
3	–	–
4	$.13 \times .94176$	.93
5	–	–
6	$.33 \times .94176$	.76
7	$.26 \times .94176$	.92
8	$.00 \times .94176$	.88

- The regression equation here is $y = 2.038 - 3.335x + 1.356x^2$ and again we use this to estimate the continuation value and decide on an early exercise strategy.
- The next table compares the two values and the final table denotes the early exercise or stopping rule.

Optimal early exercise decision at time 1

Path	Exercise	Continuation
1	.01	.0139
2	–	–
3	–	–
4	.17	.1092
5	–	–
6	.34	.2866
7	.18	.1175
8	.22	.1533

Early exercise strategy

- The early exercise strategy is as follows:

	Stopping rule		
Path	$t=1$	$t=2$	$t=3$
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

- From the early exercise strategy we can then value the option, by forming the final cash flow matrix from this rule.

Option cash flow matrix

Path	$t = 1$	$t = 2$	$t = 3$
1	.00	.00	.00
2	.00	.00	.00
3	.00	.00	.07
4	.17	.00	.00
5	.00	.00	.00
6	.34	.00	.00
7	.18	.00	.00
8	.22	.00	.00

- So the option value is the average of the discounted cash flows, so in this case:

$$V_0 = \tfrac{1}{8}(0 + 0 + 0.07e^{-3r} + 0.127e^{-r} + 0.34e^{-r} + 0.18e^{-r} + 0.22e^{-r}) = 0.1144$$

OUTLINE

Algorithm

- Choose:
  - number of sample paths N,
  - number of basis functions for the regression, M
  - type of basis functions $F_j(x)$
  - number of observation dates d.

ALGORITHM

- Choose:
  - number of sample paths N,
  - number of basis functions for the regression, M
  - type of basis functions $F_j(x)$
  - number of observation dates d.
- Draw Nd Normally distributed random numbers and simulate the sample paths for the underlying asset at each point in time $S^n_{t_i}$ $1 \leq i \leq d, 1 \leq n \leq N$
- At expiry $t = t_d$, record the cash flow values $CF^n(t_d)$ which for a put are $\max(X - S^n_{t_d}, 0)$
- Move back to $t = t_{d-1}$ for each path where $S^n_{t_{d-1}} < X$ calculate the continuation value as $CV^n(t_{d-1}) = e^{-r(t_d - t_{d-1})} CF^n(t_d)$.
- perform the regression to determine the functional form of the continuation value, $y(S)$

- Recalculate the continuation value as $CV^n(t_{d-1}) = y(S^n_{td-1})$
- For every path calculate the cash flow value where if the continuation value $CV^n(t_{d-1}) < X - S^n_{t_{d-1}}$ then $CF^n(t_{d-1}) = X - S^n_{t_{d-1}}$ and $CF^n(t_i) = 0$ for $i > d-1$ otherwise $CF^n(td-1) = 0$
- Repeat this process for the previous time step until you have $CF^n(t_i)$ for all i and n. Note that in general to calculate $CV^n(t_i)$ before the regression

$$CV^n(t_i) = \sum_{n=i+1}^{d} e^{-r(t_n - t_i)} CF^n(t_n)$$

- The option value V_0 is then

$$V_0 = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{M} e^{-rt_j} CF^i(t_j)$$

NOTES...

- Regression analysis will become computationally more expensive as the number of underlying assets increase
- When you have more than one underlying asset in order to perform the regression you need to have basis functions in all of the underlying assets as well as in the cross terms between them (i.e. in S_1, S_2 and S_1S_2).
- This means that the number of basis functions will increase exponentially as you increase the number of underlying assets,
- although it is not necessary in practise to have too large a number of basis functions.

OUTLINE

How well does it perform?

- Longstaff and Schwartz provide proofs that as $M \to \infty$ and $N \to \infty$ the option value obtained from their scheme converges to the theoretical value.
- You will have to limit M and N because of computation times
- The method's performance is mixed and can often incur unknown approximation errors.
- See Moreno and Navas (2003) for an investigation into the use of various polynomial fits and numbers of basis functions.

How well does it perform?

- It is not clear that increasing the number of basis functions actually increases accuracy
- For complex derivative pricing problems sometimes errors can increase as you add more basis functions
- In general, the method will provide good estimates but will be difficult to assess exactly how accurate it is.
- See Duck et al (2005) for some improvements on the basic Longstaff and Schwartz scheme

OUTLINE

- We have introduced a method for valuing options with early exercise features using simulation.
- The main idea is to estimate the continuation value (as a function of the current underlying asset price) by performing a least squares regression.
- The method converges to the correct option price
- Research shows that it is unclear how well the method performs