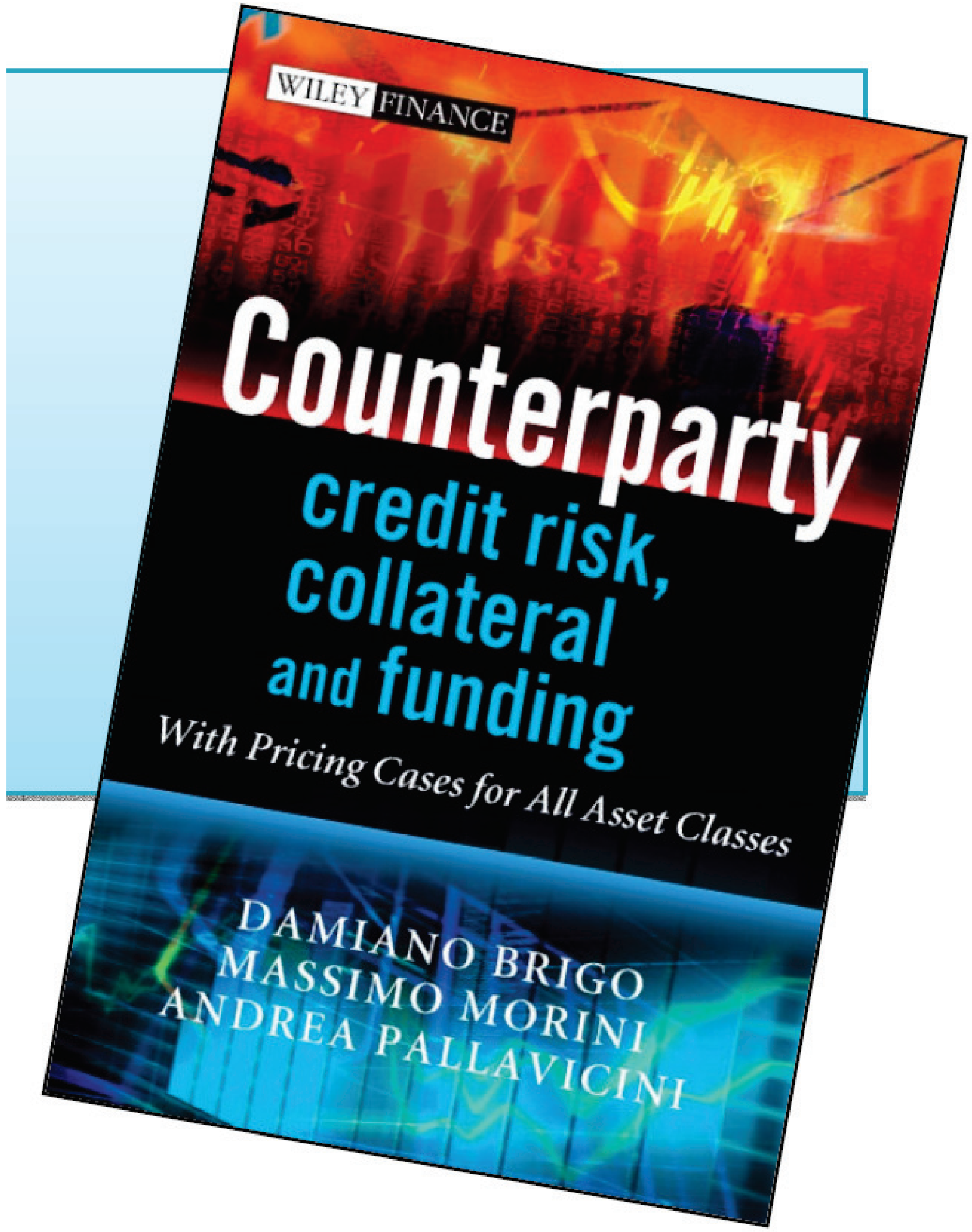
PART II: PRICING CREDIT RISK, COLLATERAL AND FUNDING

In this Part we look at how we may include Counterparty Credit Risk into the Valuation from the start rather than through unexplained ad-hoc discount (multiple) curves.

This leads to the notions of Credit and Debit Valuation Adjustments (CVA DVA).

We also hint at Funding Valuation Adjustments (FVA).

Presentation based on the Forthcoming Book



Intro to Basic Credit Risk Products and Models

Before dealing with the current topical issues of Counterparty Credit Risk, CVA, DVA and Funding, we need to introduce some basic elements of Credit Risk Products and Credit Risk Modelling.

We now briefly look at:

- Products: Credit Default Swaps (CDS) and Defaultable Bonds
- Payoffs and prices of such products
- Market implied $\mathbb{Q}$ probabilities of default defined by such models
- Intensity models and probabilities of defaults as credit spreads
- Credit spreads as possibly constant, curved or even stochastic
- Credit spread volatility (stochastic credit spreads)

Defaultable (corporate) zero coupon bonds

We started this course by defining the zero coupon bond price $P(t, T)$. Similarly to $P(t, T)$ being one of the possible fundamental quantities for describing the interest-rate curve, we now consider a defaultable bond $\bar{P}(t, T)$ as a possible fundamental variable for describing the defaultable market.

DEFAULT FREE		
time t		time T
:	$\longleftarrow$	:
$P(t, T)$		1

with	DEFAULT	
time t		time T :
:	$\longleftarrow$	NO DEFAULT: 1
$\bar{P}(t, T)$		DEFAULT: 0

When considering default, we have a random time τ representing the time at which the bond issuer defaults. $\boxed{\tau : \text{Default time of the issuer}}$

Defaultable (corporate) zero coupon bonds I

The value of a bond issued by the company and promising the payment of 1 at time T, as seen from time t, is the risk neutral expectation of the discounted payoff
BondPrice = Expectation[Discount x Payoff]

$$P(t,T) = \mathbb{E}\{D(t,T)\ 1\ |\mathcal{F}_t\}, \qquad \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T) := \mathbb{E}\{D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\}$$

where $\mathcal{G}_t$ represents the flow of information on whether default occurred before t and if so at what time exactly, and on the default free market variables (like for example the risk free rate r_t) up to t. The filtration of default-free market variables is denoted by $\mathcal{F}_t$. Formally, we assume

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \le u\},\ 0 \le u \le t).$$

D is the stochastic discount factor between two dates, depending on interest rates, and represents discounting.

Defaultable (corporate) zero coupon bonds II

The “indicator” function $\mathbf{1}_{\text{condition}}$ is 1 if “condition” is satisfied and 0 otherwise. In particular, $\mathbf{1}_{\{\tau>T\}}$ reads 1 if default τ did not occur before T, and 0 in the other case.

We understand then that (ignoring recovery) $\mathbf{1}_{\{\tau>T\}}$ is the correct payoff for a corporate bond at time T: the contract pays 1 if the company has not defaulted, and 0 if it defaulted before T, according to our earlier stylized description.

Defaultable (corporate) zero coupon bonds

If we include a recovery amount REC to be paid at default τ in case of early default, we have as discounted payoff at time t

$$D(t,T)\mathbf{1}_{\{\tau>T\}} + \mathrm{R{\scriptstyle EC}}\, D(t,\tau)\mathbf{1}_{\{\tau\le T\}}$$

If we include a recovery amount REC paid at maturity T, we have as discounted payoff

$$D(t,T)\mathbf{1}_{\{\tau>T\}} + \mathrm{R{\scriptstyle EC}}\, D(t,T)\mathbf{1}_{\{\tau\le T\}}$$

Taking $\mathbb{E}[\cdot|\mathcal{G}_t]$ on the above expressions gives the price of the bond.

Fundamental Credit Derivatives: Credit Default Swaps

Credit Default Swaps are basic protection contracts that became quite liquid on a large number of entities after their introduction.

CDS's are now actively traded and have become a sort of basic product of the credit derivatives area, analogously to interest-rate swaps and FRA's being basic products in the interest-rate derivatives world.

As a consequence, the need is not to have a model to be used to value CDS's, but rather to consider a model that can be *calibrated* to CDS's, i.e. to take CDS's as model inputs (rather than outputs), in order to price more complex derivatives.

As for options, single name CDS options have never been liquid, as there is more liquidity in the CDS index options. We may expect models will have to incorporate CDS index options quotes rather than price them, similarly to what happened to CDS themselves.

Fundamental Credit Derivatives: CDS's

A CDS contract ensures protection against default. Two companies "A" (Protection buyer) and "B" (Protection seller) agree on the following. If a third company "C" (Reference Credit) defaults at time τ, with $T_a < \tau < T_b$, "B" pays to "A" a certain (deterministic) cash amount $\mathrm{L_{GD}}$. In turn, "A" pays to "B" a rate R at times $T_{a+1}, \ldots, T_b$ or until default. Set $\alpha_i = T_i - T_{i-1}$ and $T_0 = 0$.

Protection Seller B	$\rightarrow$	protection $\mathrm{L_{GD}}$ at default τ_C if $T_a < \tau_C \leq T_b$	$\rightarrow$ Protection Buyer A
	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\leftarrow$

(protection leg and premium leg respectively). The cash amount $\mathrm{L_{GD}}$ is a *protection* for "A" in case "C" defaults. Typically $\mathrm{L_{GD}} =$ notional, or "notional - recovery" $= 1 - \mathrm{R_{EC}}$.

Fundamental Credit Derivatives: CDS's

A typical stylized case occurs when "A" has bought a corporate bond issued by "C" and is waiting for the coupons and final notional payment from "C": If "C" defaults before the corporate bond maturity, "A" does not receive such payments. "A" then goes to "B" and buys some protection against this risk, asking "B" a payment that roughly amounts to the loss on the bond (e.g. notional minus deterministic recovery) that A would face in case "C" defaults.

Or again "A" has a portfolio of several instruments with a large exposure to counterparty "C". To partly hedge such exposure, "A" enters into a CDS where it buys protection from a bank "B" against the default of "C".

Fundamental Credit Derivatives: CDS's

Protection Seller B	$\rightarrow$ $\leftarrow$	protection L_{GD} at default τ_C if $T_a < \tau_C \leq T_b$ rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\rightarrow$ $\leftarrow$	Protection Buyer A

Formally we may write the (Running) CDS discounted payoff to "B" at time $t < T_a$ as

$$\Pi_{\mathrm{RCDS}_{a,b}}(t) := D(t,\tau)(\tau - T_{\beta(\tau)-1}) R \mathbf{1}_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^{b} D(t, T_i)\alpha_i R \mathbf{1}_{\{\tau > T_i\}} - \mathbf{1}_{\{T_a < \tau \leq T_b\}} D(t,\tau)\, \mathrm{L}_{\mathrm{GD}}$$

where $T_{\beta(\tau)}$ is the first of the T_i's following τ.

CDS payout to Protection seller (receiver CDS)

The 3 terms in the payout are as follows (they are seen from the protection seller, receiver CDS):

- Discounted Accrued rate at default : This is supposed to compensate the protection seller for the protection he provided from the last T_i before default until default τ:

$$D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}$$

- CDS Rate premium payments if no default: This is the premium received by the protection seller for the protection being provided

$$\sum_{i=a+1}^{b} D(t, T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}}$$

- Payment of protection at default if this happens before final T_b

$$-\mathbf{1}_{\{T_a<\tau\leq T_b\}} D(t,\tau)\ \mathrm{L}_{\mathrm{GD}}$$

These are random discounted cash flows, not yet the CDS price.

CDS's: Risk Neutral Valuation Formula

Denote by $\mathrm{CDS}_{a,b}(t, R, \mathrm{L_{GD}})$ the time t *price* of the above Running standard CDS's *payoffs*.

As usual, the price associated to a discounted payoff is its *risk neutral expectation*.

The resulting pricing formula depends on the assumptions on interest-rate dynamics and on the default time τ (reduced form models, structural models...).

CDS's: Risk Neutral Valuation

In general by risk-neutral valuation we can compute the CDS price at time 0 (or at any other time similarly):

$$\mathrm{CDS}_{a,b}(0, R, \mathrm{L}_{\mathrm{GD}}) = \mathbb{E}\{\Pi\mathrm{RCDS}_{a,b}(0)\},$$

with the CDS discounted payoffs defined earlier. As usual, $\mathbb{E}$ denotes the risk-neutral expectation, the related measure being denoted by $\mathbb{Q}$.

However, we will not use the formulas resulting from this approach to price CDS that are already quoted in the market. *Rather, we will invert these formulas in correspondence of market CDS quotes to calibrate our models to the CDS quotes themselves. We will give examples of this later.*

Now let us have a look at some particular formulas resulting from the general risk neutral approach through some simplifying assumptions.

CDS Model-independent formulas

Assume the stochastic discount factors $D(s,t)$ **to be independent of the default time** τ **for all possible** $0 < s < t$**.** The price of the premium leg of the CDS at time 0 is:

$$\begin{aligned}\text{PremiumLeg}_{a,b}(R) &= \mathbb{E}[D(0,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}] + \\ &\quad + \sum_{i=a+1}^{b} \mathbb{E}[D(0,T_i)\alpha_i R\mathbf{1}_{\{\tau \ge T_i\}}] \\ &= \mathbb{E}\left[\int_{t=0}^{\infty} D(0,t)(t - T_{\beta(t)-1})R\mathbf{1}_{\{T_a<t<T_b\}}\delta_\tau(t)dt\right] \\ &\quad + \sum_{i=a+1}^{b} \mathbb{E}[D(0,T_i)]\alpha_i R\,\mathbb{E}[\mathbf{1}_{\{\tau \ge T_i\}}] =\end{aligned}$$

For those who don't know the theory of distributions (Dirac's delta etc), read $\delta_\tau(t)dt = \mathbf{1}_{\{\tau\in[t,t+dt]\}}$.

CDS Model-independent formulas

$$\text{PremiumLeg}_{a,b}(R) = \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)(t - T_{\beta(t)-1}) R\, \delta_\tau(t) dt] +$$
$$+ \sum_{i=a+1}^{b} P(0,T_i)\alpha_i R\, \mathbb{Q}(\tau \geq T_i) =$$
$$= \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)](t - T_{\beta(t)-1}) R\, \mathbb{E}[\delta_\tau(t) dt] + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i R\, \mathbb{Q}(\tau \geq T_i)$$
$$= R \int_{t=T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) \mathbb{Q}(\tau \in [t, t+dt)) +$$
$$+ R \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \mathbb{Q}(\tau \geq T_i),$$

where we have used independence in factoring terms. Again, read $\delta_\tau(t)dt = \mathbf{1}_{\{\tau \in [t, t+dt]\}}$

CDS Model-independent formulas

We have thus, by rearranging terms and introducing a “unit-premium” premium leg (sometimes called “DV01”, “Risky duration” or “annuity”):

$$\text{PremiumLeg}_{a,b}(R; P(0,\cdot), \mathbb{Q}(\tau > \cdot)) = R\ \text{PremiumLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot))$$

$$\text{PremiumLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)) := -\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) d_t \boxed{\mathbb{Q}(\tau \geq t)} + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)}$$

This model independent formula uses the initial market zero coupon curve (bonds) at time 0 (i.e. $P(0,\cdot)$) and the survival probabilities $\mathbb{Q}(\tau \geq \cdot)$ at time 0 (terms in the boxes).

A similar formula holds for the protection leg, again under independence between default τ and interest rates.

CDS Model-independent formulas

$$
\begin{aligned}
\mathrm{ProtecLeg}_{a,b}(\mathrm{L_{GD}}) &= \mathbb{E}[\mathbf{1}_{\{T_a<\tau\le T_b\}} D(0,\tau)\ \mathrm{L_{GD}}] \\
&= \mathrm{L_{GD}}\ \mathbb{E}\left[\int_{t=0}^{\infty} \mathbf{1}_{\{T_a<t\le T_b\}} D(0,t)\delta_\tau(t)dt\right] \\
&= \mathrm{L_{GD}}\left[\int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)\delta_\tau(t)dt]\right] \\
&= \mathrm{L_{GD}}\int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)]\mathbb{E}[\delta_\tau(t)dt] \\
&= \mathrm{L_{GD}}\int_{t=T_a}^{T_b} P(0,t)\mathbb{Q}(\tau\in[t,t+dt))
\end{aligned}
$$

(again interpret $\delta_\tau(t)dt = \mathbf{1}_{\{\tau\in[t,t+dt]\}}$)

CDS Model-independent formulas

so that we have, by introducing a "unit-notional" protection leg:

$$\text{ProtecLeg}_{a,b}(\text{L}_{\text{GD}}; P(0,\cdot), \mathbb{Q}(\tau > \cdot)) = \text{L}_{\text{GD}}\, \text{ProtecLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)),$$

$$\text{ProtecLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)) := -\int_{T_a}^{T_b} P(0,t)\, d_t \boxed{\mathbb{Q}(\tau \geq t)}$$

This formula too is model independent given the initial zero coupon curve (bonds) at time 0 observed in the market and given the survival probabilities at time 0 (term in the box).

CDS Model-independent formulas

The total (Receiver) CDS price can be written as

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \mathbb{Q}(\tau > \cdot)) = R\,\mathrm{PremiumLeg1}_{a,b}(\mathbb{Q}(\tau > \cdot))$$

$$-\mathrm{L}_{\mathrm{GD}}\ \mathrm{ProtecLeg1}_{a,b}(\mathbb{Q}(\tau > \cdot))$$

$$= R\left[-\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})d_t \boxed{\mathbb{Q}(\tau \geq t)} + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)}\right]$$

$$+\mathrm{L}_{\mathrm{GD}}\left[\int_{T_a}^{T_b} P(0,t)\ d_t \boxed{\mathbb{Q}(\tau \geq t)}\right]$$

(Receiver) CDS Model-independent formulas

We may also use that $d_t \mathbb{Q}(\tau > t) = d_t(1 - \mathbb{Q}(\tau \le t)) = -d_t\, \mathbb{Q}(\tau \le t)$.
We have

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L_{GD}}; \mathbb{Q}(\tau \le \cdot)) = -\mathrm{L_{GD}} \left[\int_{T_a}^{T_b} P(0,t)\, d_t \boxed{\mathbb{Q}(\tau \le t)} \right] +$$

$$R \left[\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) d_t \boxed{\mathbb{Q}(\tau \le t)} + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \ge T_i)} \right]$$

The integrals in the survival probabilities given in the above formulas can be valued as Stieltjes integrals in the survival probabilities themselves, and can easily be approximated numerically by summations through Riemann-Stieltjes sums, considering a low enough discretization time step.

CDS Model-independent formulas

The market quotes, at time 0, the fair $R = R_{0,b}^{\text{mkt MID}}(0)$ coming from bid and ask quotes for this fair R.

This fair R equates the two legs for a set of CDS with initial protection time $T_a = 0$ and final protection time $T_b \in \{1y, 2y, 3y, 4y, 5y, 6y.7y, 8y, 9y, 10y\}$, although often only a subset of the maturities $\{1y, 3y, 5y, 7y, 10y\}$ is available.

Solve then

$$\text{CDS}_{0,b}(t, R_{0,b}^{\text{mktMID}}(0), \text{L}_{\text{GD}}; \mathbb{Q}(\tau > \cdot)) = 0$$

in portions of $\mathbb{Q}(\tau > \cdot)$ starting from $T_b = 1y$, finding the market implied survival $\{\mathbb{Q}(\tau \geq t), t \leq 1y\}$; plugging this into the $T_b = 2y$ CDS legs formulas, and then solving the same equation with $T_b = 2y$, we find the market implied survival $\{\mathbb{Q}(\tau \geq t), t \in (1y, 2y]\}$, and so on up to $T_b = 10y$.

CDS Model-independent formulas

This is a way to strip survival (or equivalently default) probabilities from CDS quotes in a model independent way. No need to assume an intensity or a structural model for default here.

However, the market in doing the above stripping typically resorts to intensities (also called hazard rates), assuming existence of intensities associated with the default time.

We will refer to the method just highlighted as **”CDS stripping”.**

CDS and Defaultable Bonds: Intensity Models

In intensity models the random default time τ is assumed to be exponentially distributed.

A strictly positive stochastic process $t \mapsto \lambda_t$ called *default intensity* (or hazard rate) is given for the bond issuer or the CDS reference name.

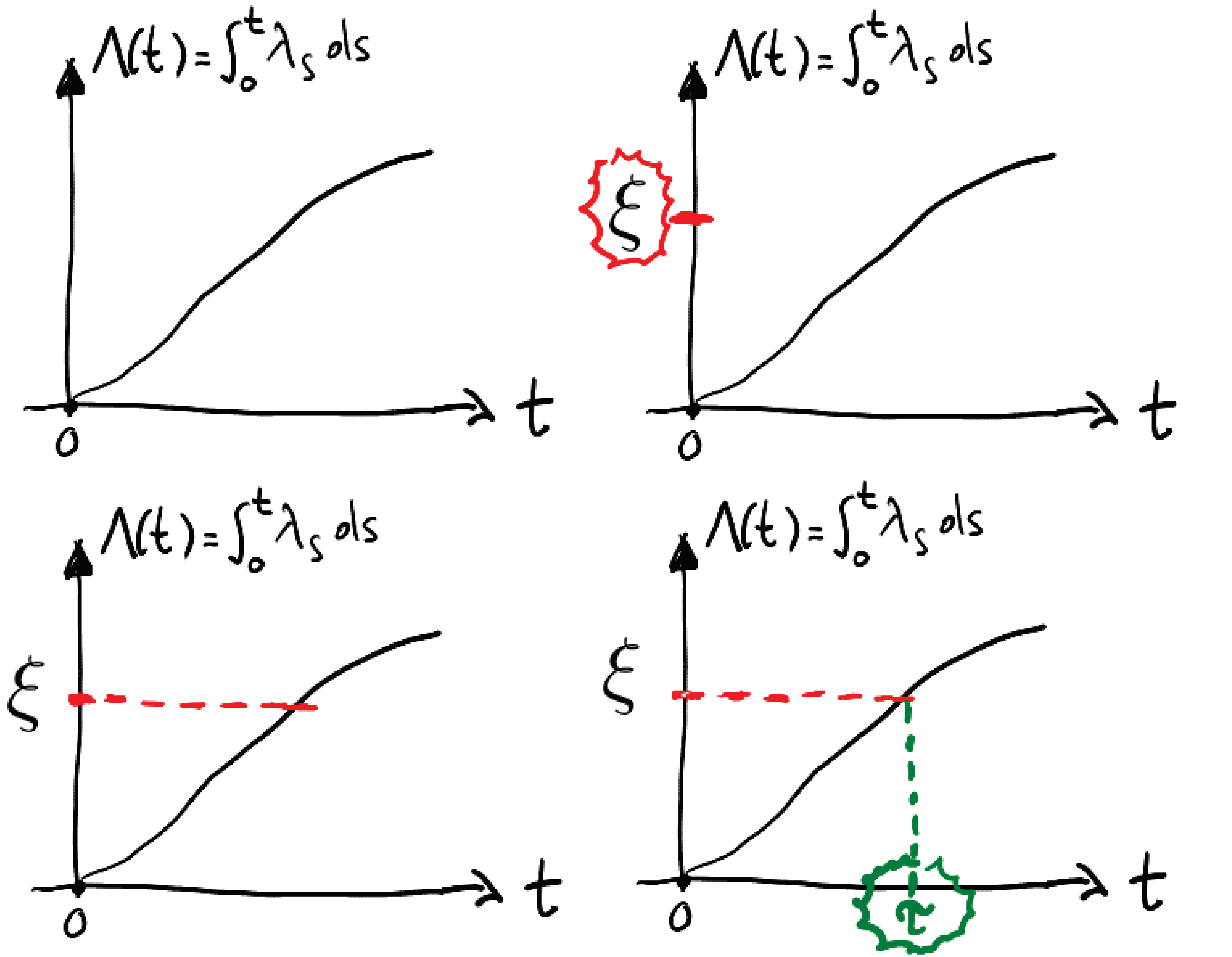
The *cumulated intensity* (or hazard function) is the process $t \mapsto \int_0^t \lambda_s\, ds =: \Lambda_t$. Since λ is positive, Λ is increasing in time.

The default time is defined as the inverse of the cumulative intensity on an exponential random variable ξ with mean 1 and independent of λ

$$\tau = \Lambda^{-1}(\xi).$$

Recall that

$$\mathbb{Q}(\xi > u) = e^{-u}, \quad \mathbb{Q}(\xi < u) = 1 - e^{-u}, \quad \mathbb{E}(\xi) = 1.$$


$\Lambda(t)=\int_0^t \lambda_s ds$
$\Lambda(t)=\int_0^t \lambda_s ds$
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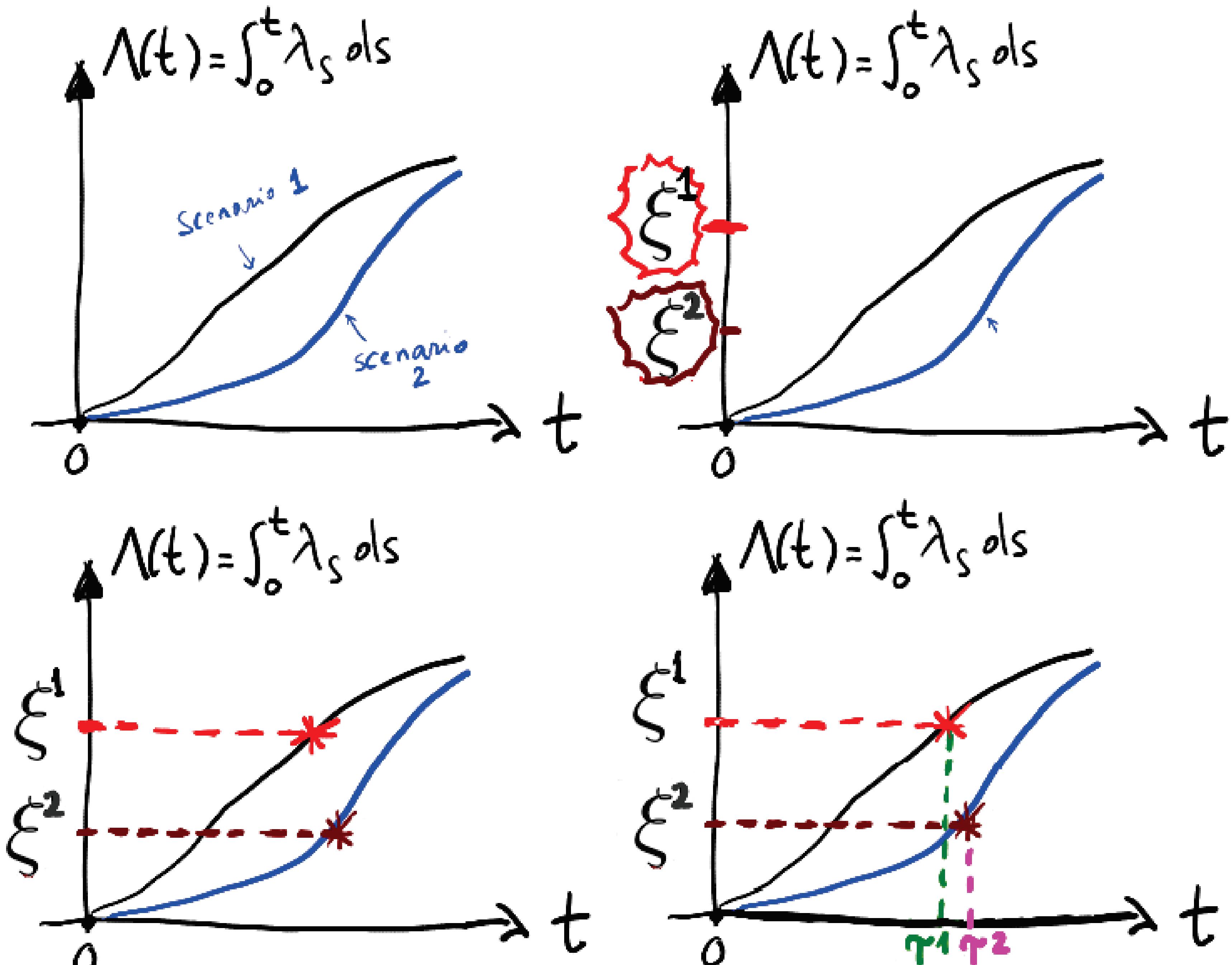


$\Lambda(t) = \int_0^t \lambda_s \, ds$
Scenario 1
scenario 2
0
t
$\Lambda(t) = \int_0^t \lambda_s \, ds$
ξ^1
ξ^2
0
t
$\Lambda(t) = \int_0^t \lambda_s \, ds$
ξ^1
ξ^2
0
t
$\Lambda(t) = \int_0^t \lambda_s \, ds$
ξ^1
ξ^2
0
τ^1
τ^2
t

CDS and Defaultable Bonds: Intensity Models

A few calculations: Probability of surviving time t:

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\Lambda^{-1}(\xi) > t) = \mathbb{Q}(\xi > \Lambda(t)) = \rightarrow$$

Let's use the tower property of conditional expectation and the fact that Λ is independent of ξ:

$$\rightarrow = \mathbb{E}[\mathbb{Q}(\xi > \Lambda(t)|\Lambda(t))] = \mathbb{E}[e^{-\Lambda(t)}] = \mathbb{E}[e^{-\int_0^t \lambda_s\, ds}]$$

This looks exactly like a bond price if we replace r by λ!

CDS and Defaultable Bonds: Intensity Models

Let's price a defaultable zero coupon bond with zero recovery. Assume that ξ is also independent of r.

$$\bar{P}(0,T) = \mathbb{E}[D(0,T)1_{\{\tau>T\}}] = \mathbb{E}[e^{-\int_0^T r_s\,ds}1_{\{\Lambda^{-1}(\xi)>T\}}] =$$

$$= \mathbb{E}[e^{-\int_0^T r_s\,ds}1_{\{\xi>\Lambda(T)\}}] = \mathbb{E}[\mathbb{E}\{e^{-\int_0^T r_s\,ds}1_{\{\xi>\Lambda(T)\}}|\Lambda, r\}]$$

$$= \mathbb{E}[e^{-\int_0^T r_s\,ds}\mathbb{E}\{1_{\{\xi>\Lambda(T)\}}|\Lambda, r\}]$$

$$= \mathbb{E}[e^{-\int_0^T r_s\,ds}\mathbb{Q}\{\xi > \Lambda(T)|\Lambda\}] = \mathbb{E}[e^{-\int_0^T r_s\,ds}e^{-\Lambda(T)}] =$$

$$= \mathbb{E}[e^{-\int_0^T r_s\,ds - \int_0^T \lambda_s\,ds} = \mathbb{E}[e^{-\int_0^T \boxed{(r_s+\lambda_s)}\,ds}]$$

So the price of a defaultable bond is like the price of a default-free bond *where the risk free discount short rate r has been replaced by r plus a spread* λ.

CDS and Defaultable Bonds: Intensity Models

This is why in intensity models, the intensity is interpreted as a credit spread.

Because of properties of the exponential random variable, one can also prove that

$$\mathbb{Q}(\tau \in [t, t+dt) | \tau > t, \text{''}\lambda[0,t]\text{''}) = \lambda_t \, dt$$

and the intensity $\lambda_t \, dt$ is also a local probability of defaulting around t.

So:
λ is an instantaneous credit spread or local default probability

ξ is pure jump to default risk

Intensity models and Interest Rate Models

As is now clear, the exponential structure of τ in intensity models makes the modeling of credit risk very similar to interest rate models.

The spread/intensity λ behaves exactly like an interest rate in discounting

Then it is possible to use a lot of techniques from interest rate modeling (short rate models for r, first choice seen earlier) for credit as well.

Intensity: Constant, time dependent or stochastic

- Constant λ_t: in this case $\lambda_t = \gamma$ for a deterministic constant credit spread (intensity);
- Time dependent deterministic intensity λ_t: in this case $\lambda_t = \gamma(t)$ for a deterministic curve in time $\gamma(t)$. This is a model with a term structure of credit spreads but without credit spread volatility.
- Time dependent and stochastic intensity λ_t: in this case λ_t is a full stochastic process. This allows us to model the term structue of credit spreads but also their volatility.

The case with constant intensity $\lambda_t = \gamma$: CDS

Assume as an approximation that the CDS premium leg pays continuously.

Instead of paying $(T_i - T_{i-1})R$ at T_i as the standard CDS, given that there has been no default before T_i, we approximate this premium leg by assuming that it pays "$dt\ R$" in $[t, t+dt)$ it there has been no default before $t + dt$.

The case with constant intensity $\lambda_t = \gamma$: CDS

This amounts to replace the original pricing formula of a CDS (receiver case, spot CDS with $T_a = 0 =$ today)

$$\mathrm{CDS}_{0,b}(0, R, \mathrm{L_{GD}}; \mathbb{Q}(\tau > \cdot)) = R\left[-\int_0^{T_b} P(0,t)(t - T_{\beta(t)-1})d_t\mathbb{Q}(\tau \geq t)\right.$$

$$\left.+\sum_{i=1}^{b} P(0,T_i)\alpha_i\mathbb{Q}(\tau \geq T_i)\right] + \mathrm{L_{GD}}\left[\int_0^{T_b} P(0,t)\, d_t\, \mathbb{Q}(\tau \geq t)\right]$$

with (accrual term vanishes because payments continuous now)

$$R\int_0^{T_b} P(0,t)\mathbb{Q}(\tau \geq t)dt + \mathrm{L_{GD}}\int_0^{T_b} P(0,t)\, d_t\mathbb{Q}(\tau \geq t)$$

The case with constant intensity $\lambda_t = \gamma$: CDS

If the intensity is a constant γ we have

$$\mathbb{Q}(\tau > t) = e^{-\gamma t}, \quad d_t\mathbb{Q}(\tau > t) = -\gamma e^{-\gamma t} dt = -\gamma \mathbb{Q}(\tau > t) dt,$$

and the receiver CDS price we have seen earlier becomes

$$\mathrm{CDS}_{0,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \mathbb{Q}(\tau > \cdot)) = -\mathrm{L}_{\mathrm{GD}} \left[\int_0^{T_b} P(0,t) \gamma \mathbb{Q}(\tau \geq t) dt \right]$$
$$+ R \left[\int_0^{T_b} P(0,t) \mathbb{Q}(\tau \geq t) dt \right]$$

If we insert the market CDS rate $R = R_{0,b}^{\mathrm{mkt\ MID}}(0)$ in the premium leg, then the CDS present value should be zero. Solve

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \mathbb{Q}(\tau > \cdot)) = 0 \quad \text{in} \quad R$$

to obtain

$$\boxed{\gamma = \frac{R_{0,b}^{\mathrm{mkt\ MID}}(0)}{\mathrm{L}_{\mathrm{GD}}}}$$

The case with constant intensity $\lambda_t = \gamma$: CDS

from which we see that also **the CDS premium rate R is indeed a sort of CREDIT SPREAD, or INTENSITY.**

We can play with this formula with a few examples.

CDS of FIAT trades at 300bps for 5y, with recovery 0.3

What is a quick rough calcul for the risk neutral probability that FIAT survives 10 years?

$$\boxed{\gamma = \frac{R_{0,b}^{\text{mkt FIAT}}(0)}{\text{L}_{\text{GD}}}} = \frac{300/10000}{1-0.3} = 4.29\%$$

The case with constant intensity $\lambda_t = \gamma$: CDS

Survive 10 years:

$$\mathbb{Q}(\tau > 10y) = \exp(-\gamma 10) = \exp(-0.0429 * 10) = 65.1\%$$

Default between 3 and 5 years:

$$\mathbb{Q}(\tau > 3y) - \mathbb{Q}(\tau > 5y) = \exp(-\gamma 3) - \exp(-\gamma 5)$$

$$= \exp(-0.0429 * 3) - \exp(-0.0429 * 5) = 7.2\%$$

If R_{CDS} goes up and REC remains the same, γ goes up and survival probabilities go down (default probs go up)

If REC goes up and R_{CDS} remains the same, $\mathrm{L_{GD}}$ goes down and γ goes up - default probabilities go up

The case with time dependent intensity $\lambda_t = \gamma(t)$: CDS

We consider now **deterministic time-varying** intensity $\gamma(t)$, which we assume to be a positive and piecewise continuous function. We define

$$\Gamma(t) := \int_0^t \gamma(u)du,$$

the **cumulated intensity, cumulated hazard rate,** or also **Hazard function**.

From the exponential assumption, we have easily

$$\mathbb{Q}\{s < \tau \le t\} = \mathbb{Q}\{s < \Gamma^{-1}(\xi) \le t\} = \mathbb{Q}\{\Gamma(s) < \xi \le \Gamma(t)\} =$$

$$= \mathbb{Q}\{\xi > \Gamma(s)\} - \mathbb{Q}\{\xi > \Gamma(t)\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)) \text{ i.e.}$$

"prob of default between s and t is "$e^{-\int_0^s \gamma(u)du} - e^{-\int_0^t \gamma(u)du} \approx \int_s^t \gamma(u)du$" (where the final approximation is good ONLY for small exponents).

CDS Calibration and Implied Hazard Rates/Intensities

Reduced form models are the models that are most commonly used in the market to infer implied default probabilities from market quotes.

Market instruments from which these probabilities are drawn are especially CDS and Bonds.

We just implement the stripping algorithm sketched earlier for ”CDS stripping”, but now taking into account that the probabilities are expressed as exponentials of the deterministic intensity γ, that is assumed to be piecewise constant.

By adding iteratively CDS with longer and longer maturities, at each step we will strip the new part of the intensity $\gamma(t)$ associated with the last added CDS, while keeping the previous values of γ, for earlier times, that were used to fit CDS with shorter maturities.

A Case Study of CDS stripping: Lehman Brothers

Here we show an intensity model with piecewise constant λ obtained by CDS stripping.

We also show the AT1P structural / firm value model by Brigo et al (2004-2010). This will not be subject for this course, but in case of interest, for details on AT1P see

`http://arxiv.org/abs/0912.3028`
`http://arxiv.org/abs/0912.3031`
`http://arxiv.org/abs/0912.4404`

Otherwise ignore the AT1P and σ_i parts of the tables.

- **August 23, 2007**: Lehman announces that it is going to shut one of its home lending units (*BNC Mortgage*) and lay off 1,200 employees. The bank says it would take a $52 million charge to third-quarter earnings.
- **March 18, 2008**: Lehman announces better than expected first-quarter results (but profits have more than halved).
- **June 9, 2008**: Lehman confirms the booking of a $2.8 billion loss and announces plans to raise $6 billion in fresh capital by selling stock. Lehman shares lose more than 9% in afternoon trade.
- **June 12, 2008**: Lehman shakes up its management; its chief operating officer and president, and its chief financial officer are removed from their posts.
- **August 28, 2008**: Lehman prepares to lay off 1,500 people. The Lehman executives have been knocking on doors all over the world seeking a capital infusion.
- **September 9, 2008**: Lehman shares fall 45%.
- **September 14, 2008**: Lehman files for bankruptcy protection and hurtles toward liquidation after it failed to find a buyer.

Lehman Brothers CDS Calibration: July 10th, 2007

On the left part of this Table we report the values of the quoted CDS spreads before the beginning of the crisis. We see that the spreads are very low. In the middle of Table 7 we have the results of the exact calibration obtained using a *piecewise constant* intensity model.

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
10 Jul 2007			100.0%		100.0%
1y	16	0.267%	99.7%	29.2%	99.7%
3y	29	0.601%	98.5%	14.0%	98.5%
5y	45	1.217%	96.2%	14.5%	96.1%
7y	50	1.096%	94.1%	12.0%	94.1%
10y	58	1.407%	90.2%	12.7%	90.2%

Table: Results of calibration for July 10th, 2007.

Lehman Brothers CDS Calibration: June 12th, 2008

We are in the middle of the crisis. We see that the CDS spreads R_i have increased with respect to the previous case, but are not very high, indicating the fact that the market is aware of the difficulties suffered by Lehman but thinks that it can come out of the crisis. Notice that now the term structure of both R and *intensities* is inverted. This is typical of names in crisis

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
12 Jun 2008			100.0%		100.0%
1y	397	6.563%	93.6%	45.0%	93.5%
3y	315	4.440%	85.7%	21.9%	85.6%
5y	277	3.411%	80.0%	18.6%	79.9%
7y	258	3.207%	75.1%	18.1%	75.0%
10y	240	2.907%	68.8%	17.5%	68.7%

Table: Results of calibration for June 12th, 2008.

Lehman Brothers CDS Calibration: Sept 12th, 2008

In this Table we report the results of the calibration on September 12th, 2008, just before Lehman's default. We see that the spreads are now very high, corresponding to lower survival probability and higher intensities than before.

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
12 Sep 2008			100.0%		100.0%
1y	1437	23.260%	79.2%	62.2%	78.4%
3y	902	9.248%	65.9%	30.8%	65.5%
5y	710	5.245%	59.3%	24.3%	59.1%
7y	636	5.947%	52.7%	26.9%	52.5%
10y	588	6.422%	43.4%	29.5%	43.4%

Table: Results of calibration for September 12th, 2008.

Stochastic Intensity. The CIR++ model

We have seen in detail CDS calibration in presence of **deterministic** and **time varying** intensity or hazard rates, $\gamma(t)dt = \mathbb{Q}\{\tau \in dt | \tau > t\}$

As explained, this accounts for credit spread structure but not for **volatility**.

The latter is obtained moving to stochastic intensity (Cox process). The deterministic function $t \mapsto \gamma(t)$ is replaced by a stochastic process $t \mapsto \lambda(t) = \lambda_t$. The Hazard function $\Gamma(t) = \int_0^t \gamma(u)du$ is replaced by the Hazard process (or cumulated intensity) $\Lambda(t) = \int_0^t \lambda(u)du$.

CIR++ stochastic intensity λ

We model the stochastic intensity as follows: consider

$$\lambda_t = y_t + \psi(t;\beta)\ ,\quad t \geq 0,$$

where the intensity has a random component y and a deterministic component ψ to fit the CDS term structure. For y we take a Jump-CIR model

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t + dJ_t,\ \ \beta = (\kappa, \mu, \nu, y_0),\ \ 2\kappa\mu > \nu^2.$$

Jumps are taken themselves independent of anything else, with exponential arrival times with intensity η and exponential jump size with a given parameter.

In this course we will focus on the case with no jumps J, see B and El-Bachir (2006) or B and M (2006) for the case with jumps.

CIR++ stochastic intensity λ.

Calibrating Implied Default Probabilities

With no jumps, y follows a noncentral chi-square distribution; Very important: $y > 0$ as must be for an intensity model (Vasicek would not work). This is the CIR++ model we have seen earlier for interest rates.

About the parameters of CIR:

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t$$

κ: speed of mean reversion
μ: long term mean reversion level
ν: volatility.

CIR++ stochastic intensity λ. I

Calibrating Implied Default Probabilities

$$E[\lambda_t] = \lambda_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$$

$$\mathrm{VAR}(\lambda_t) = \lambda_0 \frac{\nu^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa t})^2$$

After a long time the process reaches (asymptotically) a stationary distribution around the mean μ and with a corridor of variance $\mu\nu^2/2\kappa$. The largest κ, the fastest the process converges to the stationary state. So, ceteris paribus, increasing κ kills the volatility of the credit spread. The largest μ, the highest the long term mean, so the model will tend to higher spreads in the future in average.
The largest ν, the largest the volatility. Notice however that κ and ν fight each other as far as the influence on volatility is concerned.

CIR++ stochastic intensity λ. II

Calibrating Implied Default Probabilities

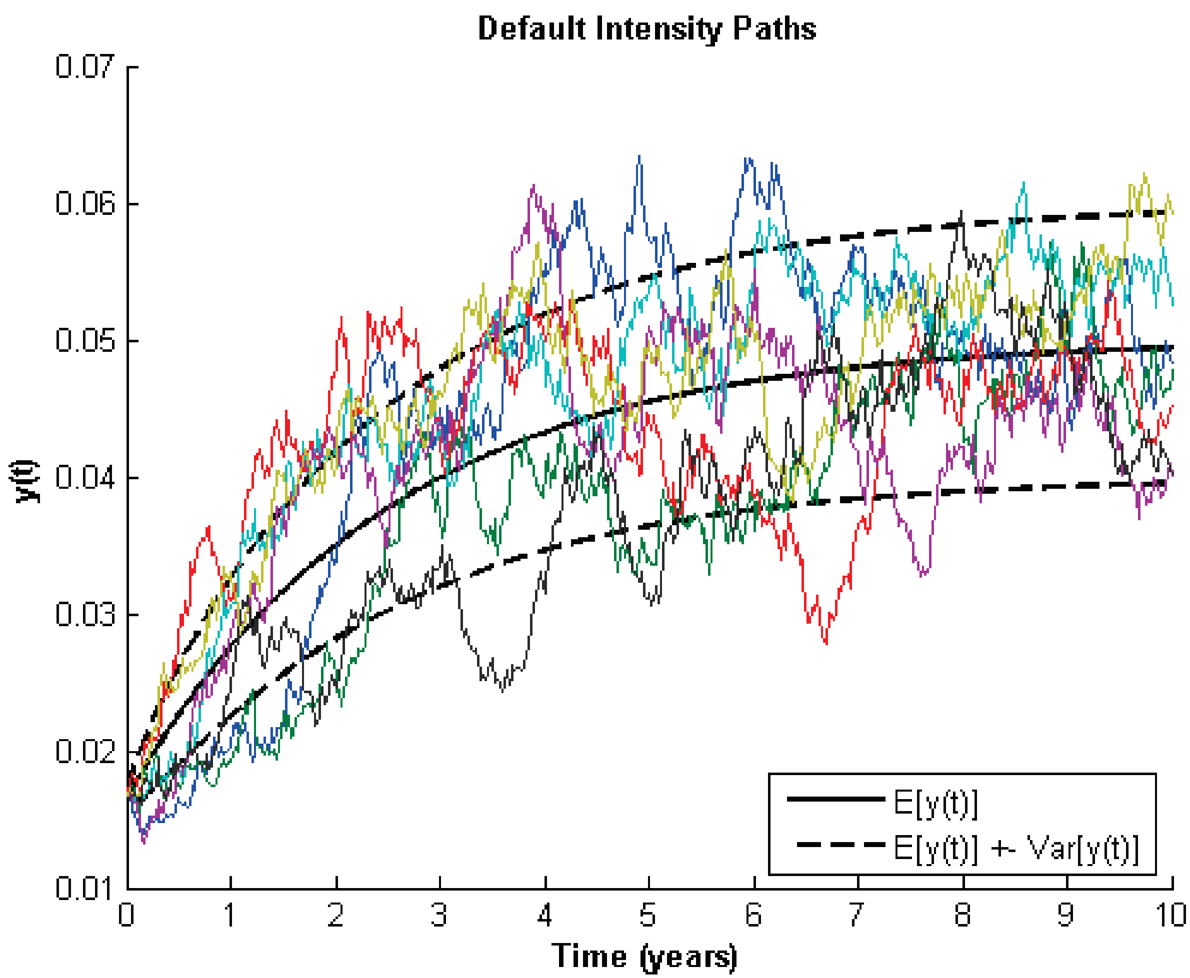


Figure: $y0 = 0.0165, \kappa = 0.4, \mu = 0.05, \nu = 0.04$

EXERCISE: The CIR model

Assume we are given a stochastic intensity process of CIR type,

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW(t)$$

where y_0, κ, μ, ν are positive constants. W is a brownian motion under the risk neutral measure.

a) Increasing κ increases or decreases randomness in the intensity? And ν?

b) The mean of the intensity at future times is affected by k? And by ν?

c) What happens to mean of the intensity when time grows to infinity?

d) Is it true that, because of mean reversion, the variance of the intensity goes to zero (no randomness left) when time grows to infinity?

e) Can you compute a rough approximation of the percentage volatility in the intensity?

EXERCISE: The CIR model

f) Suppose that $y_0 = 400bps = 0.04$, $\kappa = 0.3$, $\nu = 0.001$ and $\mu = 400bps$. Can you guess the behaviour of the future random trajectories of the stochastic intensity after time 0?
g) Can you guess the spread of a CDS with 10y maturity with the above stochastic intensity when the recovery is 0.35?

EXERCISE Solutions. I

a) We can refer to the formulas for the mean and variance of y_T in a CIR model as seen from time 0, at a given T. The formula for the variance is known to be (see for Example Brigo and Mercurio (2006))

$$\mathrm{VAR}(y_T) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2$$

whereas the mean is

$$E[y_T] = y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

We can see that for k becoming large the variance becomes small, since the exponentials decrease in k and the division by k gives a small value for large k. In the limit

$$\lim_{\kappa \to +\infty} \mathrm{VAR}(y_T) = 0$$

so that for very large κ there is no randomness left.

EXERCISE Solutions. II

We can instead see that VAR(y_T) is proportional to ν^2, so that if ν increases randomness increases, as is obvious from $\nu\sqrt{y_t}$ being the instantaneous volatility in the process y.

b) As the mean is

$$E[y_T] = y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

we clearly see that this is impacted by κ (indeed, ”speed of mean reversion”) and by μ clearly (”long term mean”) but not by the instantaneous volatility parameter ν.
c) As T goes to infinity, we get for the mean

$$\lim_{T\to+\infty} y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T}) = \mu$$

so that the mean tends to μ (this is why μ is called ”long term mean”).

EXERCISE Solutions. III

d) In the limit where time goes to infinity we get, for the variance

$$\lim_{T\to+\infty}[y_0\frac{\nu^2}{\kappa}(e^{-\kappa T}-e^{-2\kappa T})+\mu\frac{\nu^2}{2\kappa}(1-e^{-\kappa T})^2]=\mu\frac{\nu^2}{2\kappa}$$

So this does not go to zero. Indeed, mean reversion here implies that as time goes to infinite the mean tends to μ and the variance to the constant value $\mu\frac{\nu^2}{2\kappa}$, but not to zero.

EXERCISE Solutions. IV

e) Rough approximations of the percentage volatilities in the intensity would be as follows. The instantaneous variance in dy_t, conditional on the information up to t, is (remember that $VAR(dW(t)) = dt$)

$$VAR(dy_t) = \nu^2 y_t dt$$

The percentage variance is

$$VAR\left(\frac{dy_t}{y_t}\right) = \frac{\nu^2 y_t}{y_t^2} dt = \frac{\nu^2}{y_t} dt$$

and is state dependent, as it depends on y_t. We may replace y_t with either its initial value y_0 or with the long term mean μ, both known. The two rough percentage volatilities estimates will then be, for $dt = 1$,

$$\sqrt{\frac{\nu^2}{y_0}} = \frac{\nu}{\sqrt{y_0}}, \quad \sqrt{\frac{\nu^2}{\mu}} = \frac{\nu}{\sqrt{\mu}}$$

EXERCISE Solutions. V

These however do not take into account the important impact of κ in the overall volatility of finite (as opposed to instantaneous) credit spreads and are therefore relatively useless.

EXERCISE Solutions. VI

f) First we check if the positivity condition is met.

$$2\kappa\mu = 2 \cdot 0.3 \cdot 0.04 = 0.024; \quad \nu^2 = 0.001^2 = 0.000001$$

hence $2\kappa\mu > \nu^2$ and trajectories are positive. Then we observe that the variance is very small: Take $T = 5y$,

$$\mathrm{VAR}(y_T) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \theta \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2 \approx 0.0000006.$$

Take the standard deviation, given by the square root of the variance:

$$\mathrm{STDEV}(y_T) \approx \sqrt{0.0000006} = 0.00077.$$

which is much smaller of the level 0.04 at which the intensity refers both in terms of initial value and long term mean. Therefore there is almost no randomness in the system as the variance is very small compared to the initial point and the long term mean.

EXERCISE Solutions. VII

Hence there is almost no randomness, and since the initial condition y_0 is the same as the long term mean $\mu_0 = 0.04$, the intensity will behave as if it had the value 0.04 all the time. All future trajectories will be very close to the constant value 0.04.
g) In a constant intensity model the CDS spread can be approximated by

$$y = \frac{R_{CDS}}{1 - REC} \Rightarrow R_{CDS} = y(1 - REC) = 0.04(1 - 0.35) = 260bps$$

CIR++ stochastic intensity λ. I

Calibrating Implied Default Probabilities

For restrictions on the β's that keep ψ and hence λ positive, **as is required in intensity models**, we may use the results in B. and M. (2001) or (2006). We will often use the hazard process $\Lambda(t) = \int_0^t \lambda_s ds$, and also $Y(t) = \int_0^t y_s ds$ and $\Psi(t, \beta) = \int_0^t \psi(s, \beta) ds$.

If we can read from the market some implied risk-neutral default probabilities, and associate to them implied hazard functions Γ^{Mkt} (as we have done in the Lehman example), we may wish our stochastic intensity model to agree with them. By recalling that survival probabilities look exactly like bonds formulas in short rate models for r, we see that our model agrees with the market if

$$\exp(-\Gamma^{\text{Mkt}}(t)) = \exp\left(-\Psi(t, \beta)\right) \mathbb{E}[e^{-\int_0^t y_s ds}]$$

CIR++ stochastic intensity λ. II

Calibrating Implied Default Probabilities

IMPORTANT 1: This is possible only if λ is strictly positive;
IMPORTANT 2: It is fundamental, if we aim at calibrating default probabilities, that the last expected value can be computed analytically.
The only known diffusion model used in interest rates satisfying both constraints is CIR++

CIR++ stochastic intensity λ

Calibrating Implied Default Probabilities

$$\exp(-\Gamma^{\mathrm{Mkt}}(t)) = \mathbb{Q}\{\tau > t\} = \exp\left(-\Psi(t,\beta)\right)\mathbb{E}[e^{-\int_0^t y_s ds}]$$

Now notice that $\mathbb{E}[e^{-\int_0^t y_s ds}]$ is simply the bond price for a CIR interest rate model with short rate given by y, so that it is known analytically. We denote it by $P^y(0,t,y_0;\beta)$.

Similarly to the interest-rate case, λ is calibrated to the market implied hazard function Γ^{Mkt} if we set

$$\Psi(t,\beta) := \Gamma^{\mathrm{Mkt}}(t) + \ln(P^y(0,t,y_0;\beta))$$

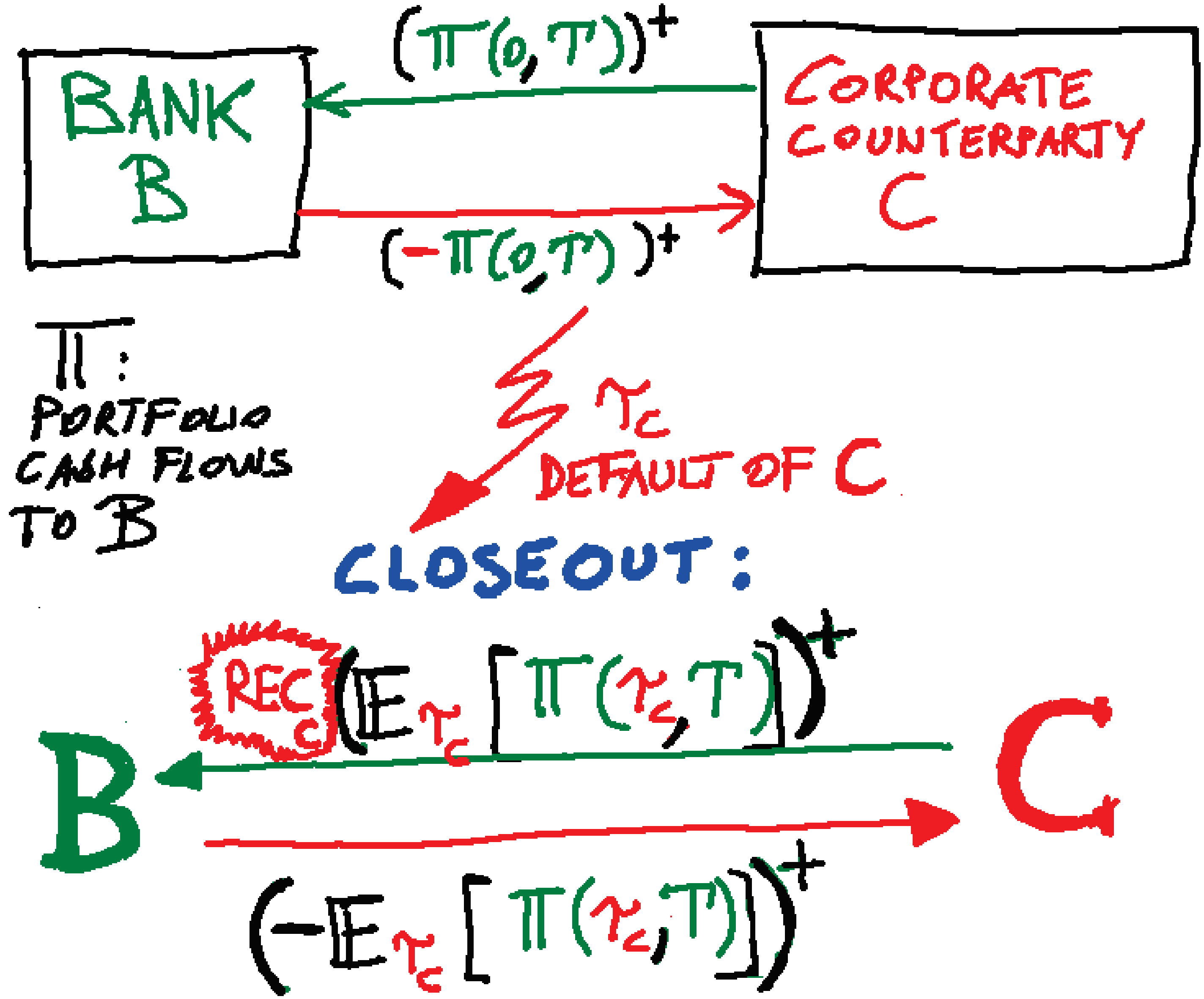
where we choose the parameters β in order to have a positive function ψ, by resorting to the condition seen earlier.

This concludes our introduction to Defaultable Bonds, CDS, credit spreads and intensity models.

We now turn to using such tools in one of the problems the industry is facing right now:

Pricing of counterparty credit risk, leading to the notion of Credit Valuation Adjustment (CVA)

Context


BANK
B
CORPORATE COUNTERPARTY
C
$(\Pi(0,T))^+$
$(-\Pi(0,T))^+$
Π: PORTFOLIO CASH FLOWS TO B
τ_C
DEFAULT OF C
CLOSEOUT:
$REC_C(\mathbb{E}_{\tau_C}[\Pi(\tau_C,T)])^+$
B
C
$(-\mathbb{E}_{\tau_C}[\Pi(\tau_C,T)])^+$

Q & A: What is Counterparty Credit Risk?

Q What is counterparty risk in general?

A *The risk taken on by an entity entering an OTC contract with a counterparty having a relevant default probability. As such, the counterparty might not respect its payment obligations.*

> *The counterparty credit risk is defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flows. An economic loss would occur if the transactions or portfolio of transactions with the counterparty has a positive economic value at the time of default.*
>
> [Basel II, Annex IV, 2/A]

Q & A: Credit VaR and CVA

Q What is the difference between Credit VaR and CVA?

A *They are both related to credit risk.*

- *Credit VaR is a Value at Risk type measure, a Risk Measure. it measures a potential loss due to counterparty default.*
- *CVA is a price, it stands for Credit Valuation Adjustment and is a price adjustment. CVA is obtained by pricing the counterparty risk component of a deal, similarly to how one would price a credit derivative.*

Q & A: Credit VaR and CVA

Q What is the difference in practical use?

A *Credit VaR answers the question:*

- *"How much can I lose of this portfolio, within (say) one year, at a confidence level of 99%, due to default risk and exposure?"*
- *CVA instead answers the question:*
  "How much discount do I get on the price of this deal due to the fact that you, my counterparty, can default? I would trade this product with a default free party. To trade it with you, who are default risky, I require a discount."

Clearly, a price needs to be more precise than a risk measure, so the techniques will be different.

Q & A: Credit VaR and CVA

Q Different? Are the methodologies for Credit VaR and CVA not similar?

A *There are analogies but CVA needs to be more precise in general. Also, Credit VaR should use statistics under the physical measure whereas CVA should use statistics under the pricing measure*

Q What are the regulatory bodies involved?

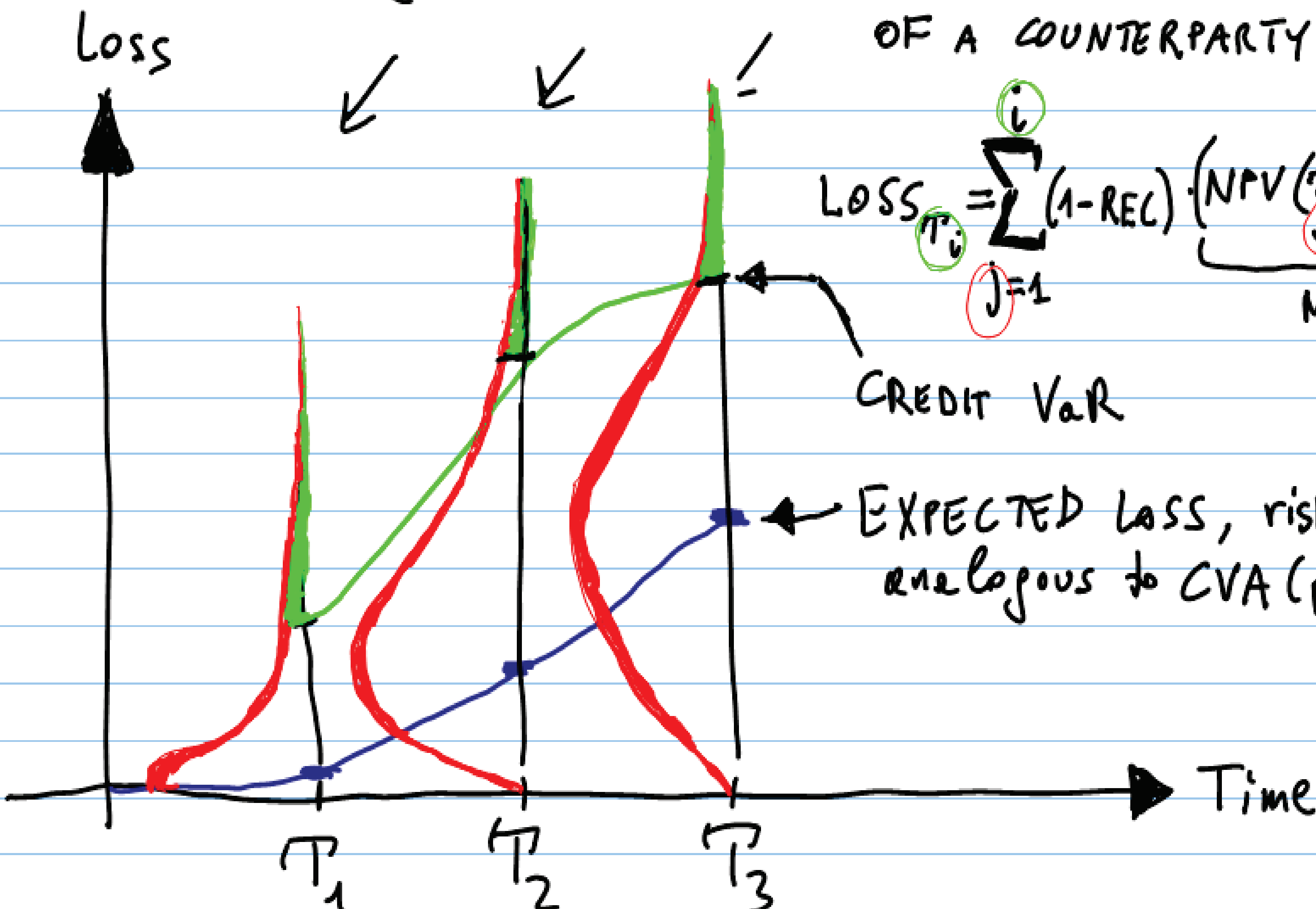
A *There are many, for Credit VaR type measures it is mostly Basel II and now III, whereas for CVA we have IAS, FASB and ISDA. But the picture is now blurring since Basel III is quite interested in CVA too*

Q What is the focus of this presentation?

A *We will focus on CVA.*

Q & A: Credit VaR and CVA

LOSS DISTRIBUTIONS DUE TO DEFAULTS OF A COUNTERPARTY



$$\mathrm{LOSS}_{T_i} = \sum_{j=1}^{i} (1-\mathrm{REC}) \cdot \underbrace{\left(\mathrm{NPV}(T_j, T_{fin})\right)^+}_{\text{MARKET RISK}} \underbrace{\mathbf{1}_{\{\tau \in [T_{j-1}, T_j]\}}}_{\text{DEFAULT RISK}}$$

LOSSES ARE BUILT UNDER THE HISTORICAL MEASURE.

Q & A: CVA and Model Risk, WWR

Q What impacts counterparty risk CVA?

A *The OTC contract's underlying volatility, the correlation between the underlying and default of the counterparty, and the counterparty credit spreads volatility.*

Q Is it model dependent?

A *It is highly model dependent even if the original portfolio without counterparty risk was not. There is a lot of model risk.*

Q What about *wrong way risk*?

A *The amplified risk when the reference underlying and the counterparty are strongly correlated in the wrong direction.*

Q & A: Collateral

Q What is collateral?

A *It is a guarantee (liquid and secure asset, cash) that is deposited in a collateral account in favour of the investor party facing the exposure. If the depositing counterparty defaults, thus not being able to fulfill payments associated to the above mentioned exposure, Collateral can be used by the investor to offset its loss.*

Q & A: Netting

Q What is netting?

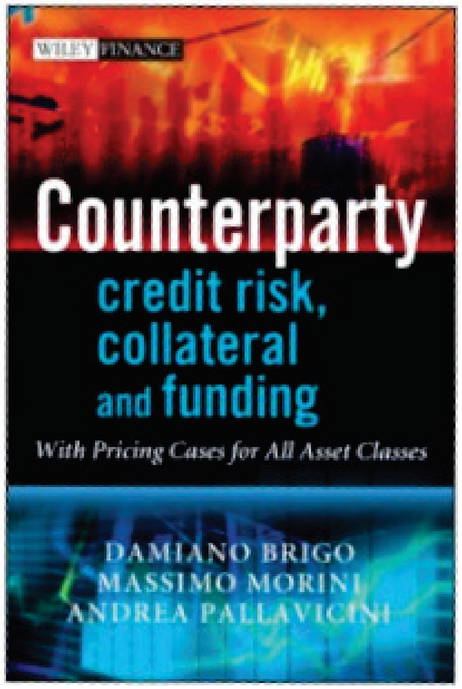
A *This is the agreement to net all positions towards a counterparty in the event of the counterparty default. This way positions with negative PV can be offset by positions with positive PV and counterparty risk is reduced. This has to do with the option on a sum being smaller than the sum of the options. CVA is typically computed on netting sets.*

For an introductory dialogue on Counterparty Risk see

CVA Q&A

D. Brigo (2012). Counterparty Risk Q&A: Credit VaR, CVA, DVA, Closeout, Netting, Collateral, Re-hypothecation, Wrong Way Risk, Basel, Funding, and Margin Lending. SSRN.com and arXiv.org.

Check also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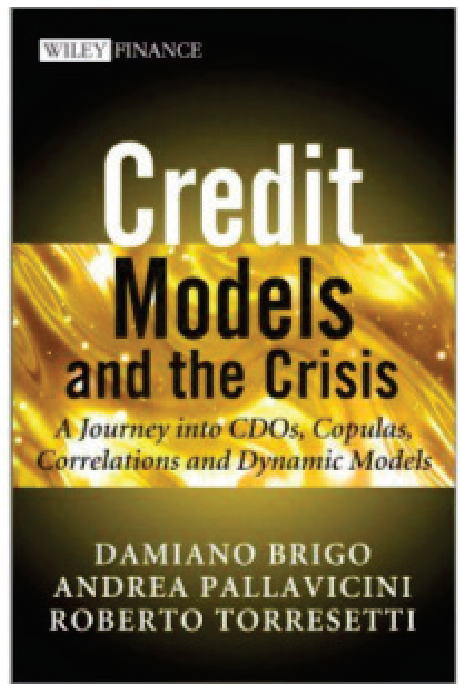
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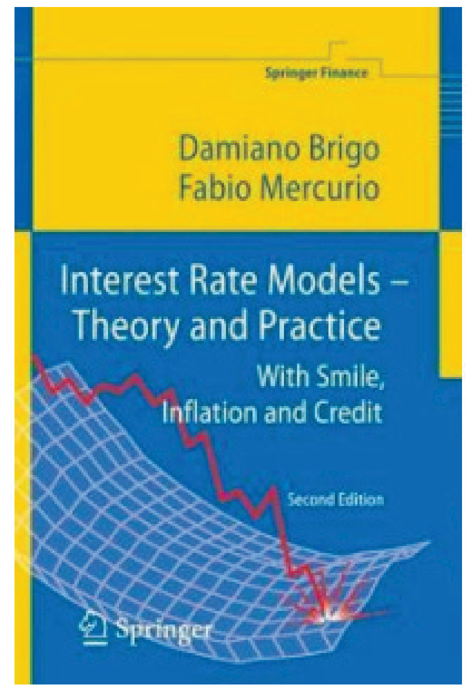

Springer Finance
Damiano Brigo
Fabio Mercurio
Interest Rate Models –
Theory and Practice
With Smile,
Inflation and Credit
Second Edition
Springer

General Notation

- We will call “Bank” or sometimes the ”investor” the party interested in the counterparty adjustment. This is denoted by “B”
- We will call “counterparty” the party with whom the Bank is trading, and whose default may affect negatively the Bank. This is denoted by “C”.
- “1” will be used for the underlying name/risk factor(s) of the contract
- The counterparty’s default time is denoted with τ_C and the recovery rate for unsecured claims with $\text{R}_{\text{EC}C}$ (we often use $\text{L}_{\text{GD}C} := 1 - \text{R}_{\text{EC}C}$).
- $\Pi_B(t, T)$ is the discounted payout without default risk seen by ‘B’ (sum of all future cash flows between t and T, discounted back at t). $\Pi_C(t, T) = -\Pi_B(t, T)$ is the same quantity but seen from the point of view of ‘C’. When we omit the index B or C we mean ‘B’.

Examples of products Π

If "B" enters an interest rate swap where "B" pays fixed K and receives from "C" LIBOR L with tenor $T_\alpha, T_{\alpha+1}, \ldots, T_\beta$, then the payout is written, as we have seen earlier, as

$$\Pi(0, T_\beta) = \sum_{i=\alpha+1}^{\beta} D(0, T_i)(T_i - T_{i-1})(L(T_{i-1}, T_i) - K).$$

The majority of the instruments that are subject to Counterparty risk is given by Interest Rate Swaps.

General Notation

- We define $NPV_B(t,T) = \mathbb{E}_t[\Pi(t,T)]$. When T is clear from the context we omit it and write NPV(t).
- $$\Pi(s,t) + D(s,t)\Pi(t,u) = \Pi(s,u)$$
- $$\mathbb{E}_0[D(0,u)NPV(u,T)] = \mathbb{E}_0[D(0,u)\mathbb{E}_u[\Pi(u,T)]] =$$
$$= \mathbb{E}_0[D(0,u)\Pi(u,T)] = NPV(0,T) - \mathbb{E}_0[\Pi(0,u)]$$
$$= NPV(0,T) - NPV(0,u)$$

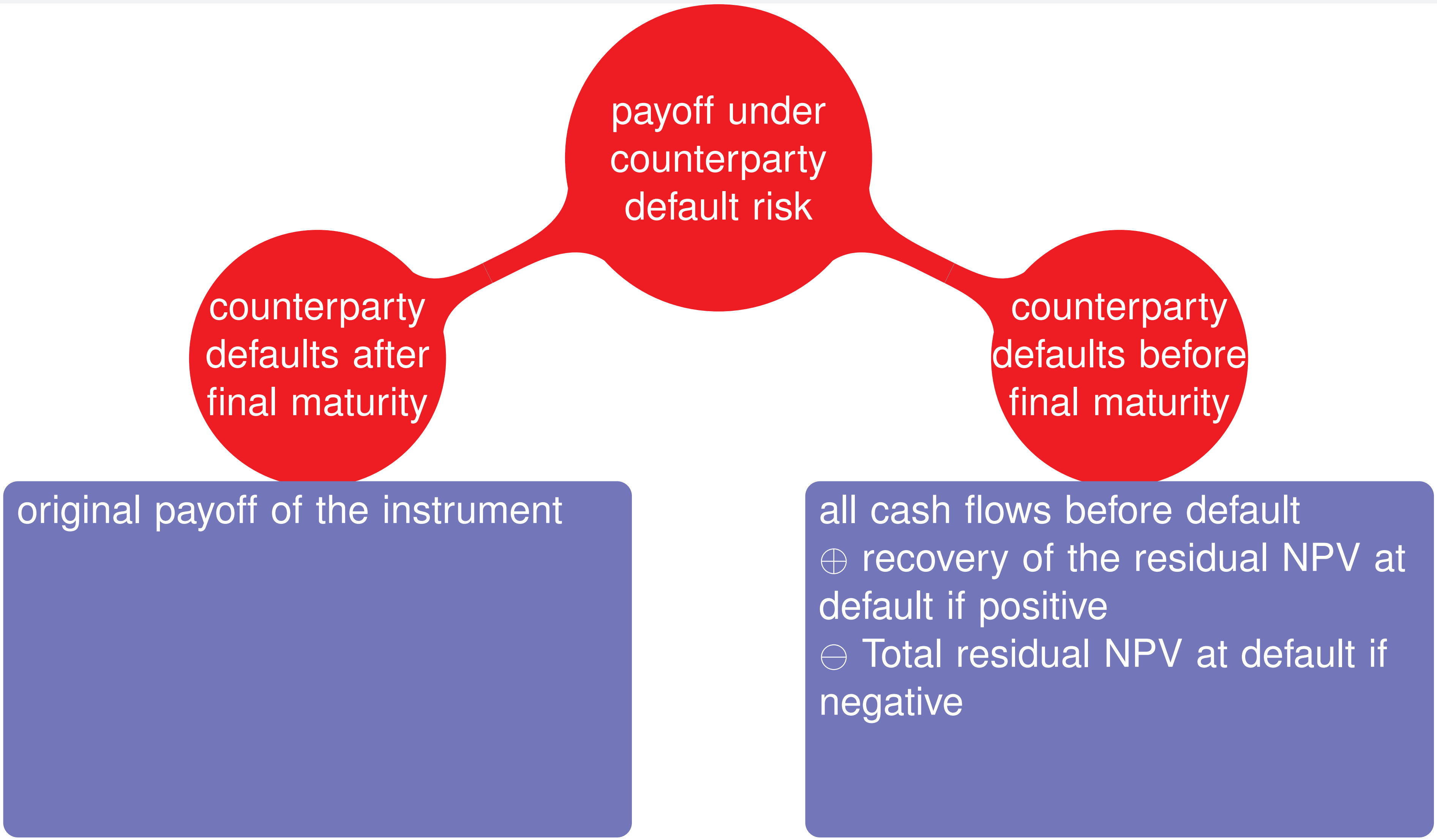
Unilateral counterparty risk

We now look into unilateral counterparty risk.

This is a situation where counterparty risk pricing is computed by assuming that only the counterparty can default, whereas the investor or bank doing the calculation is assumed to be default free.

Hence we will only consider here the default time τ_C of the counterparty. We will address the bilateral case later on.

The mechanics of Evaluating unilateral counterparty risk



General Formulation under Asymmetry

$$\Pi_B^D(t,T) = \mathbf{1}_{\tau_C > T}\Pi_B(t,T)$$
$$+\mathbf{1}_{t<\tau_C\le T}\left[\Pi_B(t,\tau_C) + D(t,\tau_C)\left(REC_C\,(\mathrm{NPV}_B(\tau_C))^+ - (-\mathrm{NPV}_B(\tau_C))^+\right.\right.$$

This last expression is the general payoff seen from the point of view of 'B' (Π_B, NPV_B) under unilateral counterparty default risk. Indeed,

1. if there is no early default, this expression reduces to first term on the right hand side, which is the payoff of a default-free claim.
2. In case of early default of the counterparty, the payments due before default occurs are received (second term)
3. and then if the residual net present value is positive only the recovery value of the counterparty REC_C is received (third term),
4. whereas if it is negative it is paid in full by the investor/ Bank (fourth term).

General Formulation under Asymmetry

If one simplifies the cash flows and takes the risk neutral expectation, one obtains the fundamental formula for the valuation of counterparty risk when the investor/ Bank B is default free:

$$\mathbb{E}_t \left\{\Pi_B^D(t,T)\right\} =$$
$$\mathbf{1}_{\{\tau_C > t\}} \mathbb{E}_t \left\{\Pi_B(t,T)\right\} - \mathbb{E}_t \left\{ \mathrm{LGD_C} \mathbf{1}_{\{t < \tau_C \le T\}} D(t,\tau_C) \left[\mathrm{NPV_B}(\tau_C)\right]^+ \right\} \quad (*)$$

- First term : Value without counterparty risk.
- Second term : Unilateral Counterparty Valuation Adjustment
- $\mathrm{NPV}(\tau_C) = \mathbb{E}_{\tau_C}\left[\Pi(\tau_C, T)\right]$ is the value of the transaction on the counterparty default date. LGD = 1 - REC_counterparty.

$$\mathrm{UCVA}_0 = \mathbb{E}_t \left\{ \mathrm{LGD_C} \mathbf{1}_{\{t < \tau_C \le T\}} D(t,\tau_C) \left[\mathrm{NPV_B}(\tau_C)\right]^+ \right\}$$

Proof of the formula

In the proof we omit indices: $\tau = \tau_C$, REC=REC$_C$, LGD=LGD$_C$, NPV=NPV$_B$, $\Pi = \Pi_B$. The proof is obtained easily putting together the following steps. Since

$$\mathbf{1}_{\{\tau > t\}} \Pi(t,T) = \mathbf{1}_{\{\tau > T\}} \Pi(t,T) + \mathbf{1}_{\{t < \tau \leq T\}} \Pi(t,T)$$

we can rewrite the terms inside the expectation in the right hand side of the simplified formula (*) as

$$\begin{aligned}
& \mathbf{1}_{\{\tau > t\}} \Pi(t,T) - \left\{ LGD \mathbf{1}_{\{t < \tau \leq T\}} D(t,\tau) \left[\mathrm{NPV}(\tau)\right]^+ \right\} \\
& = \mathbf{1}_{\{\tau > T\}} \Pi(t,T) + \mathbf{1}_{\{t < \tau \leq T\}} \Pi(t,T) \\
+ \quad & \{(\mathrm{R}_{\mathrm{EC}} - 1)[\mathbf{1}_{\{t < \tau \leq T\}} D(t,\tau)(\mathrm{NPV}(\tau))^+]\} \\
= \quad & \mathbf{1}_{\{\tau > T\}} \Pi(t,T) + \mathbf{1}_{\{t < \tau \leq T\}} \Pi(t,T) \\
+ \quad & \mathrm{R}_{\mathrm{EC}} \ \mathbf{1}_{\{t < \tau \leq T\}} D(t,\tau)(\mathrm{NPV}(\tau))^+ - \mathbf{1}_{\{t < \tau \leq T\}} D(t,\tau)(\mathrm{NPV}(\tau))^+
\end{aligned}$$

Conditional on the information at τ the second and the fourth terms are equal to

Proof (cont'd)

$$
\begin{aligned}
& E_\tau[1_{\{t<\tau\le T\}}\Pi(t,T) - 1_{\{t<\tau\le T\}}D(t,\tau)(\mathrm{NPV}(\tau))^+] \\
= \; & E_\tau[1_{\{t<\tau\le T\}}[\Pi(t,\tau) + D(t,\tau)\Pi(\tau,T) - D(t,\tau)(E_\tau[\Pi(\tau,T)])^+]] \\
= \; & 1_{\{t<\tau\le T\}}[\Pi(t,\tau) + D(t,\tau)E_\tau[\Pi(\tau,T)] - D(t,\tau)(E_\tau[\Pi(\tau,T)])^+] \\
= \; & 1_{\{t<\tau\le T\}}[\Pi(t,\tau) - D(t,\tau)(E_\tau[\Pi(\tau,T)])^-] \\
= \; & 1_{\{t<\tau\le T\}}[\Pi(t,\tau) - D(t,\tau)(E_\tau[-\Pi(\tau,T)])^+] \\
= \; & 1_{\{t<\tau\le T\}}[\Pi(t,\tau) - D(t,\tau)(-\mathrm{NPV}(\tau))^+]
\end{aligned}
$$

since

$$
1_{\{t<\tau\le T\}}\Pi(t,T) = 1_{\{t<\tau\le T\}}\{\Pi(t,\tau) + D(t,\tau)\Pi(\tau,T)\}
$$

and $f = f^+ - f^- = f^+ - (-f)^+$.

Proof (cont'd)

Then we can see that after conditioning the whole expression of the original long payoff on the information at time τ and substituting the second and the fourth terms just derived above, the expected value with respect to $\mathcal{F}_t$ coincides exactly with the one in our simplified formula (*) by the properties of iterated expectations by which $\mathbb{E}_t[X] = \mathbb{E}_t[\mathbb{E}_\tau[X]]$.

What we can observe

- Including counterparty risk in the valuation of an otherwise default-free derivative $\Longrightarrow$ credit (hybrid) derivative.

- The inclusion of counterparty risk adds a level of optionality to the payoff.
  In particular, model independent products become model dependent also in the underlying market.
  $\Longrightarrow$ **Counterparty Risk analysis incorporates an opinion about the underlying market dynamics and volatility.**

The point of view of the counterparty “C”

The deal from the point of view of ‘C’, while staying in a world where only ‘C” may default.

$$\Pi_C^D(t,T) = \mathbf{1}_{\tau_C > T}\Pi_C(t,T)$$
$$+\mathbf{1}_{t<\tau_C\le T}\left[\Pi_C(t,\tau_C) + D(t,\tau_C)\left((\mathrm{NPV}_C(\tau_C))^+ - REC_C\,(-\mathrm{NPV}_C(\tau_C))^+\right)\right]$$

This last expression is the general payoff seen from the point of view of ‘C’ (Π_C, NPV_C) under unilateral counterparty default risk. Indeed,

1. if there is no early default, this expression reduces to first term on the right hand side, which is the payoff of a default-free claim.
2. In case of early default of the counterparty ‘C”, the payments due before default occurs go through (second term)
3. and then if the residual net present value is positive to the defaulted ‘C’, it is received in full from ‘B’ (third term),
4. whereas if it is negative, only the recovery fraction REC_C it is paid to ‘B’ (fourth term).

The point of view of the counterparty "C"

The above formula simplifies to

$$\mathbb{E}_t\left\{\Pi^D_C(t,T)\right\} =$$
$$\mathbf{1}_{\tau_C>t}\mathbb{E}_t\left\{\Pi_C(t,T)\right\} + \mathbb{E}_t\left\{\mathrm{LGD}_C \mathbf{1}_{t<\tau_C\le T} D(t,\tau_C)\left[-\mathrm{NPV}_C(\tau_C)\right]^+\right\}$$

and the adjustment term with respect to the risk free price $\mathbb{E}_t\left\{\Pi_C(t,T)\right\}$ is called

UNILATERAL DEBIT VALUATION ADJUSTMENT

$$\mathrm{UDVA}_C(t) = \mathbb{E}_t\left\{\mathrm{LGD}_C \mathbf{1}_{\{t<\tau_C\le T\}} D(t,\tau_C)\left[-\mathrm{NPV}_C(\tau_C)\right]^+\right\}$$

We note that $\mathrm{UDVA}_C = \mathrm{UCVA}_B$.
Notice also that in this universe $\mathrm{UDVA}_B = \mathrm{UCVA}_C = 0$.

Including the investor/ Bank default or not?

Often the investor, when computing a counterparty risk adjustment, considers itself to be default-free. This can be either a unrealistic assumption or an approximation for the case when the counterparty has a much higher default probability than the investor.

If this assumption is made when no party is actually default-free, the unilateral valuation adjustment is asymmetric: if “C” were to consider itself as default free and “B” as counterparty, and if “C” computed the counterparty risk adjustment, this would not be the opposite of the one computed by “B” in the straight case.

Also, the total NPV including counterparty risk is similarly asymmetric, in that the total value of the position to “B” is not the opposite of the total value of the position to “C”. There is no *cash conservation.*

Including the investor/ Bank default or not?

We get back symmetry if we allow for default of the investor/ Bank in computing counterparty risk. This also results in an adjustment that is cheaper to the counterparty “C”.

The counterparty “C” may then be willing to ask the investor/ Bank “B” to include the investor default event into the model, when the Counterparty risk adjustment is computed by the investor

The case of symmetric counterparty risk

Suppose now that we allow for both parties to default. Counterparty risk adjustment allowing for default of "B"?
"B" : the investor; "C": the counterparty;
("1": the underlying name/risk factor of the contract).
τ_B, τ_C: default times of "B" and "C". T: final maturity
We consider the following events, forming a partition

Four events ordering the default times

$$\begin{aligned} \mathcal{A} &= \{\tau_B \le \tau_C \le T\} \qquad E = \{T \le \tau_B \le \tau_C\} \\ \mathcal{B} &= \{\tau_B \le T \le \tau_C\} \qquad F = \{T \le \tau_C \le \tau_B\} \\ \mathcal{C} &= \{\tau_C \le \tau_B \le T\} \\ \mathcal{D} &= \{\tau_C \le T \le \tau_B\} \end{aligned}$$

Define $\mathrm{NPV}_{\{B,C\}}(t) := \mathbb{E}_t[\Pi_{\{B,C\}}(t,T)]$, and recall $\Pi_B = -\Pi_C$.

The case of symmetric counterparty risk

$$\Pi^D_B(t,T) = \mathbf{1}_{E\cup F}\Pi_B(t,T)$$
$$+\mathbf{1}_{\mathcal{C}\cup\mathcal{D}}\left[\Pi_B(t,\tau_C) + D(t,\tau_C)\left(REC_C\,(\mathrm{NPV}_B(\tau_C))^+ - (-\mathrm{NPV}_B(\tau_C))^+\right)\right]$$
$$+\mathbf{1}_{\mathcal{A}\cup\mathcal{B}}\left[\Pi_B(t,\tau_B) + D(t,\tau_B)\left((\mathrm{NPV}_B(\tau_B))^+ - REC_B\,(-\mathrm{NPV}_B(\tau_B))^+\right)\right]$$

1. If no early default $\Rightarrow$ payoff of a default-free claim (1st term).
2. In case of early default of the counterparty, the payments due before default occurs are received (second term),
3. and then if the residual net present value is positive only the recovery value of the counterparty REC_C is received (third term),
4. whereas if negative, it is paid in full by the investor/ Bank (4th term).
5. In case of early default of the investor, the payments due before default occurs are received (fifth term),
6. and then if the residual net present value is positive it is paid in full by the counterparty to the investor/ Bank (sixth term),
7. whereas if it is negative only the recovery value of the investor/ Bank REC_B is paid to the counterparty (seventh term).

The case of symmetric counterparty risk

$$\mathbb{E}_t\left\{\Pi^D_B(t,T)\right\} = \mathbb{E}_t\left\{\Pi_B(t,T)\right\} + \mathrm{DVA}_B(t) - \mathrm{CVA}_B(t)$$

$$\mathrm{DVA}_B(t) = \mathbb{E}_t\left\{\mathrm{LGD_B}\cdot \mathbf{1}(t < \tau^{1st} = \tau_B < T)\cdot D(t,\tau_B)\cdot[-\mathrm{NPV_B}(\tau_B)]^+\right\}$$

$$\mathrm{CVA}_B(t) = \mathbb{E}_t\left\{\mathrm{LGD_C}\cdot \mathbf{1}(t < \tau^{1st} = \tau_C < T)\cdot D(t,\tau_C)\cdot[\mathrm{NPV_B}(\tau_C)]^+\right\}$$

$$1(A\cup B) = 1(t<\tau^{1st}=\tau_B<T),\quad 1(C\cup D) = 1(t<\tau^{1st}=\tau_C<T)$$

- Obtained simplifying the previous formula and taking expectation.
- 2nd term : adj due to scenarios $\tau_B < \tau_C$. This is positive to the investor/ Bank B and is called "Debit Valuation Adjustment" (DVA)
- 3d term : Counterparty risk adj due to scenarios $\tau_C < \tau_B$
- Bilateral Valuation Adjustment as seen from B: $\mathrm{BVA}_B = \mathrm{DVA}_B - \mathrm{CVA}_B$.
- If computed from the opposite point of view of "C" having counterparty "B", $\mathrm{BVA}_C = -\mathrm{BVA}_B$. Symmetry.

The case of symmetric counterparty risk

Strange consequences of the formula new mid term, i.e. DVA

- credit quality of investor WORSENS $\Rightarrow$ books POSITIVE MARK TO MKT
- credit quality of investor IMPROVES $\Rightarrow$ books NEGATIVE MARK TO MKT
- Citigroup in its press release on the first quarter revenues of 2009 reported a *positive* mark to market due to its *worsened* credit quality: “Revenues also included [...] a net 2.5$ billion positive CVA on derivative positions, excluding monolines, mainly due to the widening of Citi’s CDS spreads”

The case of symmetric counterparty risk: DVA?

`October 18, 2011, 3:59 PM ET, WSJ. Goldman Sachs Hedges Its Way to Less Volatile Earnings.`

Goldman's DVA gains in the third quarter totaled $450 million [...] That amount is comparatively smaller than the $1.9 billion in DVA gains that J.P. Morgan Chase and Citigroup each recorded for the third quarter. Bank of America reported $1.7 billion of DVA gains in its investment bank. Analysts estimated that Morgan Stanley will record $1.5 billion of net DVA gains when it reports earnings on Wednesday [...]

Is DVA real? **DVA Hedging**. Buying back bonds? Proxying?

DVA hedge? One should sell protection on oneself, impossible, unless one buys back bonds that he had issued earlier. Very Difficult. Most times: proxying. Instead of selling protection on oneself, one sells protection on a number of names that one thinks are highly correlated to oneself.

The case of symmetric counterparty risk: DVA?

Again from the WSJ article above:

[...] Goldman Sachs CFO David Viniar said Tuesday that the company attempts to hedge [DVA] using a basket of different financials.
A Goldman spokesman confirmed that the company did this by selling CDS on a range of financial firms. [...] Goldman wouldn't say what specific financials were in the basket, but Viniar confirmed [...] that the basket contained 'a peer group.'

This can approximately hedge the spread risk of DVA, but not the jump to default risk. Merrill hedging DVA risk by selling protection on Lehman would not have been a good idea. Worsens systemic risk.

DVA or no DVA? Accounting VS Capital Requirements

NO DVA: Basel III, page 37, July 2011 release

This CVA loss is calculated without taking into account any offsetting debit valuation adjustments which have been deducted from capital under paragraph 75.

YES DVA: FAS 157

Because nonperformance risk (the risk that the obligation will not be fulfilled) includes the reporting entitys credit risk, the reporting entity should consider the effect of its credit risk (credit standing) on the fair value of the liability in all periods in which the liability is measured at fair value under other accounting pronouncements FAS 157 (see also IAS 39)

DVA or no DVA? Accounting VS Capital Requirements

Stefan Walter says:

”The potential for perverse incentives resulting from profit being linked to decreasing creditworthiness means capital requirements cannot recognise it, says Stefan Walter, *secretary-general of the Basel Committee*: The main reason for not recognising DVA as an offset is that it would be inconsistent with the overarching supervisory prudence principle under which we do not give credit for increases in regulatory capital arising from a deterioration in the firms own credit quality.”

Funding and DVA

We will look at this more carefully when dealing with funding costs. For now:

DVA a component of FVA?

DVA is related to funding costs when the payout is uni-directional, eg shorting/issuing a bond, borrowing in a loan, or going short a call option.

Indeed, if we are short simple products that are uni-directional, we are basically borrowing.

As we shorted a bond or a call option, for example, we received cash V_0 in the beginning, and we will have to pay the product payout in the end.

This cash can be used by us to fund other activities, and allows us to spare the costs of fuding this cash V_0 from our treasury.

Funding and DVA

Our treasury usually funds in the market, and the market charges our treasury a cost of funding that is related to the borrowed amount V_0, to the period T and to our own bank credit risk $\tau_B < T$.

In this sense the funding cost we are sparing when we avoid borrowing looks similar to DVA: it is related to the price of the object we are shorting and to our own credit risk.

However quite a number of assumptions is needed to identify DVA with a pure funding benefit, as we will see below.

The case of symmetric counterparty risk: DVA?

When allowing for the investor to default: symmetry

- DVA: One more term with respect to the unilateral case.
- depending on credit spreads and correlations, the total adjustment to be subtracted (CVA-DVA) can now be either positive or negative. In the unilateral case it can only be positive.
- Ignoring the symmetry is clearly more expensive for the counterparty and cheaper for the investor.
- Hedging DVA is difficult. Hedging “by peers” ignores jump to default risk
- We assume the unilateral case in most of the numerical presentations
- WE TAKE THE POINT OF VIEW OF ‘B” from now on, so we omit the subscript ‘B’. We denote the counterparty as ‘C”.

Closeout: Replication (ISDA?) VS Risk Free

When we computed the bilateral adjustment formula from

$$
\begin{aligned}
&\Pi_B^D(t,T) = \mathbf{1}_{E\cup F}\Pi_B(t,T) \\
&+\mathbf{1}_{C\cup D}\left[\Pi_B(t,\tau_C) + D(t,\tau_C)\left(REC_C\,(\mathrm{NPV}_B(\tau_C))^+ - (-\mathrm{NPV}_B(\tau_C))^+\right)\right] \\
&+\mathbf{1}_{A\cup B}\left[\Pi_B(t,\tau_B) + D(t,\tau_B)\left((-\mathrm{NPV}_C(\tau_B))^+ - REC_B\,(\mathrm{NPV}_C(\tau_B))^+\right)\right]
\end{aligned}
$$

(where we now substituted $\mathrm{NPV}_B = -\mathrm{NPV}_C$ in the last two terms) we used the risk free NPV upon the first default, to close the deal. But what if upon default of the first entity, the deal needs to be valued by taking into account the credit quality of the surviving party? What if we make the substitutions

$$\mathrm{NPV}_B(\tau_C) \to \mathrm{NPV}_B(\tau_C) + \mathrm{UDVA}_B(\tau_C)$$

$$\mathrm{NPV}_C(\tau_B) \to \mathrm{NPV}_C(\tau_B) + \mathrm{UDVA}_C(\tau_B)?$$

Closeout: Replication (ISDA?) VS Risk Free

ISDA (2009) Close-out Amount Protocol.

”In determining a Close-out Amount, the Determining Party may consider any relevant information, including, [...] quotations (either firm or indicative) for replacement transactions supplied by one or more third parties that **may take into account the creditworthiness of the Determining Party** at the time the quotation is provided”

This makes valuation more continuous: upon default we still price including the DVA, as we were doing before default.

Closeout: Substitution (ISDA?) VS Risk Free

The final formula with subsitution closeout is quite complicated:

$$
\begin{aligned}
&\Pi^D_B(t,T) = \mathbf{1}_{E\cup F}\Pi_B(t,T) \\
&+\mathbf{1}_{C\cup D}\Big[\Pi_B(t,\tau_C) + D(t,\tau_C) \\
&\cdot \big(REC_C\,(\mathrm{NPV}_B(\tau_C) + \mathrm{UDVA}_B(\tau_C))^+ - (-\mathrm{NPV}_B(\tau_C) - \mathrm{UDVA}_B(\tau_C))^+\big) \\
&+\mathbf{1}_{A\cup B}\Big[\Pi_B(t,\tau_B) + D(t,\tau_B) \\
&\cdot \big((-\mathrm{NPV}_C(\tau_B) - \mathrm{UDVA}_C(\tau_B))^+ - REC_B\,(\mathrm{NPV}_C(\tau_B) + \mathrm{UDVA}_C(\tau_B))^+\big)
\end{aligned}
$$

Closeout: Substitution (ISDA?) VS Risk Free

B. and Morini (2010)

We analyze the Risk Free closeout formula in Comparison with the Replication Closeout formula for a Zero coupon bond when:

1. Default of 'B' and 'C" are independent
2. Default of 'B' and 'C" are co-monotonic

Suppose 'B' (the lender) holds the bond,
and 'C' (the borrower) will pay the notional 1 at maturity T.

The risk free price of the bond at time 0 to 'B' is denoted by $P(0, T)$.

Closeout: Replication (ISDA?) VS Risk Free

Suppose ‘B’ (the lender) holds the bond, and ‘C’ (the borrower) will pay the notional 1 at maturity T.
The risk free price of the bond at time 0 to ’B’ is denoted by $P(0, T)$.
If we assume deterministic interest rates, the above formulas reduce to

$$P^{D,Repl}(0,T) = P(0,T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \le T)]$$

$$\begin{aligned} P^{D,Free}(0,T) &= P(0,T)[\mathbb{Q}(\tau_C > T) + \mathbb{Q}(\tau_B < \tau_C < T) \\ &\quad + REC_C \mathbb{Q}(\tau_C \le \min(\tau_B, T))] \\ &= P(0,T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \le T) + LGD_C \mathbb{Q}(\tau_B < \tau_C < T)] \end{aligned}$$

Risk Free Closeout and Credit Risk of the Lender

The adjusted price of the bond DEPENDS ON THE CREDIT RISK OF THE LENDER ‘B’ IF WE USE THE RISK FREE CLOSEOUT. This is counterintuitive and undesirable.

Closeout: Replication (ISDA?) VS Risk Free

Co-Monotonic Case

If we assume the default of B and C to be co-monotonic, and the spread of the lender ‘B” to be larger, we have that the lender ‘B” defaults first in ALL SCENARIOS (e.g. ‘C’ is a subsidiary of ‘B’, or a company whose well being is completely driven by ‘B’: ‘C’ is a trye factory whose only client is car producer ‘B”). In this case

$$P^{D,Repl}(0,T) = P(0,T)[\mathbb{Q}(\tau_C > T) + REC_C \mathbb{Q}(\tau_C \leq T)]$$

$$P^{D,Free}(0,T) = P(0,T)[\mathbb{Q}(\tau_C > T) + \mathbb{Q}(\tau_C < T)] = P(0,T)$$

Risk free closeout is correct. Either ‘B” does not default, and then ‘C” does not default either, or if ‘B” defaults, at that precise time C is solvent, and B recovers the whole payment. Credit risk of ‘C” should not impact the deal.

Closeout: Substitution (ISDA?) VS Risk Free

Contagion. What happens at default of the Lender

$$P^{D,Subs}(t,T) = P(t,T)[\mathbb{Q}_t(\tau_C > T) + REC_C \mathbb{Q}_t(\tau_C \le T)]$$

$$P^{D,Free}(t,T) = P^{D,Subs}(t,T) + P(t,T) LGD_C \mathbb{Q}_t(\tau_B < \tau_C < T)$$

We focus on two cases:

- τ_B and τ_C are independent. Take $t < T$.

$$\mathbb{Q}_{t-\Delta t}(\tau_B < \tau_C < T) \mapsto \quad \{\tau_B = t\} \quad \mapsto \mathbb{Q}_{t+\Delta t}(\tau_C < T)$$

  and this effect can be quite sizeable.

- τ_B and τ_C are comonotonic. Take an example where $\tau_B = t < T$ implies $\tau_C = u < T$ with $u > t$. Then

$$\mathbb{Q}_{t-\Delta t}(\tau_C > T) \mapsto \quad \{\tau_B = t, \tau_C = u\} \quad \mapsto 0$$

$$\mathbb{Q}_{t-\Delta t}(\tau_C \le T) \mapsto \quad \{\tau_B = t, \tau_C = u\} \quad \mapsto 1$$

$$\mathbb{Q}_{t-\Delta t}(\tau_B < \tau_C < T) \mapsto \quad \{\tau_B = t, \tau_C = u\} \quad \mapsto 1$$

Closeout: Substitution (ISDA?) VS Risk Free

Let us put the pieces together:

- τ_B and τ_C are independent. Take $t < T$.

$$P^{D,Subs}(t - \Delta t, T) \mapsto \quad \{\tau_B = t\} \mapsto \quad \text{no change}$$

$$P^{D,Free}(t - \Delta t, T) \mapsto \quad \{\tau_B = t\} \mapsto \quad \text{add } \mathbb{Q}_{t-\Delta t}(\tau_B > \tau_C, \tau_C < T)$$

  and this effect can be quite sizeable.

- τ_B and τ_C are comonotonic. Take an example where $\tau_B = t < T$ implies $\tau_C = u < T$ with $u > t$. Then

$$P^{D,Subs}(t - \Delta t, T) \mapsto \quad \{\tau_B = t\} \mapsto \quad \text{subtract } X$$

$$X = LGD_C P(t, T)\mathbb{Q}_{t-\Delta t}(\tau_C > T)$$

$$P^{D,Free}(t - \Delta t, T) \mapsto \quad \{\tau_B = t\} \mapsto \quad \text{no change}$$

Closeout: Replication (ISDA?) VS Risk Free

The independence case: Contagion with Risk Free closeout

The Risk Free closeout shows that *upon default of the lender*, the mark to market to the lender itself jumps up, or equivalently **the mark to market to the borrower jumps down**. The effect can be quite dramatic.
The Replication closeout instead shows no such contagion, as the mark to market does not change upon default of the lender.

The co-monotonic case: Contagion with Replication closeout

The Risk Free closeout behaves nicely in the co-monotonic case, and there is no change upon default of the lender.
Instead the Replication closeout shows that *upon default of the lender* the mark to market to the lender jumps down, or equivalently **the mark to market to the borrower jumps up**.

Closeout: Replication (ISDA?) VS Risk Free

Impact of an early default of the Lender

Dependence→ Closeout↓	independence	co-monotonicity
Risk Free	Negatively affects Borrower	No contagion
Replication	No contagion	Further Negatively affects Lender

For a numerical case study and more details see Brigo and Morini (2010, 2011).

A simplified formula without τ^{1st} for bilateral VA

- The simplified formula is only a simplified representation of bilateral risk and ignores that upon the first default closeout proceedings are started, thus involving a degree of double counting
- It is attractive because it allows for the construction of a bilateral counterparty risk pricing system based only on a unilateral one.
- The correct formula involves default dependence between the two parties through τ^{1st} and allows no such incremental construction
- A simplified bilateral formula is possible also in case of substitution closeout, but it turns out to be identical to the simplified formula of the risk free closeout case.
- We analyze the impact of default dependence between investor ‘B’ and counterparty ‘C’ on the difference between the two formulas by looking at a zero coupon bond and at an equity forward.

A simplified formula without τ^{1st} for bilateral VA

One can show easily that the difference between the full correct formula and the simplified formula is

$$\begin{aligned} & E_0[1_{\{\tau_B<\tau_C<T\}} LGD_C D(0,\tau_C)(E_{\tau_C}(\Pi(\tau_C,T)))^+] \\ - \quad & E_0[1_{\{\tau_C<\tau_B<T\}} LGD_B D(0,\tau_B)(-E_{\tau_B}(\Pi(\tau_B,T)))^+]. \end{aligned} \tag{47}$$

A simplified formula without τ^{1st}: The case of a Zero Coupon Bond

We work under deterministic interest rates. We consider $P(t, T)$ held by ‘B” (lender) who will receive the notional 1 from ‘C”(borrower) at final maturity T if there has been no default of ‘C”.

The difference between the correct bilateral formula and the simplified one is, under risk free closeout,

$$LGD_C P(0, T)\mathbb{Q}(\tau_B < \tau_C < T).$$

The case with substitution closeout is instead trivial and the difference is null. For a bond, the simplified formula coincides with the full substitution closeout formula.

Therefore the difference above is the same difference between risk free closeout and substitution closeout formulas, and has been examined earlier, also in terms of contagion.

A simplified formula without τ^{1st}: The case of an Equity forward

In this case the payoff at maturity time T is given by $S_T - K$

where S_T is the price of the underlying equity at time T and K the strike price of the forward contract (typically $K = S_0$, ‘at the money’, or $K = S_0/P(0,T)$, ‘at the money forward’).

We compute the difference D^{BC} between the correct bilateral risk free closeout formula and the simplified one.

A simplified formula without τ^{1st}: The case of an Equity forward

$D^{BC} := A_1 - A_2$, where

$$A_1 = E_0 \left\{ 1_{\{\tau_B < \tau_C < T\}} LGD_C D(0, \tau_C)(S_{\tau_C} - P(\tau_C, T)K)^+ \right\}$$
$$A_2 = E_0 \left\{ 1_{\{\tau_C < \tau_B < T\}} LGD_B D(0, \tau_B)(P(\tau_B, T)K - S_{\tau_B})^+ \right\}$$

The worst cases will be the ones where the terms A_1 and A_2 do not compensate. For example assume there is a high probability that $\tau_B < \tau_C$ and that the forward contract is deep in the money. In such case A_1 will be large and A_2 will be small.

Similarly, a case where $\tau_C < \tau_B$ is very likely and where the forward is deep out of the money will lead to a large A_2 and to a small A_1.

However, we show with a numerical example that even when the forward is at the money the difference can be relevant. For more details see Brigo and Buescu (2011).

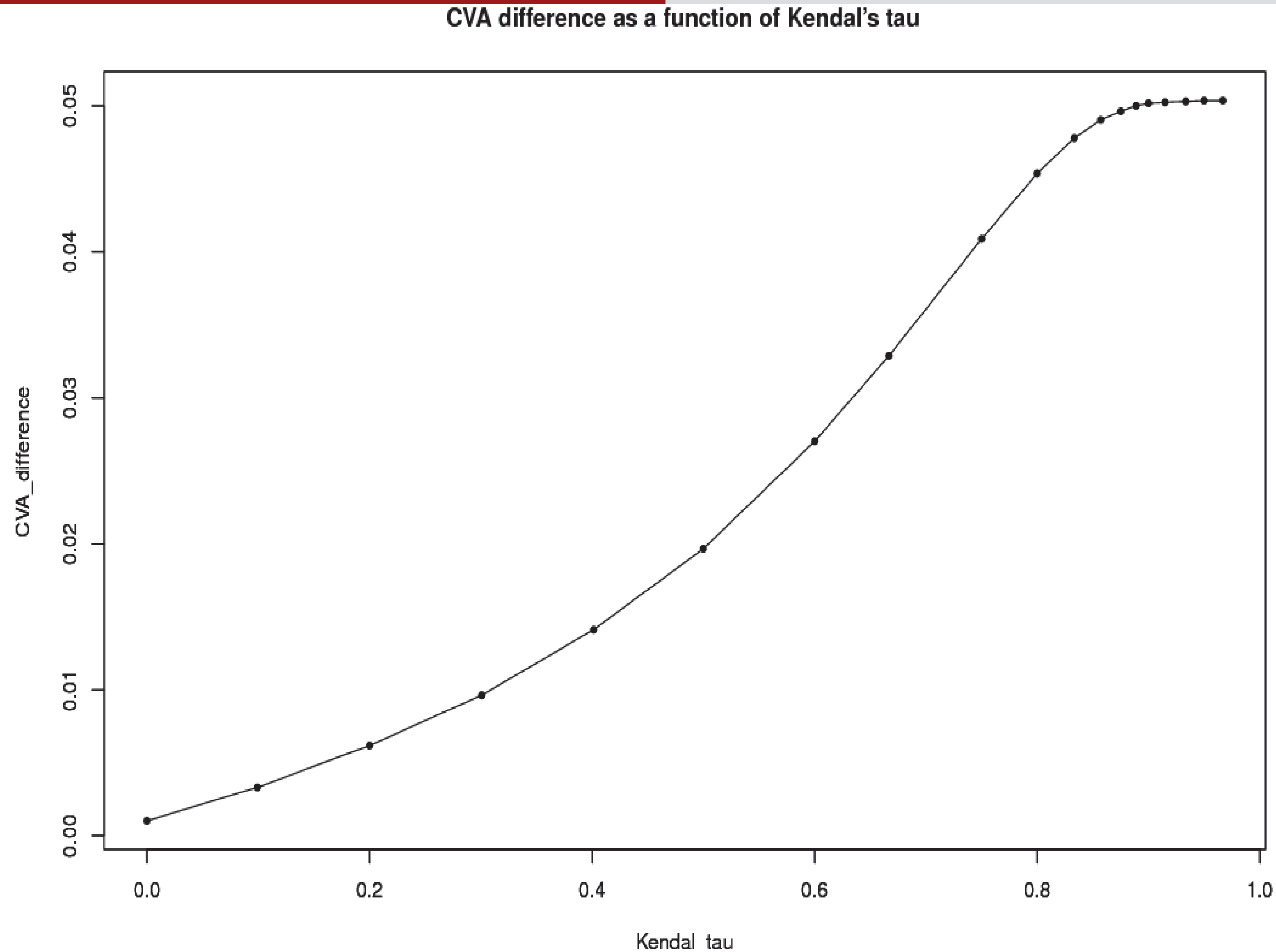


Figure: D^{BC} plotted against Kendall's tau between τ_B and τ_C, all other quantities being equal: $S_0 = 1$, $T = 5$, $\sigma = 0.4$, $K = 1$, $\lambda_B = 0.1$, $\lambda_C = 0.05$.

PAYOFF RISK

The exact payout corresponding with the Credit and Debit valuation adjustment is not clear.

- DVA or not?
- Which Closeout?
- First to default risk or not?
- How are collateral and funding accounted for exactly?

Worse than model risk: Payout risk. WHICH PAYOUT?

At a recent industry panel (WBS) on CVA it was stated that 5 banks might compute CVA in 15 different ways.

Methodology

1. Assumption: The *Bank/investor* enters a transaction with a *counterparty* and, when dealing with Unilateral Risk, the investor considers itself default free.
   Note : All the payoffs seen from the point of view of the *investor*.
2. We model and calibrate the default time of the *counterparty* using a stochastic intensity default model, except in the equity case where we will use a firm value model.
3. We model the transaction underlying and estimate the deal NPV at default.
4. We allow for the counterparty default time and the contract underlying to be correlated.
5. We start however from the case when such correlation can be neglected.

Approximation: Default Bucketing

General Formulation

1. Model (underlying) to estimate the NPV of the transaction.
2. Simulations are run allowing for correlation between the credit and underlying models, to determine the counterparty default time and the underlying deal NPV respectively.

Approximated Formulation under default bucketing

$$\mathbb{E}_0 \Pi^D(0,T) := \mathbb{E}_0 \Pi(0,T) - \text{L}_{\text{GD}} \mathbb{E}_0[1_{\{\tau < T_b\}} D(0,\tau)(\mathbb{E}_\tau \Pi(\tau,T))^+]$$

$$= \mathbb{E}_0 \Pi(0,T) - \text{L}_{\text{GD}} \mathbb{E}_0[(\sum_{j=1}^{b} 1\{\tau \in (T_{j-1},T_j]\})\, D(0,\tau)(\mathbb{E}_\tau \Pi(\tau,T))^+]$$

$$= \mathbb{E}_0 \Pi(0,T) - \text{L}_{\text{GD}} \sum_{j=1}^{b} \mathbb{E}_0[1\{\tau \in (T_{j-1},T_j]\}\, D(0,\tau)(\mathbb{E}_\tau \Pi(\tau,T))^+]$$

$$\approx \mathbb{E}_0 \Pi(0,T) - \text{L}_{\text{GD}} \sum_{j=1}^{b} \mathbb{E}_0[1\{\tau \in (T_{j-1},T_j]\}\, D(0,T_j)(\mathbb{E}_{T_j} \Pi(T_j,T))^+]$$

Approximation: Default Bucketing and Independence

1. In this formulation defaults are bucketed but we still need a joint model for τ and the underlying Π including their correlation.
2. Option model for Π is implicitly needed in τ scenarios.

Approximated Formulation under independence (and 0 correlation)

$$\mathbb{E}_0 \Pi^D(0,T) := \mathbb{E}_0 \Pi(0,T)$$
$$-\mathrm{L_{GD}} \sum_{j=1}^{b} \boxed{\mathbb{Q}\{\tau \in (T_{j-1}, T_j]\}}\, \mathbb{E}_0[D(0,T_j)(\mathbb{E}_{T_j}\Pi(T_j,T))^+]$$

1. In this formulation defaults are bucketed and only survival probabilities are needed (no default model).
2. Option model is STILL needed for the underlying of Π.

Ctrparty default model: CIR++ stochastic intensity

If we cannot assume independence, we need a default model.
Counterparty instantaneous credit spread: $\lambda(t) = y(t) + \psi(t;\beta)$

1. $y(t)$ is a CIR process with possible jumps

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW_t^y + dJ_t, \ \ \tau_C = \Lambda^{-1}(\xi), \ \ \Lambda(T) = \int_0^T \lambda(s)ds$$

2. $\psi(t;\beta)$ is the shift that matches a given CDS curve
3. ξ is standard exponential independent of all brownian driven stochastic processes
4. In CDS calibration we assume deterministic interest rates.
5. Calibration : Closed form Fitting of model survival probabilities to counterparty CDS quotes
6. B and El Bachir (2010) (Mathematical Finance) show that this model with jumps has closed form solutions for CDS options.

Literature on CVA across asset classes

Impact of dynamics, volatilities, correlations, wrong way risk

- **Interest Rate Swaps and Derivatives Portfolios** (B. Masetti (2005), B. Pallavicini 2007, 2008, B. Capponi P. Papatheodorou 2011, B. C. P. P. 2012 with collateral and gap risk)
- **Commodities swaps (Oil)** (B. and Bakkar 2009)
- **Credit: CDS on a reference credit** (B. and Chourdakis 2009, B. C. Pallavicini 2012 Mathematical Finance)
- **Equity Return Swaps** (B. and Tarenghi 2004, B. T. Morini 2011)
- Equity uses AT1P firm value model of B. and T. (2004) (barrier options with time-inhomogeneous GBM) and extensions (random barriers for risk of fraud).

Further asset classes are studied in the literature. For example see Biffis et al (2011) for CVA on **longevity swaps.**

Interest Rate and Commodities swaps and derivatives

We now examine UCVA with WWR for:

- Interest Rate Swaps and Derivatives Portfolios
- Commodities swaps

Interest rate swaps are the vast majority of market contracts on which CVA is computed.

Interest Rates Swap Case

Formulation for IRS under independence (no correlation)

$$\mathrm{IRS}^D(t,K) = \mathrm{IRS}(t.K)$$
$$-\mathrm{L{\small GD}} \sum_{i=a+1}^{b-1} \mathbb{Q}\{\tau \in (T_{i-1}, T_i]\}\, \mathrm{S{\small WAPTION}}_{i,b}(t; K, S_{i,b}(t), \sigma_{i,b})$$

Modeling Approach with corr.

Gaussian 2-factor G2++ short-rate $r(t)$ model:
$r(t) = x(t) + z(t) + \varphi(t;\alpha), r(0) = r_0$

$$dx(t) = -ax(t)dt + \sigma dW_x$$
$$dz(t) = -bz(t)dt + \eta dW_z$$

$$dW_x dW_z = \rho_{x,z} dt$$

$$\alpha = [r_0, a, b, \sigma, \eta, \rho_{1,2}]$$

$$dW_x dW_y = \rho_{x,y} dt, dW_z dW_y = \rho_{z,y} dt$$

Calibration

- The function $\varphi(\cdot;\alpha)$ is deterministic and is used to calibrate the initial curve observed in the market.
- We use swaptions and zero curve data to calibrate the model.
- The r factors x and z and the intensity are taken to be correlated.

Interest Rates Swap Case

Total Correlation Counterparty default / rates

$$\bar{\rho} = \mathrm{Corr}(dr_t, d\lambda_t) = \frac{\sigma\rho_{x,y} + \eta\rho_{z,y}}{\sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho_{x,z}}\sqrt{1 + \frac{2\beta\gamma^2}{\nu^2 y_t}}}.$$

where β is the intensity of arrival of λ jumps and γ is the mean of the exponentially distributed jump sizes.

Without jumps ($\beta = 0$)

$$\bar{\rho} = \mathrm{Corr}(dr_t, d\lambda_t) = \frac{\sigma\rho_{x,y} + \eta\rho_{z,y}}{\sqrt{\sigma^2 + \eta^2 + 2\sigma\eta\rho_{x,z}}}.$$

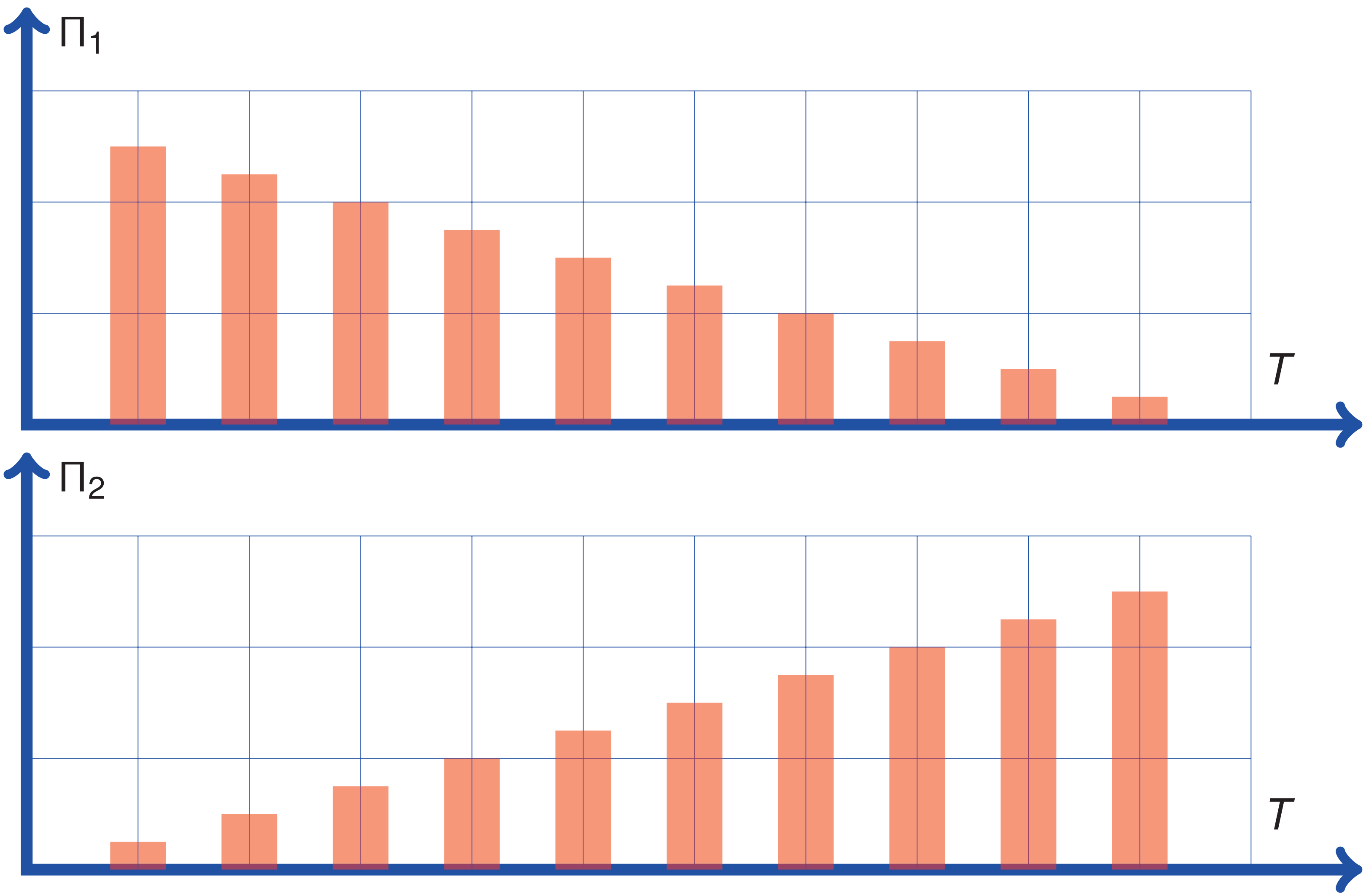
IRS: Case Study

1) Single Interest Rate Swaps (IRS)
At-the-money fix-receiver forward interest-rate-swap (IRS) paying on the EUR market.
The IRS's fixed legs pay annually a 30E/360 strike rate, while the floating legs pay LIBOR twice per year.
2) Netted portfolios of IRS.
- Portfolios of at-the-money IRS either with different starting dates or with different maturities.

1. (Π1) annually spaced dates $\{T_i : i = 0 \dots N\}$, T_0 two business days from trade date; portfolio of swaps maturing at each T_i, with $i > 0$, all starting at T_0.
2. (Π2) portfolio of swaps starting at each T_i all maturing at T_N.

Can also do exotics (Ratchets, CMS spreads, Bermudan)

IRS Case Study: Payment schedules



IRS Results

Counterparty risk price for netted receiver IRS portfolios Π1 and Π2 and simple IRS (maturity 10Y). Every IRS, constituting the portfolios, has unit notional and is at equilibrium. Prices are in bps.

λ	correlation $\bar{\rho}$	Π1	Π2	IRS
3%	-1	-140	-294	-36
	0	-84	-190	-22
	1	-47	-115	-13
5%	-1	-181	-377	-46
	0	-132	-290	-34
	1	-99	-227	-26
7%	-1	-218	-447	-54
	0	-173	-369	-44
	1	-143	-316	-37

Compare with ”Basel 2” deduced adjustments

Basel 2, under the ”Internal Model Method”, models wrong way risk by means of a 1.4 multiplying factor to be applied to the zero correlation case, even if banks have the option to compute their own estimate of the multiplier, which can never go below 1.2 anyway.

Is this confirmed by our model?

$$(140 - 84)/84 \approx 66\% > 40\%$$

$$(54 - 44)/44 \approx 23\% < 40\%$$

So this really depends on the portfolio and on the situation.

A bilateral example and correlation risk

Finally, in the bilateral case for Receiver IRS, 10y maturity, high risk counterparty and mid risk investor, we notice that depending on the correlations

$$\bar{\rho}_0 = \mathrm{Corr}(dr_t, d\lambda_t^0), \quad \bar{\rho}_2 = \mathrm{Corr}(dr_t, d\lambda_t^2), \quad \rho_{0,2}^{Copula} = 0$$

the DVA - CVA or Bilateral CVA does change sign, and in particular for portfolios Π1 and IRS the sign of the adjustment follows the sign of the correlations.

$\bar{\rho}_2$	$\bar{\rho}_0$	**Π1**	**Π2**	**10×IRS**
-60%	**0%**	-117(7)	-382(12)	-237(16)
-40%	**0%**	-74(6)	-297(11)	-138(15)
-20%	**0%**	-32(6)	-210(10)	-40(14)
0%	**0%**	-1(5)	-148(9)	31(13)
20%	**0%**	24(5)	-96(9)	87(12)
40%	**0%**	44(4)	-50(8)	131(11)
60%	**0%**	57(4)	-22(7)	159(11)

Payer vs Receiver

- Counterparty Risk (CR) has a relevant impact on interest-rate payoffs prices and, in turn, correlation between interest-rates and default (intensity) has a relevant impact on the CR adjustment.
- The (positive) CR adjustment to be subtracted from the default free price **decreases with correlation for receiver payoffs**. Natural: If default intensities increase, with high positive correlation their correlated interest rates will increase more than with low correlation, and thus a receiver swaption embedded in the adjustment decreases more, reducing the adjustment.
- The adjustment for payer payoffs increases with correlation.

Further Stylized Facts

- As the default probability implied by the counterparty CDS increases, the size of the adjustment increases as well, but the impact of correlation on it decreases.
- Financially reasonable: Given large default probabilities for the counterparty, fine details on the dynamics such as the correlation with interest rates become less relevant
- **The conclusion is that we should take into account interest-rate/ default correlation in valuing CR interest-rate payoffs.**
- In the bilateral case correlation risk can cause the adjustment to change sign

Exotics

For examples on exotics, including Bermudan Swaptions and CMS spread Options, see

Papers with Exotics and Bilateral Risk

- Brigo, D., and Pallavicini, A. (2007). Counterparty Risk under Correlation between Default and Interest Rates. In: Miller, J., Edelman, D., and Appleby, J. (Editors), Numerical Methods for Finance, Chapman Hall.
- Brigo, D., Pallavicini, A., and Papatheodorou, V. (2009). Bilateral counterparty risk valuation for interest-rate products: impact of volatilities and correlations. Available at Defaultrisk.com, SSRN and arXiv

Commodities and WWR

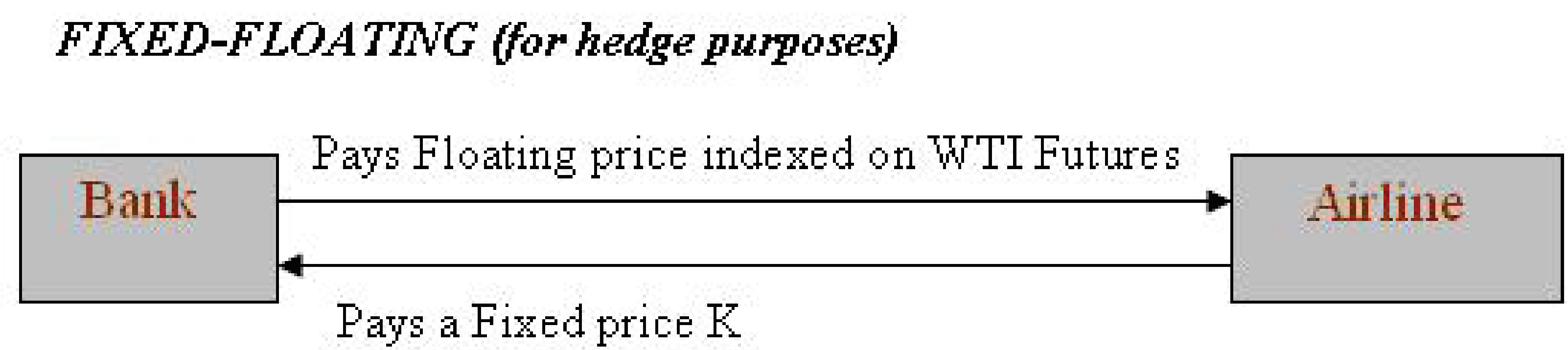
The correlation between interest rates dr_t (LIBOR, OIS) and credit intensities $d\lambda_t$, if measured historically, if often quite small in absolute value. Hence interest rates are a case where including correlation is good for stress tests and conservative hedging of CVA, but a number of market participant think that CVA can be computed by assuming zero correlations.

Whether one agrees or not, there are other asset classes on which CVA can be computed and where there is agreement on the necessity of including correlation in CVA pricing. We provide an example: Oil swaps traded with an airline.

It's natural to think that the future credit quality of the airline will be correlated with prices of oil.

Commodities: Futures, Forwards and Swaps

- **Forward**: OTC contract to buy a commodity to be delivered at a maturity date T at a price specified today. The cash/commodity exchange happens at time T.
- **Future**: Listed Contract to buy a commodity to be delivered at a maturity date T. Each day between today and T margins are called and there are payments to adjust the position.
- **Commodity Swap: Oil Example**:



Commodities: Modeling Approach

Schwartz-Smith Model

$$\begin{aligned} ln(S_t) &= x_t + l_t + \varphi(t) \\ dx_t &= -kx_t dt + \sigma_x dW_x \\ dl_t &= \mu dt + \sigma_l dW_l \\ dW_x \, dW_l &= \rho_{x,l} dt \end{aligned}$$

Variables

S_t: Spot oil price;
x_t, l_t: short and long term components of S_t;
This can be re-cast in a classic convenience yield model

Correlation with credit

$$\begin{aligned} dW_x \, dW_y &= \rho_{x,y} dt, \\ dW_l \, dW_y &= \rho_{l,y} dt \end{aligned}$$

Calibration

φ: defined to exactly fit the oil forward curve.
Dynamic parameters k, μ, σ, ρ are calibrated to At the money implied volatilities on Futures options.

Commodities

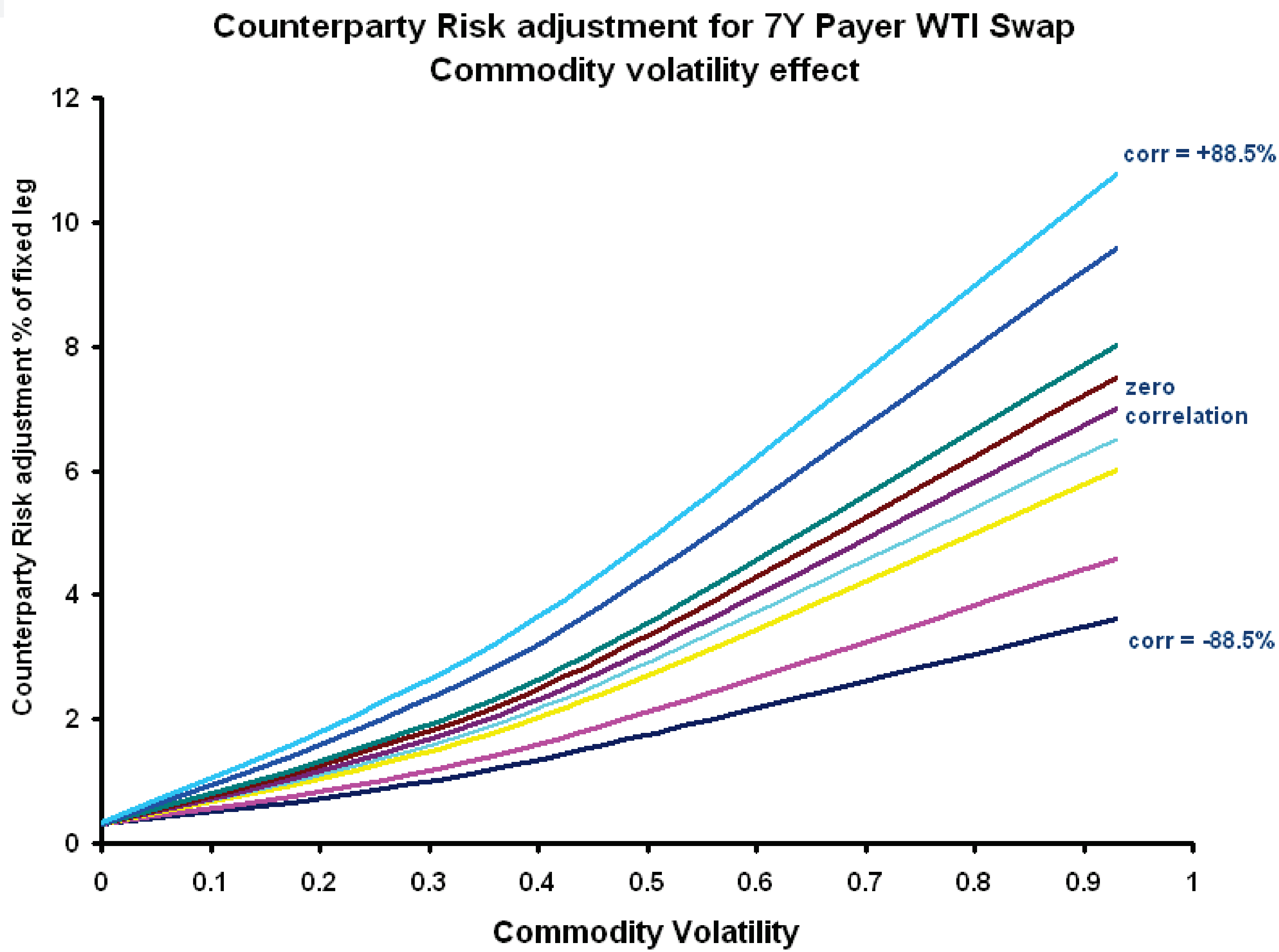
Total correlation Commodities - Counterparty default

$$\bar{\rho} = \mathrm{corr}(d\lambda_t,\ dS_t) = \frac{\sigma_x \rho_{x,y} + \sigma_L \rho_{L,y}}{\sqrt{\sigma_x^2 + \sigma_L^2 + 2\rho_{x,L}\sigma_x\sigma_L}}$$

We assumed no jumps in the intensity

We show the counterparty risk CVA computed by the AIRLINE on the BANK. This is because after 2008 a number of bank's credit quality deteriorated and an airline might have checked CVA on the bank with whom the swap was negotiated.

Commodities: Commodity Volatility Effect



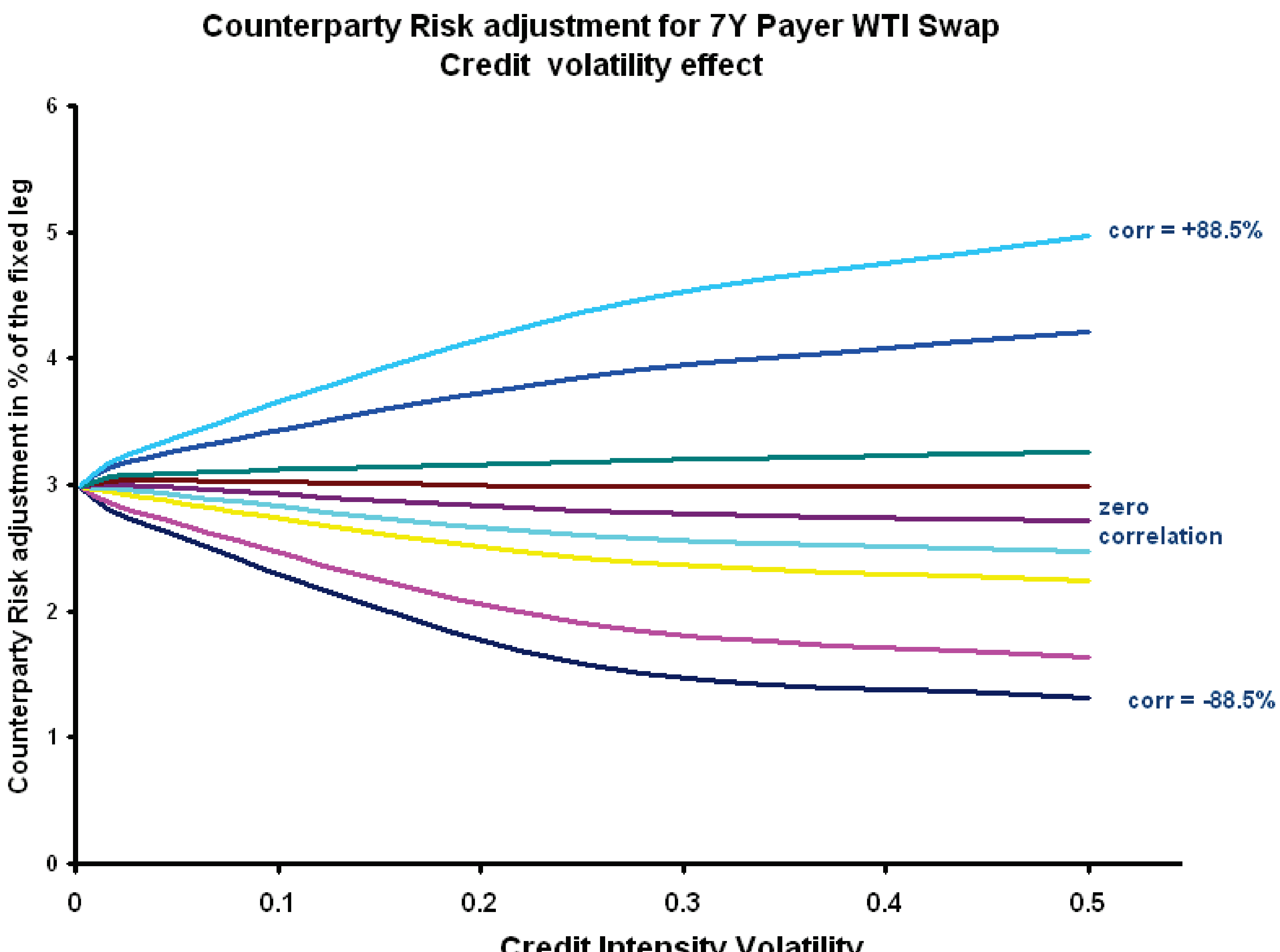
Commodities: Commodity Volatility Effect

Notice: In this example where CVA is calculated by the AIRLINE, positive correlation implies larger CVA.

This is natural: if the Bank credit spread widens, and the bank default becomes more likely, with positive correlation also OIL goes up.

Now CVA computed by the airline is an option, with maturity the default of the bank=counterparty, on the residual value of a Payer swap. As the price of OIL will go up at default due to the positive correlation above, the *payer* oil-swap will move in-the-money and the OIL option embedded in CVA will become more in-the-money, so that CVA will increase.

Commodities: Credit Volatility Effect



Commodities[1] : Credit volatility effect

$\bar{\rho}$	intensity volatility ν_R	0.025	0.25	0.50
-88.5	Payer adj	2.742	1.584	1.307
	Receiver adj	1.878	2.546	3.066
-63.2	Payer adj	2.813	1.902	1.63
	Receiver adj	1.858	2.282	2.632
-25.3	Payer adj	2.92	2.419	2.238
	Receiver adj	1.813	1.911	2.0242
-12.6	Payer adj	2.96	2.602	2.471
	Receiver adj	1.802	1.792	1.863
0	Payer adj	2.999	2.79	2.719
	Receiver adj	1.79	1.676	1.691
+12.6	Payer adj	3.036	2.985	2.981
	Receiver adj	1.775	1.562	1.527
+25.3	Payer adj	3.071	3.184	3.258
	Receiver adj	1.758	1.45	1.371
+63.2	Payer adj	3.184	3.852	4.205
	Receiver adj	1.717	1.154	0.977
+88.5	Payer adj	3.229	4.368	4.973
	Receiver adj	1.664	0.988	0.798

Fixed Leg Price maturity 7Y: 7345.39 USD for a notional of 1 Barrel per Month

[1] adjusment expressed as % of the fixed leg price

Commodities[2] : Commodity volatility effect

$\bar{\rho}$	Commodity spot volatility σ_S	0.0005	0.232	0.46	0.93
-88.5	Payer adj	0.322	0.795	1.584	3.607
	Receiver adj	0	1.268	2.546	4.495
-63.2	Payer adj	0.322	0.94	1.902	4.577
	Receiver adj	0	1.165	2.282	4.137
-25.3	Payer adj	0.323	1.164	2.419	6.015
	Receiver adj	0	0.977	1.911	3.527
-12.6	Payer adj	0.323	1.246	2.602	6.508
	Receiver adj	0	0.917	1.792	3.325
0	Payer adj	0.324	1.332	2.79	6.999
	Receiver adj	0	0.857	1.676	3.115
+12.6	Payer adj	0.324	1.422	2.985	7.501
	Receiver adj	0	0.799	1.562	2.907
+25.3	Payer adj	0.324	1.516	3.184	8.011
	Receiver adj	0	0.742	1.45	2.702
+63.2	Payer adj	0.325	1.818	3.8525	9.581
	Receiver adj	0	0.573	1.154	2.107
+88.5	Payer adj	0.326	2.05	4.368	10.771
	Receiver adj	0	0.457	0.988	1.715

Fixed Leg Price maturity 7Y: 7345.39 USD for a notional of 1 Barrel per Month

[2] adjusment expressed as % of the fixed leg price

Wrong Way Risk?

Basel 2, under the ”Internal Model Method”, models wrong way risk by means of a 1.4 multiplying factor to be applied to the zero correlation case, even if banks have the option to compute their own estimate of the multiplier, which can never go below 1.2 anyway.
What did we get in our cases? Two examples:

$$(4.973 - 2.719)/2.719 = 82\% >> 40\%$$

$$(1.878 - 1.79)/1.79 \approx 5\% << 20\%$$

Collateral Management and Gap Risk I

Collateral (CSA) is considered to be the solution to counterparty risk.

Periodically, the position is re-valued (”marked to market”) and a quantity related to the change in value is posted on the collateral account from the party who is penalized by the change in value.

This way, the collateral account, at the periodic dates, contains an amount that is close to the actual value of the portfolio and if one counterparty were to default, the amount would be used by the surviving party as a guarantee (and viceversa).
Gap Risk is the residual risk that is left due to the fact that the realingment is only periodical. If the market were to move a lot between two realigning (”margining”) dates, a significant loss would still be faced.

Folklore: Collateral completely kills CVA and gap risk is negligible.

Collateral Management and Gap Risk I

Folklore: Collateral completely kills CVA and gap risk is negligible.

We are going to show that there are cases where this is not the case at all (B. Capponi and Pallavicini 2012, Mathematical Finance)

- Risk-neutral evaluation of counterparty risk in presence of collateral management can be a difficult task, due to the complexity of clauses.
- Only few papers in the literature deal with it. Among them we cite Cherubini (2005), Alavian *et al.* (2008), Yi (2009), Assefa *et al.* (2009), Brigo et al (2011) and citations therein.
- Example: Collateralized bilateral CVA for a netted portfolio of IRS with 10y maturity and 1y coupon tenor for different default-time correlations with (and without) collateral re-hypothecation. See B, Capponi, Pallavicini and Papatheodorou (2011)

Collateral Management and Gap Risk II

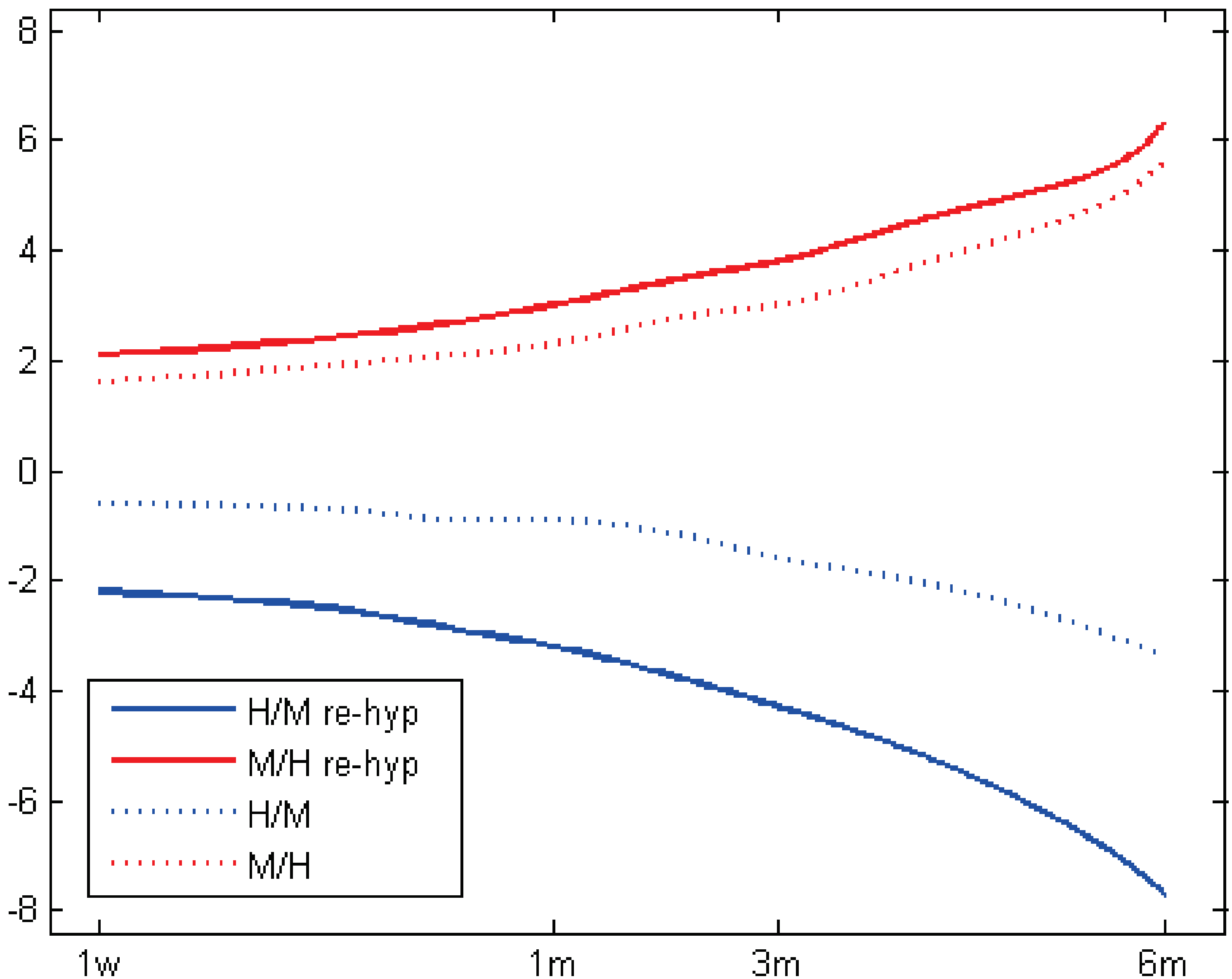


Figure explanation

Bilateral valuation adjustment, margining and rehypotecation

The figure shows the BVA(DVA-CVA) for a ten-year IRS under collateralization through margining as a function of the update frequency δ with zero correlation between rates and counterparty spread, zero correlation between rates and investor spread, and zero correlation between the counterparty and the investor defaults. The model allows for nonzero correlations as well.

Continuous lines represent the re-hypothecation case, while **dotted lines** represent the opposite case. The *red line* represents an investor riskier than the counterparty, while the *blue line* represents an investor less risky than the counterparty. All values are in basis points.

See the full paper by Brigo, Capponi, Pallavicini and Papatheodorou ‘Collateral Margining in Arbitrage-Free Counterparty Valuation Adjustment including Re-Hypotecation and Netting” available at `http://arxiv.org/abs/1101.3926`

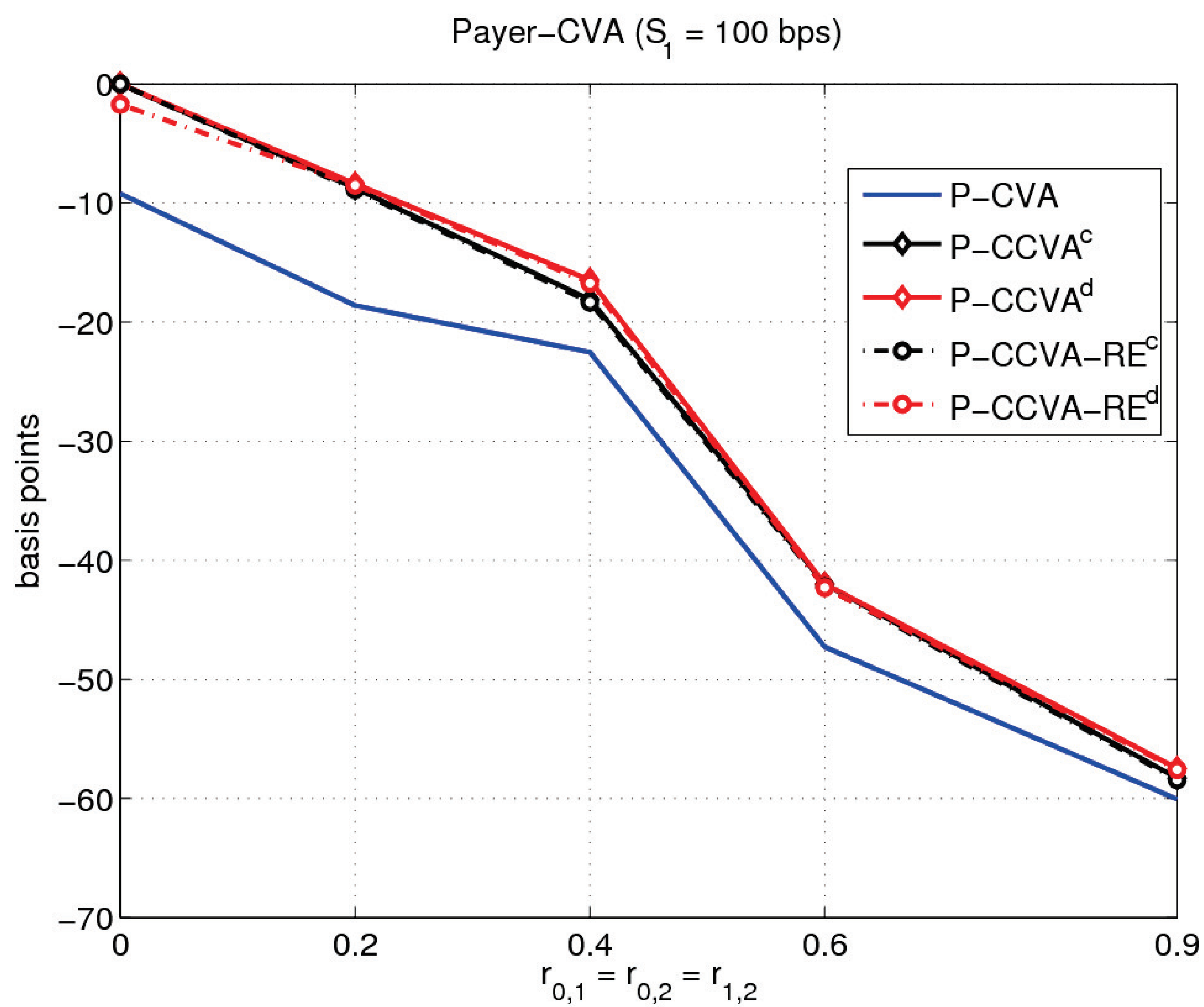
Figure explanation

From the fig, we see that the case of an investor riskier than the counterparty (M/H) leads to positive value for DVA-CVA, while the case of an investor less risky than the counterparty has the opposite behaviour. If we inspect the DVA and CVA terms as in the paper we see that when the investor is riskier the DVA part of the correction dominates, while when the investor is less risky the counterparty has the opposite behaviour.

Re-hypothecation enhances the absolute size of the correction, a reasonable behaviour, since, in such case, each party has a greater risk because of being unsecured on the collateral amount posted to the other party in case of default.

Let us now look at a case with more contagion: a CDS.

Collate


Payer−CVA (S_1 = 100 bps)
basis points
0
−10
−20
−30
−40
−50
−60
−70
0
0.2
0.4
0.6
0.9
$r_{0,1} = r_{0,2} = r_{1,2}$
P−CVA
P−CCVA^c
P−CCVA^d
P−CCVA−RE^c
P−CCVA−RE^d

Collateral Management and Gap Risk II

The figure refers to a payer CDS contract as underlying. See the full paper by Brigo, Capponi and Pallavicini (2011) for more cases.

If the investor holds a payer CDS, he is buying protection from the counterparty, i.e. he is a protection buyer.

We assume that the spread in the fixed leg of the CDS is 100 while the initial equilibrium spread is about 250.

Given that the payer CDS will be positive in most scenarios, when the investor defaults it is quite unlikely that the net present value be in favor of the counterparty.

We then expect the CVA term to be relevant, given that the related option will be mostly in the money. This is confirmed by our outputs.

Collateral Management and Gap Risk III

We see in the figure a relevant CVA component (part of the bilateral DVA - CVA) starting at 10 and ending up at 60 bps when under high correlation.

We also see that, for zero correlation, collateralization succeeds in completely removing CVA, which goes from 10 to 0 basis points.

However, collateralization seems to become less effective as default dependence grows, in that collateralized and uncollateralized CVA become closer and closer, and for high correlations we still get 60 basis points of CVA, even under collateralization.

The reason for this is the instantaneous default contagion that, under positive dependency, pushes up the intensity of the survived entities, as soon as there is a default of the counterparty.

Collateral Management and Gap Risk IV

Indeed, the term structure of the on-default survival probabilities (see paper) lies significantly below the one of the pre-default survival probabilities conditioned on $\mathcal{G}_{\tau-}$, especially for large default correlation.

The result is that the default leg of the CDS will increase in value due to contagion, and instantaneously the Payer CDS will be worth more. This will instantly increase the loss to the investor, and most of the CVA value will come from this jump.

Given the instantaneous nature of the jump, the value at default will be quite different from the value at the last date of collateral posting, before the jump, and this explains the limited effectiveness of collateral under significantly positive default dependence.

Collateral Management and Gap Risk V

The precise payout of residual CVA and DVA adjustment cash flows after collateralization will be introduced in the Funding Costs modeling part below, and will be called $\Pi_{CVAcoll}$ *and* $\Pi_{DVAcoll}$. *These are the terms that have been priced in the above examples.*

Inclusion of Funding Cost

We now move to the inclusion of funding costs.

This is an important part of valuation, as shown by the financial news concerning JPMorgan as from January 2014, showing that Funding costs impacted the firm for 1.5 Billion $.

More details on JPMorgan are given below.

Where does the problem of funding costs originate from?

Inclusion of Funding Cost

When managing a trading position, one needs to obtain cash in order to do a number of operations:

- hedging the position,
- posting collateral,
- paying coupons or notionals, or interest on received collateral
- set reserves in place

and so on. Where are such founds obtained from?

- Obtain cash/assets from Treasury department or market.
- receive cash as a consequence of being in the position:
  - a coupon or notional reimbursement,
  - a positive mark to market move,
  - getting some collateral or interest on posted collateral,
  - a closeout payment.

All such flows need to be remunerated:

- if one is ”borrowing”, this will have a cost,
- and if one is ”lending”, this will provide revenues.

Inclusion of Funding Cost

Funding is not just different discounting

- CVA and DVA are not obtained just by adding a spread to the discount factor of assets cash flows
- Similarly, a hypothetical FVA is not simply applying spreads to borrowing and lending cash flows.

One has to carefully and properly analyze and price the real cash flows rather than add an artificial spread. The simple spread may emerge for very simple deals and under simplifying assumptions (no correlations, uni-directional cash flows, etc)

Funding Valuation Adjustment? Can FVA be additive?

A fundamental point is including funding consistently with counterparty risk. Industry wishes for a “Funding Valuation Adjustment”, or FVA, that would be additive:

TOTAL PRICE =

= RISK FREE PRICE + DVA - CVA + FVA

Since I need to pay the funding costs to my treasury desk or to the market party that is funding me, or perhaps since I am receiving interest on collateral I posted, the real value of the deal is affected.

But is the effect just additive and decomposable with CVA and DVA?

It is not so simple

Funding, credit and market risk interact in a nonlinear and recursive way and they cannot be decomposed additively.

Funding and DVA

We discussed DVA and funding earlier. We repeat the point now.

DVA a component of FVA?

DVA is related to funding costs when the payout is uni-directional, eg shorting a bond, borrowing in a loan, or shorting a call option.

Indeed, if we are short simple products that are uni-directional, we are basically borrowing.

As we shorted a bond or a call option, for example, we received cash V_0 in the beginning, and we have to pay the product payout in the end.

This cash can be used by us to fund other activities, and allows us to spare the costs of fuding this cash V_0 from our treasury.

Funding and DVA

Our treasury usually funds in the market, and the market charges our treasury a cost of funding that is related to the borrowed amount V_0, to the period T and to our own bank credit risk $\tau_B < T$.

In this sense the funding cost we are sparing when we avoid borrowing looks similar to DVA: it is related to the price of the object we are shorting and to our own credit risk.

However quite a number of assumptions is needed to identify DVA with a pure funding benefit, as we will see below.

Introduction to Quant. Analysis of Funding Costs I

We now present an introduction to funding costs modeling.

This is important as it is related to a recent effort in the industry.

Funding Value Adjustment Proves Costly to J.P. Morgans 4Q Results (Michael Rapoport, Wall St Journal, Jan 14, 2014)

"[...] So what is a funding valuation adjustment, and why did it cost J.P. Morgan Chase $1.5 billion?

The giant bank recorded a $1.5 billion charge in its fourth-quarter earnings announced Tuesday because of the adjustment the result of a complex change in J.P. Morgans approach to valuing some of the derivatives on its books.

Introduction to Quant. Analysis of Funding Costs II

J.P. Morgan was persuaded to make the FVA change by an industry migration toward such a move, the bank said in an investor presentation. A handful of other large banks, mostly in the U.K. and Europe, have already made a similar change.

If J.P. Morgan is correct, its possible other banks with large derivatives holdings, like Citigroup and Bank of America could make the same sort of change in the future and potentially incur losses, though perhaps not as large as J.P. Morgans. Both Citigroup and Bank of America declined to comment.

What J.P. Morgan did was to reduce the carrying value of its over-the-counter derivatives and structured notes. The bank did this to account for the costs to fund transactions involving the derivatives and notes.

Introduction to Quant. Analysis of Funding Costs III

In recent years, there has been a lively if highly technical debate in academic and accounting circles about whether derivatives holders should be doing that. The debate was prompted by the financial crisis, and the changes in its wake by regulators and lawmakers to make the derivatives market safer by putting more trading on centralized systems.

Heres more detail on J.P. Morgans change:

Like other derivatives holders, J.P. Morgan has to raise funds to serve as collateral when it hedges the risks of its transactions involving uncollateralized derivatives those not secured by any assets.

Introduction to Quant. Analysis of Funding Costs IV

Until relatively recently, investors and other market participants found it hard to gauge the magnitude of those funding costs, because the derivatives market was so opaque. But in the wake of the crisis, derivatives has become more regulated, and more derivatives trading is moving onto open, more-transparent forums.

Traders also have shifted away from using the London Interbank Offered Rate, or Libor, to price derivatives trades, raising costs to derivatives holders.

All that makes the funding costs easier to see and has raised questions about whether they should be factored into a derivatives value. **For the first time this quarter, we were able to clearly observe the existence of funding costs in market clearing levels, J.P. Morgan said in its investor presentation.**

Introduction to Quant. Analysis of Funding Costs V

Some banks already recognize funding valuation adjustments, like Royal Bank of Scotland, which recognized FVA losses of 174 million pounds in 2012 and 493 million pounds in 2011. A spokeswoman for RBS didnt have any immediate comment.

Goldman Sachs says in its Securities and Exchange Commission filings that its derivatives valuations incorporate funding valuation adjustments, though the investment bank doesnt provide any further detail. [...]"

We now approach funding costs modeling by incorporating funding costs into valuation.

We restart from scratch from the product cash flows and add collateralization, cost of collateral, CVA and DVA after collateral, and funding costs for collateral and for the replication of the product.

Introduction to Quant. Analysis of Funding Costs VI

In the following τ_I denotes the default time of the investor doing the calculation of the price. This used to be bank B in previous slides so $\tau_I = \tau_B$ usually.

Basic Payout plus Credit and Collateral: Cash Flows I

- We calculate prices by discounting cash-flows under the pricing measure. Collateral and funding are modeled as additional cashflows (as for CVA and DVA)
- We start from derivative's basic cash flows without credit, collateral of funding risks

$$\bar{V}_t \quad := \quad \mathbb{E}_t[\,\Pi(t, T \wedge \tau) + \ldots]$$

where

$\longrightarrow$ $\tau := \tau_C \wedge \tau_I$ is the first default time, and
$\longrightarrow$ $\Pi(t, u)$ is the sum of all discounted payoff terms up from t to u,

Cash flows are stopped either at the first default or at portfolio's expiry if defaults happen later.

Basic Payout plus Credit and Collateral: Cash Flows II

- As second contribution we consider the collateralization procedure and we add its cash flows.

$$\bar{V}_t := \mathbb{E}_t[\,\Pi(t, T \wedge \tau)\,] + \mathbb{E}_t[\,\gamma(t, T \wedge \tau; C) + \dots\,]$$

where

$\longrightarrow$ C_t is the collateral account defined by the CSA,

$\longrightarrow$ $\gamma(t, u; C)$ are the collateral margining costs up to time u.

- The second expected value originates what is occasionally called Liquidity Valuation Adjustment (LVA) in simplified versions of this analysis. We will show this in detail later.
- If $C > 0$ collateral has been overall posted by the counterparty to protect us, and we have to pay interest c^+.
- If $C < 0$ we posted collateral for the counterparty (and we are remunerated at interest c^-).

Basic Payout plus Credit and Collateral: Cash Flows III

- The cash flows due to the margining procedure on the time grid $\{t_k\}$ are equal to

$$\gamma(t,u;C) := -\sum_{k=1}^{n-1} 1_{\{t \le t_k < u\}} D(t,t_k) C_{t_k} \left(P_{t_k}(t_{k+1})(1+\alpha_k \tilde{c}_{t_k}(t_{k+1})) - 1 \right)$$

where $\alpha_k = t_{k+1} - t_k$ and the collateral accrual rates are given by

$$\tilde{c}_t := c_t^+ 1_{\{C_t>0\}} + c_t^- 1_{\{C_t<0\}}$$

- Linearization of exponential bond formulas in the continuously compounded rates yields

$$\gamma(t,u;C) \approx -\sum_{k=1}^{n-1} 1_{\{t \le t_k < u\}} D(t,t_k) C_{t_k} \alpha_k (\tilde{c}_{t_k}(t_{k+1}) - r_{t_k}(t_{k+1}))$$

Note that if the collateral rates in $\tilde{c}$ are both equal to the risk free rate, then this term is zero.

Close-Out: Trading-CVA/DVA under Collateral – I

- As third contribution we consider the cash flow happening at 1st default, and we have

$$\begin{aligned}\bar{V}_t &:= \mathbb{E}_t[\,\Pi(t, T\wedge\tau)\,] \\ &+ \mathbb{E}_t[\,\gamma(t, T\wedge\tau; C)\,] \\ &+ \mathbb{E}_t\left[\,1_{\{\tau<T\}} D(t,\tau)\theta_\tau(C,\varepsilon) + \dots\right]\end{aligned}$$

where

$\longrightarrow$ ε_τ is the close-out amount, or residual value of the deal at default, which we called NPV earlier, and

$\longrightarrow$ $\theta_\tau(C,\varepsilon)$ is the on-default cash flow.

- θ_τ will contain collateral adjusted CVA and DVA payouts for the instument cash flows
- We define θ_τ including the pre-default value of the collateral account since it is used by the close-out netting rule to reduce exposure

Close-Out: Trading-CVA/DVA under Collateral – II

- The close-out amount is not a symmetric quantity w.r.t. the exchange of the role of two parties, since it is valued by one party after the default of the other one.

$$\varepsilon_\tau := 1_{\{\tau=\tau_C\}}\varepsilon_{I,\tau} + 1_{\{\tau=\tau_I\}}\varepsilon_{C,\tau}$$

- Without entering into the detail of close-out valuation we can assume a close-out amount equal to the risk-free price of remaining cash flows inclusive of collateralization and funding costs. More details in the examples.
  - $\longrightarrow$ See ISDA document “Market Review of OTC Derivative Bilateral Collateralization Practices” (2010).
  - $\longrightarrow$ See, for detailed examples, Parker and McGarry (2009) or Weeber and Robson (2009)
  - $\longrightarrow$ See, for a review, Brigo, Morini, Pallavicini (2013).

Close-Out: Trading-CVA/DVA under Collateral – III

- At transaction maturity, or after applying close-out netting, the originating party expects to get back the remaining collateral.
- Yet, prevailing legislation's may give to the Collateral Taker some rights on the collateral itself.
  - $\longrightarrow$ In presence of re-hypothecation the collateral account may be used for funding, so that cash requirements are reduced, but counterparty risk may increase.
  - $\longrightarrow$ See Brigo, Capponi, Pallavicini and Papatheodorou (2011).
- In case of collateral re-hypothecation the surviving party must consider the possibility to recover only a fraction of his collateral.
  - $\longrightarrow$ We name such recovery rate Rec'_I, if the investor is the Collateral Taker, or Rec'_C in the other case.
  - $\longrightarrow$ In the worst case the surviving party has no precedence on other creditors to get back his collateral, so that

$$\mathrm{Rec}_I \leq \mathrm{Rec}'_I \leq 1 \ , \quad \mathrm{Rec}_C \leq \mathrm{Rec}'_C \leq 1$$

Close-Out: Trading-CVA/DVA under Collateral – IV

- The on-default cash flow $\theta_\tau(C,\varepsilon)$ can be calculated by following ISDA documentation. We obtain

$$\begin{aligned}\theta_\tau(C,\varepsilon) &:= 1_{\{\tau=\tau_C<\tau_I\}}\left(\varepsilon_{I,\tau} - \mathrm{L{\scriptstyle GD}}_C(\varepsilon^+_{I,\tau} - C^+_{\tau-})^+ - \mathrm{L{\scriptstyle GD}}'_C(\varepsilon^-_{I,\tau} - C^-_{\tau-})^+\right)\\ &+ 1_{\{\tau=\tau_I<\tau_C\}}\left(\varepsilon_{C,\tau} - \mathrm{L{\scriptstyle GD}}_I(\varepsilon^-_{C,\tau} - C^-_{\tau-})^- - \mathrm{L{\scriptstyle GD}}'_I(\varepsilon^+_{C,\tau} - C^+_{\tau-})^-\right)\end{aligned}$$

where loss-given-defaults are defined as $\mathrm{L{\scriptstyle GD}}_C := 1 - \mathrm{R{\scriptstyle EC}}_C$, and so on.

- If both parties agree on exposure, namely $\varepsilon_{I,\tau} = \varepsilon_{C,\tau} = \varepsilon_\tau$ then

$$\begin{aligned}\theta_\tau(C,\varepsilon) &:= \varepsilon_\tau - 1_{\{\tau=\tau_C<\tau_I\}}\Pi_{\text{CVAcoll}} + 1_{\{\tau=\tau_I<\tau_C\}}\Pi_{\text{DVAcoll}}\\ \Pi_{\text{CVAcoll}} &= \mathrm{L{\scriptstyle GD}}_C(\varepsilon^+_\tau - C^+_{\tau-})^+ + \mathrm{L{\scriptstyle GD}}'_C(\varepsilon^-_\tau - C^-_{\tau-})^+\\ \Pi_{\text{DVAcoll}} &= \mathrm{L{\scriptstyle GD}}_I((-\varepsilon_\tau)^+ - (-C_{\tau-})^+)^+ + \mathrm{L{\scriptstyle GD}}'_I(C^+_{\tau-} - \varepsilon^+_\tau)^+\end{aligned}$$

Close-Out: Trading-CVA/DVA under Collateral – V

- In case of re-hypothecation, when $\text{L}_{\text{GD}_C} = \text{L}_{\text{GD}'_C}$ and $\text{L}_{\text{GD}_I} = \text{L}_{\text{GD}'_I}$, we obtain a simpler relationship

$$
\begin{aligned}
\theta_\tau(C,\varepsilon) &:= \varepsilon_\tau \\
&- 1_{\{\tau=\tau_C<\tau_I\}}\text{L}_{\text{GD}_C}(\varepsilon_{I,\tau} - C_{\tau^-})^+ \\
&- 1_{\{\tau=\tau_I<\tau_C\}}\text{L}_{\text{GD}_I}(\varepsilon_{C,\tau} - C_{\tau^-})^-
\end{aligned}
$$

Funding Costs of the Replication Strategy – I

- As fourth and last contribution we consider the cost of funding for the hedging procedures and we add the relevant cash flows.

$$\begin{aligned}\bar{V}_t \; &:= \; \mathbb{E}_t[\,\Pi(t, T\wedge\tau)\,] + \mathbb{E}_t\big[\,\gamma(t, T\wedge\tau; C) + 1_{\{\tau<T\}} D(t,\tau)\theta_\tau(C,\varepsilon)\,\big] \\ &+ \; \mathbb{E}_t[\,\varphi(t, T\wedge\tau; F, H)\,]\end{aligned}$$

The last term, especially in simplified versions, is related to what is called FVA in the industry. We will point this out once we get rid of the rate r.

$\longrightarrow$ F_t is the cash account for the replication of the trade,
$\longrightarrow$ H_t is the risky-asset account in the replication,
$\longrightarrow$ $\varphi(t, u; F, H)$ are the cash F and hedging H funding costs up to u.

- In classical Black Scholes on Equity, for a call option (no credit risk, no collateral, no funding costs),

$$\bar{V}_t^{\text{Call}} = \Delta_t S_t + \eta_t B_t =: H_t + F_t, \quad \tau = +\infty, \quad C = \gamma = \varphi = 0.$$

Funding Costs of the Replication Strategy – II

- Cash flows due to funding of the replication strategy are ($\tilde{f}$ are net credit spreads, since credit is included explicitly)

$$\varphi(t,u) := \sum_{j=1}^{m-1} 1_{\{t \le t_j < u\}} D(t,t_j)(F_{t_j} + H_{t_j}) \left(1 - P_{t_j}(t_{j+1})(1 + \alpha_k \tilde{f}_{t_j}(t_{j+1}))\right)$$
$$- \sum_{j=1}^{m-1} 1_{\{t \le t_j < u\}} D(t,t_j) H_{t_j} \left(1 - P_{t_j}(t_{j+1})(1 + \alpha_k \tilde{h}_{t_j}(t_{j+1}))\right)$$

where the funding and lending rates for F and H are given by

$$\tilde{f}_t := f_t^+ 1_{\{F_t > 0\}} + f_t^- 1_{\{F_t < 0\}} \quad \tilde{h}_t := h_t^+ 1_{\{H_t > 0\}} + h_t^- 1_{\{H_t < 0\}}$$

Funding Costs of the Replication Strategy – III

- Writing bonds and rates in continuously compounding format and linearizing exponentials:

$$\begin{aligned}\varphi(t,u) \quad &:= \quad \sum_{j=1}^{m-1} 1_{\{t \le t_j < u\}} D(t,t_j)(F_{t_j} + H_{t_j})\alpha_k \left(r_{t_j}(t_{j+1}) - \tilde{f}_{t_j}(t_{j+1}) \right) \\ &- \quad \sum_{j=1}^{m-1} 1_{\{t \le t_j < u\}} D(t,t_j) H_{t_j}\alpha_k \left(r_{t_j}(t_{j+1}) - \tilde{h}_{t_j}(t_{j+1}) \right)\end{aligned}$$

- Note: the expected value of φ is related to the so called FVA. If the treasury funding rates $\tilde{f}$ are the same as the asset lending/borrowing rates $\tilde{h}$ then the funding cash flows simplify to

$$\varphi(t,u) := \sum_{j=1}^{m-1} 1_{\{t \le t_j < u\}} D(t,t_j) F_{t_j}\alpha_k \left(r_{t_j}(t_{j+1}) - \tilde{f}_{t_j}(t_{j+1}) \right)$$

Funding Costs of the Replication Strategy – IV

- If further the treasury borrows and lends at the risk free rate, $\tilde{f} = r$, then $\varphi = 0$ and FVA$= 0$.

Funding Costs of the Replication Strategy – V

Our replica consists in F cash and H risky asset.
Cash is borrowed $F > 0$ from the treasury at an interest f^+ (cost) or is lent $F < 0$ at a rate f^- (revenue)

Risky asset position in the replica is worth H. Cash needed to buy $H > 0$ is borrowed at an interest f from the treasury; in this case H can be used for asset lending (Repo for example) at a rate h^+ (revenue);

Else if risky asset in replica is worth $H < 0$, meaning that we should replicate via a short position in the asset, we may borrow cash from the repo market by posting the asset H as guarantee (rate h^-, cost), and lend the obtained cash to the treasury to be remunerated at a rate f.

It is possible to include the risk of default of the funder and funded, leading to CVA and DVA adjustments for the funding position, see PPB.

Funding rates depend on Treasury policies

- In real applications the funding rate $\tilde{f}_t$ is determined by the party managing the funding account for the investor, eg the bank's treasury:
  - $\longrightarrow$ trading positions may be netted before funding on the mkt
  - $\longrightarrow$ a Funds Transfer Pricing (FTP) process may be implemented to gauge the performances of different business units;
  - $\longrightarrow$ a maturity transformation rule can be used to link portfolios to effective maturity dates;
  - $\longrightarrow$ sources of funding can be mixed into the internal funding curve ...
- In part of the literature the role of the treasury is usually neglected, leading to controversial results particularly when the funding positions are not distinguished from the trading positions.
- See partial claims "funding costs = DVA", or "there are no funding costs", cited in the literature (Hull White, "FVA =0")

Recursive non-decomposable Nature of Pricing – I

$$(*) \quad \bar{V}_t = \mathbb{E}_t\Big[\Pi(t, T\wedge\tau) + \gamma(t, T\wedge\tau) + 1_{\{\tau<T\}}D(t,\tau)\theta_\tau(C,\varepsilon) + \varphi(t, T\wedge\tau)\Big]$$

Can we interpret:

$$\mathbb{E}_t\Big[\Pi(t, T\wedge\tau) + 1_{\{\tau<T\}}D(t,\tau)\theta_\tau(C,\varepsilon)\Big] : \quad \text{RiskFree Price + DVA - CVA?}$$

$$\mathbb{E}_t[\gamma(t, T\wedge\tau) + \varphi(t, T\wedge\tau; F, H)] : \quad \text{Funding adjustment FVA?}$$

Not really. This is not a decomposition. It is an equation. In fact since

$$\bar{V}_t = F_t + H_t + C_t \quad \text{(re–hypo)}$$

we see that the φ present value term depends on future $F_t = \bar{V}_t - H_t + C_t$ and generally the closeout θ, via ϵ and C, depends on future $\bar{V}$ too. All terms feed each other and there is no neat separation of risks. *Recursive pricing: Nonlinear PDE's / BSDEs* for $\bar{V}$

"FinalPrice = RiskFreePrice (+ DVA?) - CVA + FVA" not possible.

See Pallavicini Perini B. (2011, 2012) for $\bar{V}$ equations and algorithms.

Recursive non-decomposable Nature of Pricing – II

We can obtain a valuation PDE (and BSDE) by further steps:

1. Write the equation for $\bar{V}_{t_j}$ starting from $\bar{V}_{t_{j+1}}$, backwards.
2. Take the continuous time limit, where funding happens instantaneously and collateral is posted continuosly (still gap risk, unless you assume NPV to be left continuous)
3. Immersion hypothesis for credit risk: work under default-free filtration $\mathcal{F}_t$. Recall that we assumed earlier

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau_i \leq u\}, u \leq t)$$

with i indexing all the default times in the system. Working under $\mathcal{F}$ usually means that the risks in the basic cash flows Π are assumed not to be credit sensitive but to depend only on the filtration $\mathcal{F}$ of pre-default or default-free market information, eg default free interest rate swaps portfolio.

Recursive non-decomposable Nature of Pricing – III

It also means that we assume default times to be $\mathcal{F}$-conditionally independent. More precisely, if we define

$$\tau_I = \Lambda_I^{-1}(\xi_I), \quad \tau_C = \Lambda_C^{-1}(\xi_C),$$

this means assuming that ξ_I and ξ_C are independent. Intensities $\lambda_I(t)$ and $\lambda_C(t)$ are taken $\mathcal{F}_t$ adapted (& can be correlated) and

$$\mathbb{Q}(\tau > t) = \mathbb{Q}(\min(\tau_I, \tau_C) > t) = \mathbb{Q}(\tau_I > t \cap \tau_C > t) =$$

We use the tower property and the independence of the ξ's on each other, and the independence of the ξ on $\mathcal{F}$:

$$= \mathbb{E}[\mathbb{Q}(\tau_I > t \ \cap \ \tau_C > t|\mathcal{F}_t)] = \mathbb{E}[\mathbb{Q}(\tau_I > t|\mathcal{F}_t)\mathbb{Q}(\tau_C > t|\mathcal{F}_t)] =$$

$$= \mathbb{E}[e^{-\Lambda_I(t)}e^{-\Lambda_C(t)}] = \mathbb{E}[e^{-\Lambda_I(t)-\Lambda_C(t)}] = \mathbb{E}[e^{-\int_0^t(\lambda_I(s)+\lambda_C(s))ds}]$$

Recursive non-decomposable Nature of Pricing – IV

Similarly, one can show that in the case of independent ξ's the first to default time τ intensity λ is

$$\mathbb{Q}(\tau \in [t, t + dt) | \tau > t, \mathcal{F}_t) = \lambda_t \, dt = (\lambda_I(t) + \lambda_C(t))dt.$$

Whenever we use the immersion hypothesis, meaning that we switch filtration from $\mathcal{G}$ to $\mathcal{F}$, we assume the ξ to be conditionally independent and the basic cash flows $\Pi(s,t)$ to be $\mathcal{F}_t$ adapted for all $s \leq t$.

Switching to the filtration $\mathcal{F}$ typically transforms indicators such as $1_{\{\tau > t\}}$ into their $\mathcal{F}$ expectations $e^{-\int_0^t (\lambda_I(s)+\lambda_C(s))ds}$. This is often collected in the discount term $D(0,t;r)$ that becomes $D(0,t;r+\lambda)$. The switching also transforms $1_{\{\tau \in dt\}}$ into $\lambda_t e^{-\int_0^t \lambda_s ds} dt$.

Recursive non-decomposable Nature of Pricing – V

4. With the above steps, we obtain (here $\pi_t \, dt = \Pi(t, t + dt)$)

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t, u; r + \lambda)[\pi_u + (r_u - \tilde{c}_u)C_u + \lambda_u \theta_u$$

$$+ (r_u - \tilde{f}_u)(F_u + H_u) + (\tilde{h}_u - r_u)H_u] | \mathcal{F}_t\} du \quad \text{EQFund1}$$

5. We can also write

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t, u; r + \lambda)[\pi_u + \lambda_u \theta_u + (\tilde{f}_u - \tilde{c}_u)C_u +$$

$$+ (r_u - \tilde{f}_u)V_u + (\tilde{h}_u - r_u)H_u] | \mathcal{F}_t\} du \quad \text{EQFund2}$$

Recursive non-decomposable Nature of Pricing – VI

6. Write this last eq as a BSDEs by completing the martingale term.

$$d\bar{V}_t - (\tilde{f}_t + \lambda_t)\bar{V}_t dt + (\tilde{f}_t - \tilde{c}_t)C_t dt + \pi_t\, dt + \lambda_t \theta(C_t, \bar{V}_t)dt - (r - \tilde{h})H_t dt = dM_t,$$

$$\bar{V}_t = H_t + F_t + C_t, \ \ \varepsilon_t = \bar{V}_t \ \text{(replacement closeout)}, \quad \bar{V}_T = 0.$$

   Recall that $\tilde{f}$ depends on $\bar{V}$ nonlinearly, and so does $\tilde{c}$ on C and $\tilde{h}$ on H. M is a martingale under the pre-default filtration.

7. Assume a Markovian vector of underlying assets S (pre- credit and funding) with diffusive generator $\mathcal{L}^{r,\sigma}$ under $\mathbb{Q}$, whose 2nd order part is $\mathcal{L}_2$. Let this be associated with brownian W under $\mathbb{Q}$.

$$dS = rSdt + \sigma(t,S)SdW_t, \ \ \mathcal{L}^{r,\sigma}u(t,S) = rS\partial_S u + \frac{1}{2}\sigma(t,S)^2 S^2 \partial_S^2 u$$

Recursive non-decomposable Nature of Pricing – VII

8. Use Ito's formula on $\bar{V}(t, S)$ and match dt (and dW) terms: obtain PDE (& explicit representation for BSDE term ZdW).

Details are given in the Pallavicini Perini and B. (2011, 2012) reports. This leads to the following PDE with terminal condition $\bar{V}_T = 0$.

$$(\partial_t - \tilde{f}_t - \lambda_t + \mathcal{L}^{r,\sigma})\bar{V}_t + (\tilde{f}_t - \tilde{c}_t)C_t + \pi_t + \lambda_t \theta(C_t, \bar{V}_t) - (r - \tilde{h})H_t = 0, \quad \text{[NPDE1]}$$

$$\bar{V}_t = H_t + F_t + C_t, \quad \varepsilon_t = \bar{V}_t \quad \text{(replacement closeout)}$$

Alternatively, the funding/credit risk free price can be used for closeout (risk free closeout), simplifying calculations.

Recursive non-decomposable Nature of Pricing – VIII

The above PDE can be simplified further by assuming Delta Hedging:

$$H_t = S_t \frac{\partial \bar{V}_t}{\partial S} \text{ (delta hedging), leading to}$$

$$(\partial_t - \tilde{f}_t - \lambda_t + \mathcal{L}^{\tilde{h},\sigma})\bar{V}_t + (\tilde{f}_t - \tilde{c}_t)C_t + \pi_t + \lambda_t \theta(C_t, \bar{V}_t) = 0, \quad \text{[NPDE2]}$$

This PDE is NON-LINEAR not only because of θ, but also because $\tilde{f}$ depends on F, and $\tilde{h}$ on H, and hence both on $\bar{V}$ itself.

IMPORTANT: <u>THIS PDE DOES NOT DEPEND ON r.</u>
This is good, since r is a theoretical rate that does not correspond to any market observable.

Recursive non-decomposable Nature of Pricing – IX

We may now use nonlinear Feynman Kac to rewrite this last PDE, free from r, as an expected value. We obtain

$$\bar{V}_t = \int_t^T \mathbb{E}^{\tilde{h}}\{D(t,u;\tilde{f}+\lambda)[\pi_u + \lambda_u \theta_u + (\tilde{f}_u - \tilde{c}_u)C_u]|\mathcal{F}_t\}du$$

$$\bar{V}_t = \int_t^T \mathbb{E}^{\tilde{h}}\{D(t,u;\tilde{f})1_{\{\tau>u\}}[\pi_u + \delta_\tau(u)\theta_u + (\tilde{f}_u - \tilde{c}_u)C_u]|\mathcal{G}_t\}du \quad \text{EQFund3}$$

Here $\mathbb{E}^h$ is the expected value under a probability measure where the underlying assets evolve with a drift rate (return) of $\tilde{h}$. Remember that $\tilde{h}$ depends on H, and hence on V. Therefore the PRICING MEASURE DEPENDS ON THE FUTURE VALUES OF THE VERY PRICE V WE ARE COMPUTING. NONLINEAR EXPECTATION. THE PRICING MEASURE BECOMES DEAL DEPENDENT.

Recursive non-decomposable Nature of Pricing – X

$$\bar{V}_t = \int_t^T \mathbb{E}\{D(t,u;r)1_{\{\tau>u\}}[\pi_u + (r_u - \tilde{c}_u)C_u +$$

$$+(r_u - \tilde{f}_u)(F_u + H_u) + (\tilde{h}_u - r_u)H_u + 1_{\{\tau\in du\}}\theta_u]|\mathcal{G}_t\}du$$

where we rewrote EQFund1 under the filtration $\mathcal{G}$. Recalling

$$\theta_u = \varepsilon_u - 1_{\{u=\tau_C<\tau_I\}}\Pi_{\text{CVAcoll}}(u) + 1_{\{u=\tau_I<\tau_C\}}\Pi_{\text{DVAcoll}}(u)$$

we can write

$$\bar{V}_t = \int_t^T \mathbb{E}\Bigg\{D(t,u;r)1_{\{\tau>u\}}\Bigg[\pi_u + \delta_\tau(u)\varepsilon_u +$$

$$+(r_u - \tilde{c}_u)C_u + (r_u - \tilde{f}_u)(F_u + H_u) + (\tilde{h}_u - r_u)H_u +$$

$$-\delta_\tau(u)1_{\{u=\tau_C<\tau_I\}}\Pi_{\text{CVAcoll}}(u) + \delta_\tau(u)1_{\{u=\tau_I<\tau_C\}}\Pi_{\text{DVAcoll}}(u)\Bigg]|\mathcal{G}_t\Bigg\}du$$

Recursive non-decomposable Nature of Pricing – XI

It is tempting to set $\bar{V}$ = RiskFreePrice + LVA + FVA - CVA + DVA

$$RiskFreePrice = \int_t^T \mathbb{E}\left\{ D(t,u;r) 1_{\{\tau > u\}} \left[\pi_u + \delta_\tau(u)\varepsilon_u \right] | \mathcal{G}_t \right\} du$$

$$LVA = \int_t^T \mathbb{E}\left\{ D(t,u;r) 1_{\{\tau > u\}} (r_u - \tilde{c}_u) C_u | \mathcal{G}_t \right\} du$$

$$FVA = \int_t^T \mathbb{E}\left\{ D(t,u;r) 1_{\{\tau > u\}} \left[(r_u - \tilde{f}_u)(F_u + H_u) + (\tilde{h}_u - r_u) H_u \right] | \mathcal{G}_t \right\} du$$

$$-CVA = \int_t^T \mathbb{E}\left\{ D(t,u;r) 1_{\{\tau > u\}} \left[-1_{\{u = \tau_C < \tau_I\}} \Pi_{\mathrm{CVAcoll}}(u) \right] | \mathcal{G}_t \right\} du$$

$$DVA = \int_t^T \mathbb{E}\left\{ D(t,u;r) 1_{\{\tau > u\}} \left[1_{\{u = \tau_I < \tau_C\}} \Pi_{\mathrm{DVAcoll}}(u) \right] | \mathcal{G}_t \right\} du$$

Recursive non-decomposable Nature of Pricing – XII

If we insist in applying these equations, rather than the r-independent NPDE2 or EQFund3, then we need to find a proxy for r. This can be taken as the overnight rate (OIS discounting).

Further, if we assume $\tilde{h} = \tilde{f}$ then

$$FVA = \int_t^T \mathbb{E}\left\{ D(t,u;r) 1_{\{\tau > u\}} \left[(r_u - \tilde{f}_u) F_u \right] | \mathcal{G}_t \right\} du$$

Notice that when we are borrowing cash F, since usually $f > r$, FVA is negative and is a cost. Also LVA can be negative.

Recall that the funding and hedging rates f and h are net credit risk, so they are typically a risk free rate plus a liquidity basis.

The above decomposition however, as pointed out earlier, only makes sense a posteriori and is not a real decomposition.

Recursive non-decomposable Nature of Pricing – XIII

Going back to the r-indepdendent formula EQFund3:
If collateral is a variable fraction $\alpha_t > 0$ of mark to market, with α_t being $\mathcal{F}_t$ adapted, typically smaller than one, then the above equation becomes

$$\bar{V}_t = \int_t^T \mathbb{E}^{\tilde{h}}\{D(t,u;\tilde{f}(1-\alpha)+\lambda+\alpha\tilde{c})[\pi_u+\lambda_u\theta_u]|\mathcal{F}_t\}du$$

If we move back to the full filtration $\mathcal{G}_t$ where credit is observable we get

$$\bar{V}_t = \int_t^T \mathbb{E}^{\tilde{h}}\{1_{\{\tau>u\}}D(t,u;\tilde{f}(1-\alpha)+\alpha\tilde{c})\pi_u|\mathcal{G}_t\}du, \quad \bar{V}_{\tau\wedge T} = 1_{\{\tau<T\}}\,\theta_\tau$$

We now turn to a benchmark case: The Black Scholes model.

Recursive non-decomposable Nature of Pricing – XIV

The structure can be explored further by assuming for example $C_t = \alpha_t \bar{V}_t$, with α being $\mathcal{F}_t$ adapted and positive

$\tilde{f}_t = f_+ 1_{F \geq 0} + f_- 1_{F \leq 0}, \;\; \tilde{c}_t = c_+ 1_{\bar{V}_t \geq 0} + c_- 1_{\bar{V}_t \leq 0}, \;\;\; f_{+,-}$ and $c_{+,-}$ constants.

We further assume $\tilde{h} = \tilde{f}$. One obtains

$$\partial_t V - f_+(V - S_t \partial_S V_t - \alpha V)^+ + f_-(-V + S_t \partial_S V_t + \alpha V)^+ - \lambda_t V +$$

$$+\frac{1}{2}\sigma^2 S^2 \partial_S^2 V - c_+ \alpha_t (V_t)^+ + c_- \alpha_t (-V_t)^+ + \pi_t + \lambda_t \theta_t(V_t) = 0$$

NONLINEAR PDE (SEMILINEAR). λ is the first to default intensity, π is the ongoing dividend cash flow process of the payout, θ are the complex optional contractual cash flows at default including CVA and DVA payouts after collateral. c_+ and c_- are the borrowing and lending rates for collateral.

Recursive non-decomposable Nature of Pricing – XV

We can use Lipschitz coefficients results to investigate $\exists!$ of viscosity solutions. Classical soultions may also be found but require much stronger assumptions and regularizations.

The Black Scholes Benchmark Case I

We just recall briefly the Black Scholes PDE to see how funding costs change this basic benchmark case.

The Black Scholes framework:

- In the given economy, two securities are traded continuously from time 0 until time T. The first one (a bond/bank account/cash) is riskless and its (deterministic) price B_t follows

$$dB_t = B_t r dt, \quad B_0 = 1, \quad \Longleftrightarrow \quad B_t = e^{rt}, \quad r \geq 0. \tag{48}$$

  The short term interest rate r is constant through time.

- As for the 2nd risky asset (eg equity stock), consider the following SDE

$$dS_t = S_t[\mu dt + \sigma dW_t], \quad \mu = \text{return/local growth}, \quad \sigma = \text{volatility} \tag{49}$$

  with $0 \leq t \leq T$, initial condition $S_0 > 0$.

The Black Scholes Benchmark Case II

Modeling Assumptions: Black and Scholes' *ideal conditions* and further assumptions (see first slides of this course). In particular

- (i) there are no transaction costs in trading the stock;
- (ii) short selling is allowed without any restriction or penalty.
- (iii) No credit or default risk
- (iv) No funding costs: borrowing and lending either B or S happen at the risk free rate r
- (v) Since there is no credit risk, no collateral is present either.

We limit ourselves to price *simple* contingent claims, i.e. claims $Y = f(S_T)$ (for example call option $Y = (S_T - K)^+$).

The Black Scholes Benchmark Case III

Self financing condition for a trading strategy in B and S plus Ito's formula gives a PDE for the option value V:

$$\partial_t V(t,S) + rS\,\partial_S V(t,S) + \frac{1}{2}\sigma^2 S^2\,\partial_S^2 V(t,S) = rV(t,S), \qquad (50)$$

$$V(T,S) = (S-K)^+.$$

This PDE does not depend on μ but only on r.

This price corresponds to a heding strategy that is self–financing and is based on holding

$$\partial_S V(t,S) \text{ stock } S \text{ and } (V - \partial_S V(t,S)S)/B \text{ amount of cash } B$$

The Black Scholes Benchmark Case IV

But what would happen now if we had a different cost of borrowing and lending, say both S and B, at a rate f_+ if positive and f_- if negative?

And if we have default risk for the parties in the trade? (including us...)

We are the trader on the desk. Extend valuation: hedging strategy assets S and B are now funded by our bank treasury at a given cost corresponding to an interest rate f_+. If we have cash or risky asset we can lend to the market at a rate f_-. If $f_+ \neq f_-$ the pricing PDE changes

In presence of collateral $C_t = \alpha_t V_t$ and credit risk in the CVA/DVA term θ, and with hedge $H_t = S_t \partial_S V$ the PDE becomes

$$\partial_t V - f_+(V - S_t \partial_S V_t - \alpha V)^+ + f_-(-V + S_t \partial_S V_t + \alpha V)^+ - \lambda_t V +$$

$$+\frac{1}{2}\sigma^2 S^2 \partial_S^2 V - c_+ \alpha_t (V_t)^+ + c_- \alpha_t (-V_t)^+ + \lambda_t \theta_t (V_t) = 0$$

The Black Scholes Benchmark Case V

$$\partial_t V - f_+(V - S_t \partial_S V_t - \alpha V)^+ + f_-(-V + S_t \partial_S V_t + \alpha V)^+ - \lambda_t V +$$

$$+ \frac{1}{2}\sigma^2 S^2 \partial_S^2 V - c_+ \alpha_t (V_t)^+ + c_- \alpha_t (-V_t)^+ + \lambda_t \theta_t (V_t) = 0$$

(used immersion to move under the default-free market filtration).

NONLINEAR PDE. λ is the first to default intensity, θ are the complex optional contractual cash flows at default (residual CVA/DVA after collateral). c_+ and c_- are collateral borrowing and lending rates that are often assumed to be Overnight rates.

In real applications the rates f depend on the funding policy of the institution

The Black Scholes Benchmark Case VI

Notice that

- if $f_+ = f_- = r$ (symmetric risk free borrowing and lending),
- $\alpha = 0$ (no collateral),
- $\lambda = 0$ (no credit risk),

then we get back the Black Scholes LINEAR PDE.

Nonlinearities due to funding I

NONLINEAR PDEs cannot be solved as Feynman Kac expectations.

Backward Stochastic Differential Equations (BSDEs)

For NPDEs, the correct translation in stochastic terms are BSDEs. The equations have a recursive nature and simulation is quite complicated.

Aggregation–dependent and asymmetric valuation

Worse, the valuation of a portfolio is aggregation dependent and is different for the two parties in a deal. In the classical pricing theory a la Black Scholes, if we have 2 or more derivatives in a portfolio we can price each separately and then add up. Not so with funding. Without funding, the price to one entity is minus the price to the other one. Not so with funding.

Nonlinearities due to funding II

Aggregation levels decided a priori and somewhat arbitrarily.

Consistent global modeling across asset classes and risks

Once the level of aggregation is set, the funding valuation problem is non–separable. An holistic approach is needed and consistent modeling across trading desks and asset classes is needed. Internal competition in banks does not favour this.

Furthermore, the classical transaction-independent arbitrage free price is lost, now the price depends on the specific entities trading the product and on their policies.

Nonlinearities due to funding III

The end of Platonic pricing?

There is no Platonic measure $\mathbb{Q}$ in the sky to price all derivatives with an expectation where all assets have the risk free return r.
Now the pricing measure is product dependent, and every trade will have a specific measure. This is an implication of the PDE non-linearity (don't say things like "Relativity of the Pricing measure"!!!).

Other markets had realized a long time ago that a product price would also depend on the conditions under which the product itself is traded and on the company policies.

Finance arrived at this conclusion quite late, even if market practitioners had been doing this in a sort of implicit way.

Nonlinearities due to funding IV

When basic financial sense leads to complex mathematics

Notice that adherence to real banking policies does not make the problem ”boring, purely accounting–like and trivial”. Rather, valuation becomes aggregation dependent and holistic. We need BSDEs rather than expected values, or nonlinear PDEs rather than linear ones.

This opens many problems of operational efficiency and efficiency of implementation.

In many cases one forces symmetries and linearization so as to go back to a linear setting and have funding included as simple discounting. This is not accurate in general but allows the quick calculation of a funding valuation adjustment (FVA).

Funding Costs: Industry approximations I

We now look at how banks have been approximating the funding calculation to avoid the above problems and complexities.

Disclaimer: Banks have not agreed to a standard on FVA calculations, or even to a precise payout. To the best of my knowledge, a number of banks are applying the approaches below. The two examples I am giving are completely different. This is to give you an idea of the total lack of standards we have at the moment. Our full approach above is the most clear possibility at the moment.

Often the subtleties of analyzing the replicating portfolio and the single funding costs of its components are left aside. The bank focuses on the exposure as a whole as if the exposure had to be funded as a whole in a uniform way.

Funding Costs: Industry approximations II

We will look at two industry approaches:

- The approximate cost of funding for the offsetting hedge trade (FOHT) with a CCP, and
- The cost of Funding as explained by the Synthetic Cash (or CDS-Bond) Basis (FSCB)

Cost of Funding the collateralization: LVA I

We can analyze the following costs.

- **Cost of funding the collateralization process**. This is at times called Liquidity Valuation Adj (LVA). In our derivation above, the related term is γ. The way banks compute this is similar to ours but simpler. Say that the collateral process is C_i at margining time t_i, being positive if received and negative if posted by us (Bank B). Let Bank B credit spread be S_B. The cost is given by the following sum over margining dates, idealized as an integral

$$\mathrm{LVA} = \mathbb{E}_0\left\{\int_0^T D(0,t;r)[(C_t^+ - (-C_t)^+)(r_t - \phi_t)]\mathbb{Q}(\tau_B > t)\mathbb{Q}(\tau_C > t)dt\right\}$$

$$= \mathbb{E}_0\left\{\int_0^T D(0,t;r+S_B+S_C)[(C_t^+ - (-C_t)^+)(r_t - \phi_t)]dt\right\}$$

Cost of Funding the collateralization: LVA II

Here typically r is overnight and D OIS discounting plus the Bank and counterparty credit spreads. Rates ϕ include the funding rates and the credit spreads (recall our earlier f were net credit spreads). Rates are assumed to be symmetric. Typically, $\phi = r + S^B + \ell$, where ℓ is a possible liquidity basis (eg CDS-BOND basis).

- **Cost of funding for uncollateralized deals**. Now there is no collateral. In this case we discuss the two approaches above, FOHT and FSBC.

Cost of FOHT I

We have an uncollateralized deal.

- In this case we look at hedging our uncollateralized derivative towards a client by an opposite payout traded with Bank C via a CCP. We can charge the funding costs for this opposite deal to our counterparty as a charge for lack of collateralization. The full Funding Valuation Adjustment (FVA) term payout was φ in our previous analysis.
  The simplified formula for the cost part of FVA, that we call Fuding Cost Adjustment (FCA), becomes the following

$$-\mathrm{FCA} = -\int_0^T S_t^B \varepsilon_{0,t}^+ \mathbb{Q}(\tau_B > t)\mathbb{Q}(\tau_C > t)dt$$

Cost of FOHT II

where $\varepsilon_{0,t}^{+,-}$ are the positive and negative exposures at t for the original trade, seen from 0. In particular, under risk free closeout

$$\varepsilon_{0,t} = \mathbb{E}_0[D(0,t)\Pi(t,T)], \quad \varepsilon_{0,t}^{+} = \max(\varepsilon_{0,t}, 0), \quad \varepsilon_{0,t}^{-} = \max(-\varepsilon_{0,t}, 0).$$

$\varepsilon_{0,t}^{-}$ is positive and it is defined as the negative part of the claim at t discounted back at 0 and priced at zero.
We are assuming again that our default and the default of bank C are independent, and both are independent from exposure. By using as an approximation the constant intensity formula for CDSs (this is not correct, strictly speaking),

$$S_t^B = \lambda_t^B (1 - Rec_B)$$

Cost of FOHT III

we can write

$$-\mathrm{FCA} = -\int_0^T (1 - Rec_B)\varepsilon^+_{0,t}\mathbb{Q}(\tau_B \in dt)\mathbb{Q}(\tau_C > t)$$

We recall that we are trading with C the opposite of the original payout with the client, that is why we have a funding cost when the original exposure is positive. It means that our hedge trade is negative and we need to borrow from our treasury to post the relevant marging to the CCP. The cost of this borrowing is above and is related to our bank B spread.

Cost of FOHT IV

- Recall that in the end the total deal value will be

$$\mathit{DefaultFundingFreePrice} - \mathit{CVA} - \mathit{FCA} + \mathit{DVA}$$

Now if we go back to our earlier formulas for CVA and DVA, assume mutual independence between τ_B, τ_C and ε,

$$-\mathrm{CVA} = -\int_0^T (1 - \mathit{Rec}_C)\varepsilon^+_{0,t}\mathbb{Q}(\tau_C \in dt)\mathbb{Q}(\tau_B > t)$$

$$\mathrm{DVA} = \int_0^T (1 - \mathit{Rec}_B)\varepsilon^-_{0,t}\mathbb{Q}(\tau_B \in dt)\mathbb{Q}(\tau_C > t)$$

$$-\mathrm{FCA} = -\int_0^T (1 - \mathit{Rec}_B)\varepsilon^+_{0,t}\mathbb{Q}(\tau_B \in dt)\mathbb{Q}(\tau_C > t)$$

Cost of FOHT V

In fact what we call DVA could be considered as a funding benefit, or Funding Benefit Adjustment (FBA). Its expression is exactly the same as FCA except for the fact that the negative exposure is referenced here. Recall that we are trading with C the opposite of the original payout, so that if ε is negative we are actually trading its opposite, a positive amount, with C. Hence we will receive margins from C (via the CCP) for an amount circa ε^-, which we can immediately use to reduce our borrowing from our treasury. This saves us a rate related to our own bank cost of funding, expressed by $\mathbb{Q}(\tau_B \in dt)$.

Cost of FOHT VI

- In fact some banks view DVA as a positive funding benefit FBA offseting the funding cost -FCA above, rather than as a debit valuation adjustment, and call FVA the total

$$FVA = -FCA + DVA = -\int_0^T (1 - Rec_B)\varepsilon_{0,t}\mathbb{Q}(\tau_B \in dt)\mathbb{Q}(\tau_C > t)$$

- Other banks keep the name ”DVA” for the debit valuation adjustment and call FVA only the -FCA part above
- Notice that with these spread and exposure based definitions, funding costs become separable again and the adjustments CVA, DVA and FVA become more objectively distinct. Furthermore, all we need are credit spreads and exposures, and we can avoid exotic nonlinear pricing tools such as BSDEs or NPDEs.

Cost of FOHT VII

- However, to obtain this we need to introduce unrealistic independence assumptions between defaults and exposures. No Wrong Way Risk. Symmetric borrowing and lending rates.

FSCB approach I

In this case we use a toy model for simplicity and we consider a very simple product.

We assume the payoff is payed at time T and is a notional amount N that can be either positive or negative (long or short a zero coupon bond). We assume constant risk free interest rates r, here proxied by EONIA.

We assume we are a bank B trading with a counterparty C, without collateral. For simplicity we assume zero recovery rates for both B and C, and we call $S_B = \lambda_B$ and $S_C = \lambda_C$ the CDS (hence synthetic) spreads for B and C, which we assume to be constant across maturities.

FSCB approach II

We will call C_B and C_C the related bond (hence Cash) spreads for the same name, also assumed to be constant over maturities.

We will denote by B the synthetic-cash CDS-Bond basis and will assume it to be the same for B and C:

$$B = S_B - C_B = S_C - C_C.$$

According to the FSCB approach, one writes CVA and DVA in the usual way by using synthetic spreads, that are good proxies of credit risk without funding liquidity components, since CDS are typically collateralized too. The total value of the bond is

$$V = Ne^{-rT} - CVA^S + DVA^S$$

FSCB approach III

In other words, in this approach we write the traditional CVA/DVA adjustments without including any funding initially. The S superscript points out that CVA and DVA are computed with synthetic spreads.

Now we assume independence between defaults of B and C, deterministic rates.

FSCB approach IV

$$CVA = e^{-rT}(1 - e^{-S_C\, T})N^{+} \approx e^{-rT} S_C\, T\, N^{+},$$

$$DVA = e^{-rT}(1 - e^{-S_B\, T})N^{-} \approx e^{-rT} S_B\, T\, N^{-}$$

Here $N^{-} = max(-N, 0)$, $N = N^{+} - N^{-}$.

Now write the above in terms of cash credit spreads:

$$V = Ne^{-rT} - CVA + DVA = Ne^{-rT} - e^{-rT} S_C\, T\, N^{+} + e^{-rT} S_B\, T\, N^{-}$$

$$= Ne^{-rT} - e^{-rT}(C_C + B)\, T\, N^{+} + e^{-rT}(C_B + B)\, T\, N^{-}$$

$$= Ne^{-rT} - e^{-rT} C_C\, T\, N^{+} + e^{-rT} C_B\, T\, N^{-} - e^{-rT} B\, T\, N$$

$$=: Ne^{-rT} - CVA^{C} + DVA^{C} + FVA$$

FSCB approach V

It may be interesting to check what we obtain with the previous method, FOHT, under the zero recovery and flat spread assumptions.

$$\text{FOHT}: \; V = RiskFreePrice - CVA + DVA - FCA$$
$$\text{FOHT}: \; V = Ne^{-rT} - e^{-rT} S_C \, T \, N^+ + e^{-rT} S_B \, T \, N^- - e^{-rT} S_B \, T \, N^+$$
$$\text{FOHT}: \; V = RiskFreePrice - CVA + FVA$$
$$\text{FOHT}: \; V = Ne^{-rT} - e^{-rT} S_C \, T \, N^+ - e^{-rT} S_B \, T \, N$$

$$\text{FSCB}: \; V = RiskFreePrice - CVA^S + DVA^S$$
$$\text{FSCB}: \; V = Ne^{-rT} - e^{-rT} S_C \, T \, N^+ + e^{-rT} S_B \, T \, N^-$$
$$\text{FSCB}: \; V = RiskFreePrice - CVA^C + DVA^C + FVA$$
$$\text{FSCB}: \; V = Ne^{-rT} - e^{-rT} C_C \, T \, N^+ + e^{-rT} C_B \, T \, N^- - e^{-rT} B \, T \, N$$

FSCB approach VI

One could also implement FOHT with cash spreads, although being the offsetting trade with a CCP (collateral) we would expect to use synthetic spreads.

Which of the two methods is more reasonable?

If one compares with our earlier rigorous analysis, none of the two makes fully sense. However, if hard pressed, one would probably conclude that FOHT is more in line with the full approach above. This is because credit spreads are a key component of the funding benefit and cost, as embedded in survival indicators that drive the valuation adjustment formulas.

Moreover, the FSCB fails to explicitly recognize DVA as a funding benefit. The funding benefit interpretation is quite obvious in simplified settings, as we have seen earlier.

Funding: Price or Value

We go now back to the general discussion. An important question is

Is the funding inclusive ”price” a real price? Price and Value

Each entity computes a different funding adjusted price for the same product. The funding adjusted ”price” is not a price in the conventional sense. We may use it for cost/profitability analysis or to pay our treasury, but can we charge it to a client? Why should the client pay for our funding inefficiencies? It is more a ”value” than a ”price”.

Funding structures inside a bank?

Funding implications on a Bank structure

Including funding costs into valuation, even via a simplistic FVA, involves methodological, organisational, and structural challenges.

Many difficulties are similar to CVA's and DVA's, so Funding can be integrated in the CVA effort typically.

- Reboot IT functions, analytics, methodology, by adopting a consistent global methodology including a consistent credit-debit-collateral-funding adjustment
- Very strong investment, discontinuity, and against the "internal competition" culture
- OR include separate and inconsistent CVA and FVA adjustments, accepting simplifications and double counting.
- It can be important to analyze the global funding implications of the whole trading activity of the bank.

Coda: Multiple Curves? I

Can we make this whole machinery, based on existence of a risk free rate r (even if it disappears in the final equations) rigorously consistent with the current multiple curves environment in interest rate modeling?

From the abovementioned dialogue:

Funding and CCPs Q&A

D. Brigo, A. Pallavicini (2013). CCPs, Central Clearing, CSA, Credit Collateral and Funding Costs Valuation FAQ (`http://ssrn.com/abstract=2361697`, arXiv.org)

Q: [Three days later] Multiple curves: How do they connect with all you told me so far?

Coda: Multiple Curves? II

A: The connection is made quite explicit in the paper ”D. Brigo, A. Pallavicini (2013). Interest-Rate Modelling in Collateralized Markets: Multiple curves, credit-liquidity effects, CCPs. SSRN.com and arXiv.org”

Q: What’s the story?

A: The ”story” is that the classical relationship between forward LIBOR rates and zero coupon bonds is not working anymore. In particular, interpreting the zero-coupon curves as risk-free rate curves is no longer tenable. Likewise, interpreting the LIBOR rates as the simply compounded rates that underlie risk-free one-period swaps does not work anymore. To include risks that are now heavily affecting interest rates, one needs to design a theory that includes explicit cash flows accounting for default closeout and also for costs of funding for the hedging portfolio, and costs of funding for collateral (margining).

Coda: Multiple Curves? III

Q: Even leaving aside the LIBOR rigging scandal... what are LIBOR rates these days? I'm not sure anymore...

A: LIBOR is now a rate assigned by the market and not derived by risk-free forward rate agreements or risk-free one-period swaps.

Q: How does this connect to our discussion?

A: **By including credit, collateral and funding effects in valuation one obtains the master equation seen earlier. In the specific case where collateral is a fraction of current all-inclusive mark-to-market, one obtains a simpler pricing equation based on a hedge-funding-fees equivalent measure and on a generalized dividend process inclusive of default risk, treasury funding and collateral costs.**

Q: hedge funding fees equivalent measure???

Coda: Multiple Curves? IV

A: Now we have new pricing measures Q's where assets evolve with local returns given by funding rates for the hedging portfolios...

Q: Ah...

A: Again in B. and Pallavicini (2013) they apply these approximations to the money market, in order to evaluate collateralized interest-rate derivatives.
They look at defining new building blocks that will replace the old unobservables-based ones (such as risk free rate zero coupon bonds, risk-free one period swap rates, etc). Such new instruments are based on the collateral rate, which is an observable rate, since it is contractually defined by the CSA as the rate to be used in the margining procedure.

Q: ISDA and CCPs are helping there...

Coda: Multiple Curves? V

A: **The collateralized zero-coupon bond can be proxied by single-period Overnight Indexed Swaps (or by quantities bootstrapped from multi-period OIS) when we accept to approximate a daily compounded rate with continuously compounded rates. They then define fair OIS rates at inception in terms of collateralized zero coupon bonds.**

Q: And so forward LIBOR rates become...

A: **This setup allows us to define new forward LIBOR rates as equilibrium rates in a collateralized one-period swaps. The resulting rates depend on the collateralized coupon-bearing bonds. They also define a collateral based forward measure where forward LIBOR rates are expected values of future realized LIBOR rates.**

Q: Quite a market–based model...

Coda: Multiple Curves? VI

A: They then hint indeed at the development of a market model theory for the new collateral-inclusive forward LIBOR rates and to a forward rates theory for the OIS-based instantaneous forward rates.

Q: And this is where the multiple curve picture finally shows up I guess

A: Indeed, they have a curve with LIBOR based forward rates, that are collateral adjusted expectation of LIBOR market rates that are taken as primitive rates from the market, and they have instantaneous forward rates that are OIS based rates.

Q: Quite reasonable actually.

A: OIS rates are driven by collateral fees, whereas forward LIBOR rates are driven both by collateral rates and by the primitive LIBOR market rates.

Coda: Multiple Curves? VII

Q: Ok this is a general framework and is very interesting, but are there specific models discussed?

A: They approach this by introducing a dynamical multiple-curve model for OIS and LIBOR rates, by reformulating the parsimonious HJM model by Moreni and Pallavicini (2010, 2012) under the new pricing framework.

Q: So those are the models they enrich with the credit, collateral and funding analysis?

A: Correct. They focus on uncollateralized, partially-collateralized and over-collateralized contracts. With partial collateralization they evaluate the adjustment needed by pricing equations to include the corrections coming from Treasury cash and hedging funding. In particular, forward LIBOR rates associated to partially collateralized one-period swap contracts acquire a covariance term that can be interpreted as a convexity adjustment.

Coda: Multiple Curves? VIII

Q: What happens with CCPs?

A: In that paper CCPs initial margins are modeled too. This leads to a generalization of the general formula.
They also discuss credit spreads. It is argued that credit spreads should be calibrated via Credit Default Swaps or Defaultable Bonds. However, this should be a global calibration because CDS are collateralized and are in principle priced with the same general formula, inclusive of collateral and funding, that is used for all other deals.

Q: Another "Global Calibration"???

A: Well, bonds are not collateralized and are funded more heavily, so funding risk is also there. This global CDS or Bond calibration is not done usually even though interpreting CDS or Bonds as sources of pure credit risk calibration may lead to important errors.

Coda: Multiple Curves? IX

Q: What about funding more specifically?

A: The term structure of funding rates depends on the funding policy and is model dependent. Stripping it directly from market liquid instruments is very difficult, especially because the interbank market is no longer (fully) representative of such costs. Collateral portfolios play now a key role too. Default intensities, collateral rates and liquidity bases will be key drivers of the funding spreads. Finally, one may consider using Value-at-Risk measures to determine the effective fraction of mark-to-market one should hold as collateral or even the initial margin, as in our examples above [see the full dialogue paper], and introduce collateral haircuts.

Q: Quite a composite picture!

A: At least their final interest-rate curves are consistently explained by such effects and based on market observables.

Q & A. CCPs

Q And what about Central Counterparty Clearing houses (CCP's)?

A *CCPs are commercial entities that, ideally, would interpose themselves between the two parties in a trade.*

- *Each party will post collateral margins say daily, every time the mark to market goes against that party.*
- *Collateral will be held by the CCP as a guarantee for the other party.*
- *If a party in the deal defaults and the mark to market is in favour of the other party, then the surviving party will obtain the collateral from the CCP and will not be affected, in principle, by counterparty risk.*
- *Moreover, there is also an initial margin that is supposed to cover for additional risks like deteriorating quality of collateral, gap risk, wrong way risk, etc.*

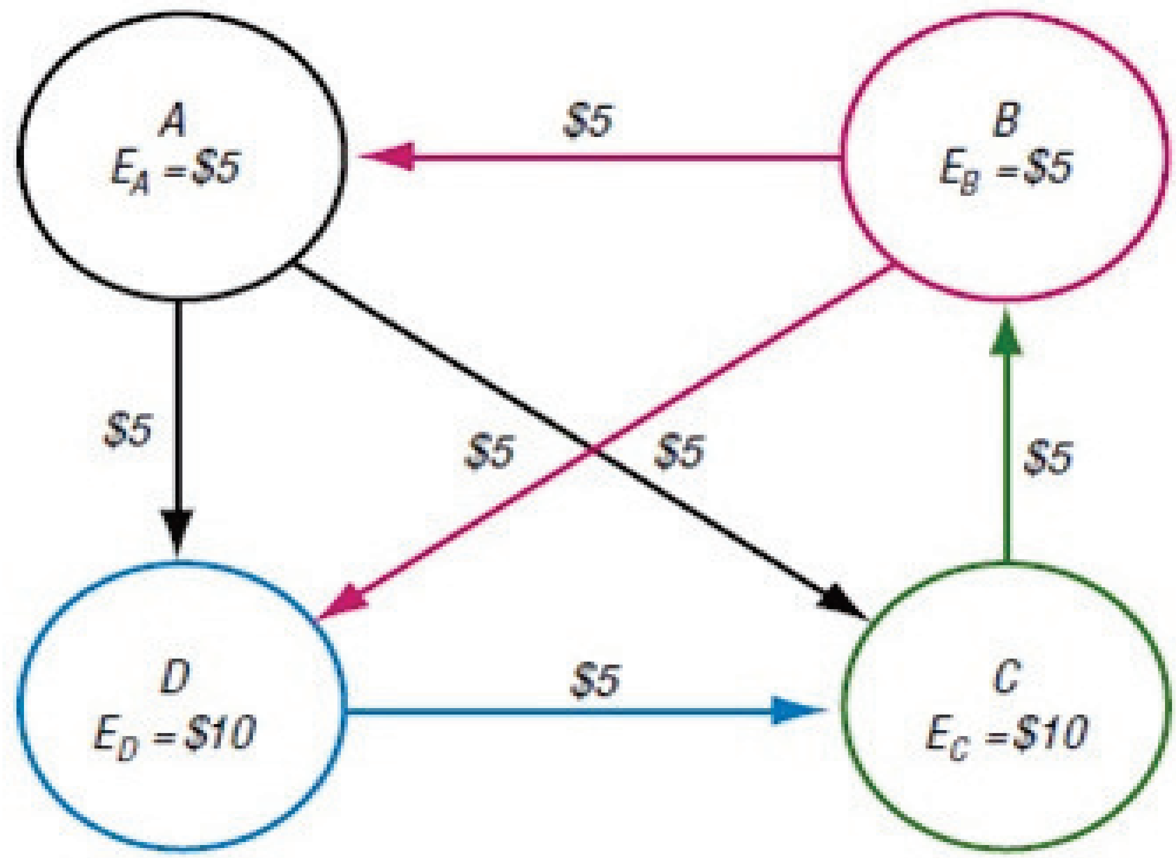


Figure: Bilateral trades and exposures without CCPs. Source: John Kiff.

http://shadowbankers.wordpress.com/2009/05/07/mitigating-counterparty-credit-risk-in-otc-markets-the-basics

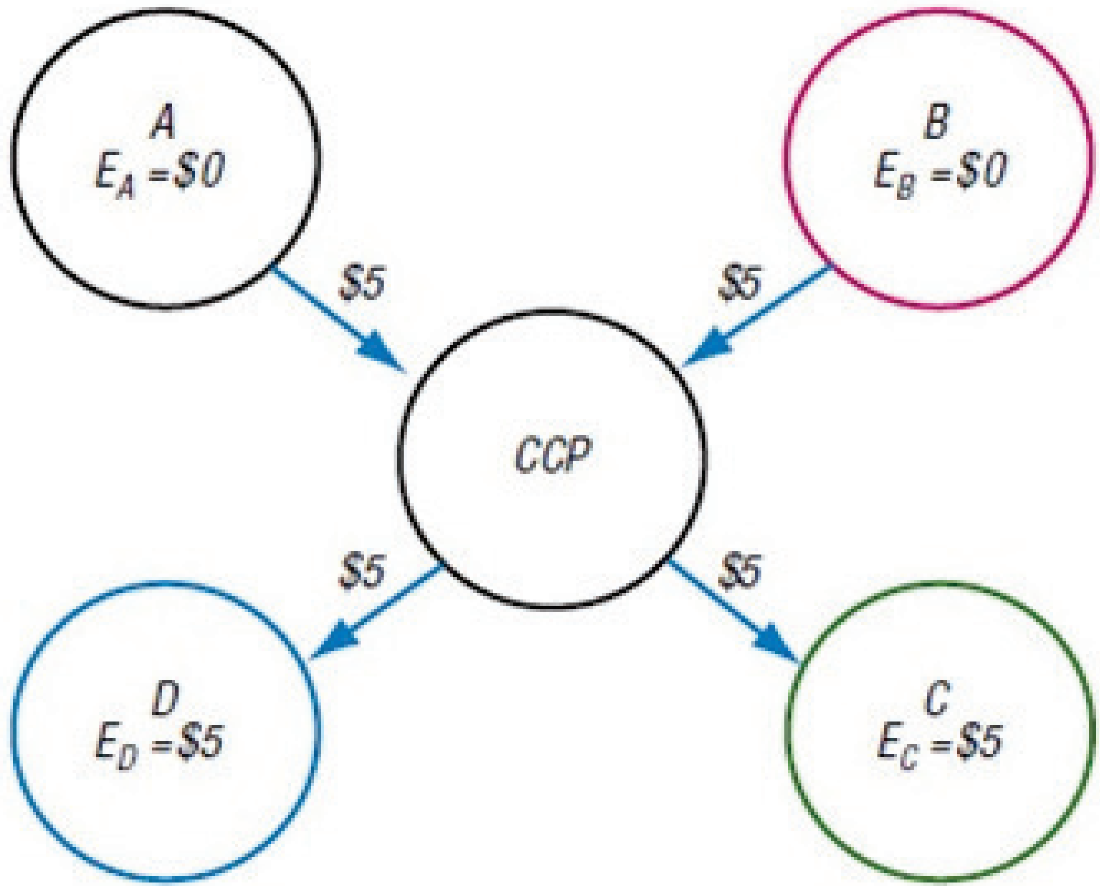


Figure: Bilateral trades and exposures with CCPs. Source: John Kiff.

http://shadowbankers.wordpress.com/2009/05/07/mitigating-counterparty-credit-risk-in-otc-markets-the-basics

Q & A: CCPs

Q It looks pretty safe. With the current regulation and law pushing firms to trade through central clearing, will all this analysis of credit, liquidity and funding risk be a moot point? Are CCP's going to be the end of CVA/DVA/FVA problems?

A *CCP's will reduce risk in many cases but are not a panacea. They also require daily margining, and one may question*

- *The pricing of the fees they apply*
- *The appropriateness of the initial margins and of overcollateralization buffers that are supposed to account for wrong way risk and collateral gap risk*
- *The default risk of CCPs themselves.*

Q & A: CCPs

Q So it is not that safe after all.

A *Valuation of the above points requires CVA type analytics, inclusive of collateral gap risk and wrong way risk, similar to those we discuss here. So unless one trusts blindly a specific clearing house, it will be still necessary to access CVA analytics and risk measures.*

Q & A: CCPs

Q So CCP's are not really a panacea. Other issues with CCPs?

A *The following points are worth keeping in mind:*[a]

- *CCPs are usually highly capitalised. All clearing members post collateral (asymmetric "CSA"). Initial margin means clearing members are overcollateralised all the time.*
- *TABB Group says extra collateral could be about 2 \$ Trillion.*[b]
- *CCPs can default and did default. Defaulted ones - 1974: Caisse de Liquidation des Affaires en Marchandises; 1983: Kuala Lumpur Commodity Clearing House; 1987: Hong Kong Futures Exchange. The ones that were close to default- 1987: CME and OCC, US; 1999: BM&F, Brazil.*

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[a] See for example Piron, B. (2012). Why collateral and CCPs can be bad for your wealth. SunGard's Adaptive White Paper.

[b] Rhode, W. (2011). European Credit and Rates Dealers 2011 – Capital, Clearing and Central Limit Order Books. TABB Group Research Report

Q & A: CCPs

Q And what about netting under CCPs? That should improve as everything goes through them.

A *Not so clear. A typical bank may have a quite large number of outstanding trades, making the netting clause quite material. With just one CCP for all asset classes across countries and continents, netting efficiency would certainly improve. However, in real life CCPs deal with specific asset classes or geographical areas, and this may even reduce netting efficiency compared to now.*

Q & A: CCPs

Q So CCPs could compete with each other?

A *Yes and one can be competitive in specific areas but hardly in all of them. Some CCPs will be profitable in specific asset classes and countries. They will deal mostly with standardised transactions. Even if CCPs could function across countries, bankruptcy laws can make collateral held in one place unusable to cover losses in other places.*[a]

[a] Singh, M. (2011) Making OTC Derivatives Safe - A Fresh Look. IMF paper

Q & A: CCPs

Q The geographical angle seems to be an issue, with no international law addressing how CCPs would connect through EMIR/CDR 4/Basel III and DFA. Is that right?

A *Indeed, there is currently ”No legal construct to satisfy both Dodd Frank Act and EMIR and allow EU clients to access non-EU CCP's”.*[a] *And there are also other conflicts in this respect. Where will CCPs be located and which countries will they serve? For example, the European Central Bank opposed LCH–Clearnet to work with Euro denominated deals because this CCP is not located in the Eurozone. This lead to a legal battle with LCH invoking the European Court of Justice.*

[a] Wayne, H. (2012). Basel 3, Dodd Frank and EMIR. Citigroup Presentation.

Q & A: CCPs

Q But competition should be a good thing?

A *To compete CCPs may lower margin requirements, which would make them riskier, remember the above CCPs defaults? In the US, where the OTC derivatives market is going through slightly more than 10 large dealers and is largely concentrated among 5, we could have a conflict of interest. If CCPs end up incorporating most trades currently occurring OTC bilaterally, then CCPs could become "too big to fail".*[a]

Q So what should a bank do in modeling CCPs counterparty risk?

A *The CCP does not post collateral directly to the entities trading with it, as the collateral agreement is not symmetric. Hence, it is like CVA but computed without collateral. On top of that, one has the overcollateralization cost to lose. Hopefully, the default probability is low, making CVA small, bar strong contagion, gap risk and WWR*

[a] Miller, R. S. (2011) Conflicts of interest in derivatives clearing. Congressional Research Service report.

Q & A: CCPs

Q It seems like even with CCPs one needs a strong analytical and numerical apparatus for pricing/hedging and risk.

A *Yes for all the reasons we illustrated, CCPs are not the end of CVA and its extensions. We need to consider and price/ risk-manage*

- *Checking initial margin charges across different CCPs to see which ones best reflect actual gap risk and contagion. This requires a strong pricing apparatus*
- *Computing counterparty risk associated with the default of the CCP itself*
- *Understanding quantitatively the consequences of the lack of coordination among CCPs across different countries and currencies.*

Q & A: CVA Desks? "Best practices"?

Q In terms of active and concrete CVA (DVA? FVA?) management, what is the "best practice" banks follow? I hear about "CVA Desks", what does that mean?

A *The idea is to move Counterparty Risk management away from classic asset classes trading desks by creating a specific counterparty risk trading desk, or "CVA desk". Under a lot of simplifying assumptions, this would allow "classical" traders to work in a counterparty risk-free world in the same way as before the counterparty risk crisis exploded.*

Q & A: CVA Desks? "Best practices"?

Q Why are CVA desks mostly a development of the crisis?

A *Roughly, CVA followed this historical path:*

- *Up to 1999/2000 no CVA. Banks manage counterparty risk through rough and static credit limits, based on exposure measurements (related to Credit VaR: Credit Metrics 1997).*
- *2000-2007 CVA was introduced to assess the cost of counterparty credit risk. However, it would be charged upfront and would be managed mostly statically, with an insurance based approach.*
- *2007 on, banks increasingly manage CVA dynamically. Banks become interested in CVA monitoring, in daily and even intraday CVA calculations, in real time CVA calculations and in more accurate CVA sensitivities, hedging and management.*
- *CVA explodes after 7[8] financials defaults occur in one month of 2008 (Fannie Mae, Freddie Mac, Washington Mutual, Lehman, [Merrill] and three Icelandic banks).*

Q & A: CVA Desks? "Best practices"?

Q Where is the CVA desk located and what does it do?

A *In most tier-1 and 2 banks it is in on the capital markets/trading floor division, being a trading desk. Occasionally it may sit on the Treasury department (eg Banca IMI). In a few cases it can be a stand-alone entity outside standard departments classifications.*

Q Why these different choices?

A *Trading floor is natural because it is a trading desk.*

The CVA desk charges classical trading desks a CVA fee in order to protect their trading activities from counterparty risk through hedging. This may happen also with collateral/CSA in place (Gap Risk, WWR, etc). The cost of implementing this hedge is the CVA fee the CVA desk charges to the classical trading desk.

Q & A: CVA Desks? "Best practices"?

Q Then why did some banks place it in the treasury department, for example?

A *Charging a fee is not easy and can make a lot of P&L sensitive traders nervous. That is one reason why some banks set the CVA desk in the treasury for example. Being outside the trading floor can avoid some "political" issues on P&L charges among traders.*

Q Would that be the only reason?

A *There is more. Given that the treasury often controls collateral flows and funding policies, this would allow to coordinate CVA and FVA calculations and charges after collateral.*

Q & A: CVA Desks? "Best practices"?

Q How would this CVA desk help classical trading desks, more in detail?[a]

A *It would free the classical traders from the need to:*

- *develop advanced credit models to be coupled with classical asset classes models (FX, equity, rates, commodities...);*
- *know the whole netting sets trading portfolios; traders would have to worry only about their specific deals and asset classes, as the CVA desk takes care of "options on whole portfolios" embedded in counterparty risk pricing and hedging;*
- *Hedge counterparty credit risk, which is very complicated.*

[a] See for example "CVA Desk in the Bank Implementation", *Global Market Solutions* white paper

Q & A: CVA Desks? "Best practices"?

Q Given all we have discussed earlier, this looks like a difficult challenge?

A *It is certainly difficult. The CVA desk has* **little/no control** *on inflowing trades, and has to:*

- *quote quickly to classical trading desks a "incremental CVA" for specific deals, mostly for pre-deal analysis with the client;*
- *For every classical trade that is done, the CVA desk needs to integrate the position into the existing netting sets and in the global CVA analysis in real time;*
- *related to pre-deal analysis, after the trade execution CVA desk needs to allocate CVA results for each trade ("marginal CVA")*
- *Manage the global CVA, and this is the core task: Hedge counterparty credit and classical risks, including credit-classical correlations (WWR), and check with the risk management department the repercussions on capital requirements.*

Q & A: CVA Desks? "Best practices"?

Q Is this working?

A *Of course the idea of being able to relegate all CVA(/DVA/FVA) issues to a single specialized trading desk is a little delusional.*

- *WWR makes isolating CVA from other activities quite difficult.*
- *In particular WWR means that the idea of hedging CVA and the pure classical risks separately is not effective.*
- *CVA calculations may depend on the collateral policy, which does not depend on the CVA desk or even on the trading floor.*
- *We have seen FVA and CVA interact*

In any case a CVA desk can have different levels of sophistication and effectiveness.

Q & A: CVA Desks? "Best practices"?

Q What do "classical traders" think about this?

A *Clearly, being P&L sensitive this is rather delicate. There are mixed feelings.*

- Because CVA is hard to hedge (especially the jump to default risk and WWR), occasionally classical traders feel that the CVA desk does not really hedge their counterparty risk effectively and question the validity of the CVA fees they pay to the CVA desk.
- Other traders are more optimistic and feel protected by the admittedly approximate hedges implemented by the CVA desk.
- There is also a psychological component of relief in delegating management of counterparty risk elsewhere.

PART III: RISK MEASURES

In this third part we look at the problem of risk measurement and management.

So far we discussed mostly valuation and hedging. This is important and is done under the risk neutral measure $\mathbb{Q}$, as we have seen earlier.

Risk Management however is partly based on historical estimation, and is interested in potential losses in the physical world, hence we need to go back to the historical/physical measure $\mathbb{P}$.

We introduce the two fundamental risk measures of Value at Risk (VaR) and Expected Shortfall (ES).

Risk Measures: A historical perspective I

This historical perspective is from Brian McHugh's review (2011)

This is an introduction into 'Risk Measures', particularly focusing on Value-at-Risk (VaR) and Expected Shortfall (ES) measures. A brief history of risk measures is given, along with a discussion of key contributions from various authors and practitioners.

What do we mean by ”Risk”? I

Risk is defined by the dictionary as ’a situation involving exposure to danger’. It is related to the randomness of uncertainty. Risk is also described as ’the possibilty of financial loss’ and this is the definition that will be discussed here.

Risk management, described by Kloman[5] as ’a discipline for living with the possibility that future events may cause adverse effects’, is of vital importance to the appropriate day to day running of financial institutions.

Here, downside risk (the probability of loss or less than expected returns) will be the focus of discussion as it is the most crucial area for risk managers. In particular, Value at Risk (VaR) and Expected Shortfall (ES) methodologies of measuring risk will be analysed.

What do we mean by ”Risk”? II

The question that comes to mind is where does this risk come from, and of course there is no single answer.

Risk can be created by a great number of sources, both directly and indirectly, it propogates from government policies, war, inflation, technological innovations, natural phenomena, and many others.

There are a number of risks faced by financial institutions everyday, these include market risk, credit risk, operational risk, liquidity risk, and model risk.

What do we mean by "Risk"? III

- Market risk includes the unexpected moves in the underlying of the financial assets (stock prices, interest rates, fx rates...)
- Credit risk propogates from the creditworthiness of a counterparty in a contract and the possibility of losses caused by its default.
- Operational risk: possibility of losses occured by internal processes, people, and systems or from other sources externally.
- Liquidity risk stems from the inability, in some cases, to buy or sell financial instruments in sufficient time as to minimise losses.
- Model risk: inaccurate use of valuation and pricing models, for instance inaccurate distributions or unrealistic assumptions. Negative interest rates? (eg Vasicek, Hull White), Models with thin tails instead of fat tails? Bad future volatility structures? Unrealistic correlation patterns? (see discussion on LMM above).

What do we mean by "Risk"? IV

- Finally, all such risks may interact in complex ways and their mutual dependence and contagion is a key aspect of modern research. As these risks are not really completely separable, this classification is purely indicative and not substantial.

[5] H. F. Kloman (1990), *Risk Management Agonistes*, Risk Analysis 10:201-205.

A brief history of VaR and Expected Shorfall I

The origins of VaR and risk measures can be traced back as far 1922 to capital requirements the New York Stock Exchange imposed on member firms according to Holton[6].

However, Markowitz's seminal paper 'Porfolio Theory' (1952), which developed a means of selecting portfolios based on an optimization of return given a certain level of risk, was the first convincing if stylized and simplistic method of measuring risk. His idea was to focus portfolio choices around this measurement.

A brief history of VaR and Expected Shorfall II

Risk management methodologies really took off from this point and over the next couple of decades new ideas, such as the Sharpe Ratio, the Capital Asset Pricing Model (CAPM) and Arbitrage Pricing Theory (APT), were being proposed and implemented.

Along with this came the introduction of the Black-Scholes option-pricing model in 1973, which lead to a great expansion of the options market, and by the early 1980s a market for over-the-counter (OTC) contracts had formed.

The related theory had important precursors in Bachelier (1900) and de Finetti (1931)[7]

A brief history of VaR and Expected Shorfall III

Perhaps the greatest consequence of the financial innovations of the 1970s and 1980s was the proliferation of leverage, and with these new financial instruments, opportunities for leverage abounded.

Think of an interest rate swap that is at the money: it costs nothing to enter this swap even on a huge notional, and yet this may lead to very large losses in the future.

Similarly for credit default swaps, oil swaps, and a number of other derivatives.

A brief history of VaR and Expected Shorfall IV

Along with academic innovation came technological advances. Information technology companies like Reuters, Telerate, and Bloomberg started compiling databases of historical prices that could be used in valuation techniques.

Financial instruments could be valued quicker with new hi-tech methods such as the Monte Carlo pricing for complex derivatives, and thus trades were being made quicker.

We have now reached super-human speed with high frequency trading, so debated that the EU is considering banning it.

However, in addition to all these innovations and advances came catastrophes in the financial world such as:

A brief history of VaR and Expected Shorfall V

- The Barings Bank collapse of 1995, which was solely down to the fraudulent dealings of one of its traders.
- Metallgesellschaft lost $1.3 billion by entering into long term oil contracts in 1993.
- Long-Term Capital Management's near collapse in 1998 and subsequent bailout overseen by the Federal Reserve. *Somewhat ironically, members of LTCM's board of directors included Scholes and Merton.*

For more information on these financial disasters and others see Jorion (2007).[8] Organizations were now more than ever increasingly in need for a single risk measure that could be applied consistently across asset categories in hope that financial disasters such as these could be prevented. However, even this wouldn't be enough, as the Lehman collapse of 2008 has shown. We'll discuss why later.

A brief history of VaR and Expected Shorfall VI

Q: ”What is Basel?”

A: ”A city in Europe? Perhaps switzerland?”

The Basel Committee on Banking Supervision was central to the introduction and implementation of VaR on a worldwide scale. The Committee itself does not possess any overall supervising authority, but rather gives standards, guidelines, and recommendations for individual national authorities to undertake.

The first Basel Accord of 1988 on Banking Supervision attempted to set an international minimum capital standard, however, according to McNeil at al.[9] this accord took an approach which was fairly coarse and measured risk in an insufficiently differentiated way.

A brief history of VaR and Expected Shorfall VII

The G-30 (consultative group on international economic and monetary affairs) report in 1993 titled 'Derivatives: Practices and Principles' addressed the growing problem of risk management in great detail.

It was created with help from J.P. Morgans' RiskMetrics system, which measured the firm's risk daily.

The report gave recommendations that portfolios be marked-to-market daily and that risk be assessed with both VaR and stress testing.

While the G-30 Report focused on derivatives, most of its recommendations were applicable to the risks associated with other traded instruments.

For this reason, the report largely came to define the new risk management of the 1990s and set the an industry-wide standard.

A brief history of VaR and Expected Shorfall VIII

The report is also interesting, as it may be the first published document to use the word ”value-at-risk”.

Expected shortfall (ES) is a seemingly more recent risk measure, however, Rappoport (1993) [10] mentions a new approach called Average Shortfall in J.P. Morgan’s Fixed Income Research Technical Document, which first noted application of the theory of Expected Shortfall in finance.

The later paper of Artzner et al. (1999)[11] introduces four properties for measures of risk and calls the measures satisfying these properties as ’coherent’.

While such ”coherent” risk measures become ill defined in presence of liquidity risk (especially the proportionality assumption), this was the catalyst for the need of a new ’coherent’ risk measure.

A brief history of VaR and Expected Shorfall IX

As ES was practically the only operationally manageable coherent risk measure, ES was proposed as a coherent alternative to VaR.

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[6] G. A. HOLTON (2002), working paper. *History of Value-at-Risk: 1922-1998*.

[7] Pressacco, F., and Ziani, L. (2010). Bruno de Finetti forerunner of modern finance. In: Convegno di studi su Economia e Incertezza, Trieste, 23 ottobre 2009, Trieste, EUT Edizioni Universit di Trieste, 2010, pp. 65-84.

[8] P. JORION *Value at Risk: The New Benchmark for Managing Financial Risk 3rd ed.* McGraw-Hill.

[9] A. MCNEIL, R. FREY AND P. EMBRECHTS (2005), *Quantitative Risk Management*, Princeton University Press.

[10] P. RAPPOPORT (1993), *A New Approach: Average Shortfall*, J.P. Morgan Fixed Income Research Technical Document.

[11] P. ARTZNER, F. DELBAEN, J. EBER AND D. HEATH , *Coherent Measures of Risk*, Mathematical Finance Vol.9 No.3.

Value at Risk I

Value at risk (VaR) is a single, summary, statistical measure of possible portfolio losses. It aggregates all of the risks in a portfolio into a single number suitable for use in the boardroom, reporting to regulators, or disclosure in an annual report, and it is the most widely used risk measure in financial institutions according to McNeil et al.

In addition to this, VaR estimates not only serve as a summary statistic, but are also often used as a tool to manage and control risk with institutions changing their market exposure to maintain their VaR at a prespecified level.

The theory behind VaR is quite simplistic, actually too simplistic: VaR is defined as

the loss level that will not be exceeded with a certain confidence level over a certain period of time.

Value at Risk II

Again, this is related to the idea of downside risk, which measures the likelihood that a financial instrument or portfolio will lose value.

Downside risk can be measured by quantiles, which are the basis of the mathematics behind VaR. We now introduce a formal definition of VaR.

Value at Risk III

VaR is related to the potential loss on our portfolio, due to downside risk, over the time horizon H. Define this loss L_H as the difference between the value of the portfolio today (time 0) and in the future H.

$$L_H = \text{Portfolio}_0 - \text{Portfolio}_H.$$

Consistently with earlier notation, we may call $\Pi(t, T)$ the sum of all future cash flows in $[t, T]$, discounted back at t, for our portfolio. These are random cash flows and not yet prices. Price of the portfolio at t is

$$\text{Portfolio}_t = \mathbb{E}_t^{\mathbb{Q}}[\Pi(t, T)].$$

T is usually the final maturity of the portfolio, and typically $H << T$.

Value at Risk IV

For example, if the portfolio is just an interest rate swap where we pay fixed K and receive LIBOR L with tenor $T_\alpha, T_{\alpha+1}, \ldots, T_\beta$, then the payout is written, as we have seen earlier, for $t \leq T_\alpha$, as

$$\Pi(t, T_\beta) = \sum_{i=\alpha+1}^{\beta} D(t, T_i)(T_i - T_{i-1})(L(T_{i-1}, T_i) - K).$$

Value at Risk V

$\text{VaR}_{H,\alpha}$ with horizon H and confidence level α is defined as that number such that

$$\mathbb{P}[L_H < \text{VaR}_{H,\alpha}] = \alpha$$

or,

$$\boxed{\mathbb{P}[\mathbb{E}_0^{\mathbb{Q}}[\Pi(0,T)] - \mathbb{E}_H^{\mathbb{Q}}[\Pi(H,T)] < \text{VaR}_{H,\alpha}] = \alpha}$$

so that our loss at time H is smaller than $\text{VaR}_{H,\alpha}$ with $\mathbb{P}$-probability α.

In other terms, it is that level of loss over a time T that we will not exceed with $\mathbb{P}$-probability α. It is the α $\mathbb{P}$-percentile of the loss distribution over T.

From this last equation, notice the interplay of the two probability measures.

Value at Risk VI

From the dialogue by Brigo (2011). ”Counterparty Risk FAQ: Credit VaR, PFE, CVA, DVA, Closeout, Netting, Collateral, Re-hypothecation, WWR, Basel, Funding, CCDS and Margin Lending”. See also the forthcoming book by Brigo, Morini and Pallavicini: ”Credit, Collateral and Funding”, Wiley, March 2013.

A: VaR is calculated through a simulation of the basic financial variables underlying the portfolio under the historical probability measure, commonly referred as $\mathbb{P}$, up to the risk horizon H. At the risk horizon, the portfolio is priced in every simulated scenario of the basic financial variables, including defaults, obtaining a number of scenarios for the portfolio value at the risk horizon.

Q: So if the risk horizon H is one year, we obtain a number of scenarios for what will be the value of the portfolio in one year, based on the eveolution of the underlying market variables and on the possible default of the counterparties.

Value at Risk VII

A: Precisely. A distribution of the losses of the portfolio is built based on these scenarios of portfolio values. When we say "priced" we mean to say that the discounted future cash flows of the portfolio after the risk horizon are averaged conditional on each scenario at the risk horizon but under another probability measure, the Pricing measure, or Risk Neutral measure, or Equivalent Martingale Measure if you want to go technical, commonly referred as $\mathbb{Q}$.

Q: Not so clear... [Looks confused]

A: [Sighing] All right, suppose your portfolio has a call option on equity, traded with a Corporate client, with a final maturity of two years. Suppose for simplicity there is no interest rate risk, so discounting is deterministic. To get the Var, roughly, you simulate the underlying equity under the P measure up to one year, and obtain a number of scenarios for the underlying equity in one year.

Value at Risk VIII

Q: Ok. We simulate under P because we want the risk statistics of the portfolio in the real world, under the physical probability measure, and not under the so called pricing measure Q.

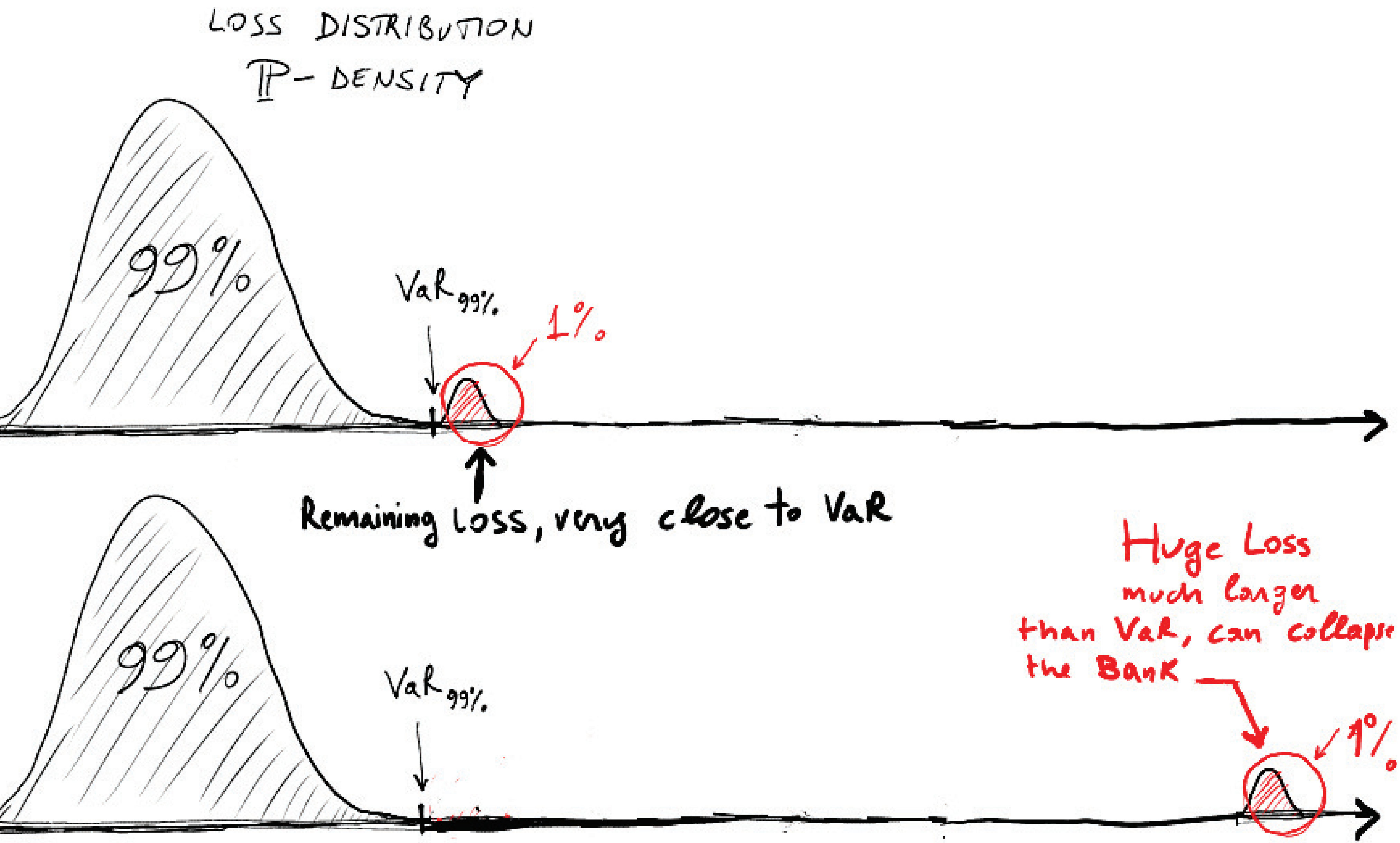
A: That's right. And then in each scenario at one year, we price the call option over the remaining year using for example a Black Scholes formula. But this price is like taking the expected value of the call option payoff in two years, conditional on each scenario for the underlying equity in one year. Because this is pricing, this expected value will be taken under the pricing measure Q, not P. This gives the Black Scholes formula if the underlying equity follows a geometric brownian motion under Q.

VaR drawbacks and Expected Shortfall I

As we explained in the introduction to risk measures, VaR has a number of drawbacks. We list two of them now, starting from the most relevant.

VaR drawback 1: VaR does not take into account the tail structure beyond the percentile.

Consider the following two cases.


LOSS DISTRIBUTION
P- DENSITY
99%
VaR 99%
1%
Remaining Loss, very close to VaR
99%
VaR 99%
Huge Loss
much larger
than VaR, can collapse
the Bank
1%

VaR drawbacks and Expected Shortfall I

From the picture above we see that we may have two situations where the VaR is the same but where the risks in the tail are dramatically different.

In the first case, the VaR singles out a 99% percentile, after which a slightly larger loss follows with 1% probability mass. The bank may be happy to know the 99% percentile in this case and to base its risk decision on that.

In the second case, the VaR singles out the same 99% percentile, after which an enormously much larger loss concentration follows with probability 1%. For example, this is now so large to easily collapse the bank. Would the bank be happy to ignore this potential huge and devastating loss, even if it has a small 1% probability?

VaR drawbacks and Expected Shortfall II

Probably not, and in this second case the bank would not base its risk analysis on VaR at 99%.

The VaR at 99% does not capture this difference in the two distributions, and if the bank does not explore the tail structure, it cannot know the real situation.

The most dangerous situation is the bank computing VaR and thinking it is in the first situation when it is actually in the second one.

VaR drawbacks and Expected Shortfall III

VaR drawback 2: VaR is not sub-additive on portfolios.
Suppose we have two portfolios P_1 and P_2, and a third portfolio $P = P_1 + P_2$ that is given by the two earlier portfolios together.
VaR at a given confidence level and horizon would be sub-additive if

$$\mathrm{VaR}(P_1 + P_2) \leq \mathrm{VaR}(P_1) + \mathrm{VaR}(P_2) \quad \text{(VaR subadditivity. Is it true?)}$$

ie the risk of the total portfolio is smaller than the sum of the risks of its sub-portfolios (benefits of diversification, among other things).

However, *this is not true*. It may happen that

$$\mathrm{VaR}(P_1 + P_2) > \mathrm{VaR}(P_1) + \mathrm{VaR}(P_2) \quad \text{in some cases.}$$

While such cases are usually difficult to see in practice, it is worth keeping this in mind.

VaR drawbacks and Expected Shortfall IV

As a remedy to this sub-additivity problem (and only partly to the first drawback) Expected Shortfall (ES) has been introduced.

ES requires to compute VaR first, and then takes the expected value on the TAIL of the loss distribution for values larger than VaR, conditional on the loss being larger than Value at Risk.

ES is sub-additive (solves drawback 2).

ES looks at the tail after VaR, but only in expectation, without analyzing the tail structure carefully. Hence, it is only a partial solution to drawback 1.

VaR drawbacks and Expected Shortfall V

Recalling that we defined the loss L_H as the difference between the value of the portfolio today (time 0) and in the future H.

$$L_H = \text{Portfolio}_0 - \text{Portfolio}_H,$$

ES for this portfolio at a confidence level α and a risk horizon H is

$$\text{ES}_{H,\alpha} = \mathbb{E}^{\mathbb{P}}[L_H | L_H > \text{VaR}_{H,\alpha}]$$

By definition, ES is always larger than the corresponding VaR.

Be aware of the fact that ES has several other names, and there are other risk measures that are defined very similarly. Names you may hear are:
Conditional value at risk (CVaR), average value at risk (AVaR), and expected tail loss (ETL).

Value at Risk and ES: An example I

- PORTFOLIO: ZERO COUPON BOND and CALL OPTION ON A STOCK.
- FIRST ASSET of the PORTFOLIO: Zero coupon bond with Maturity $T = 2$ years and notional $N_B = 1000$.
- RISK FACTOR r: The BOND value is driven by interest rates. We choose a short rate model for r_t for this risk factor.
- Short term interest rate under the physical measure $\mathbb{P}$:

$$dr_t = k_r(\bar{\theta} - r_t)dt + \sigma_r d\bar{Z}_t,$$

$$r_0 = 0.01, k_r = 0.1, \bar{\theta} = 0.1, \sigma_r = 0.004.$$

Value at Risk and ES: An example II

- Short term interest rate under the pricing measure $\mathbb{Q}$:

$$dr_t = k_r(\theta - r_t)dt + \sigma_r dZ_t,$$

$$\theta = 0.05.$$

- We need both the $\mathbb{P}$ and $\mathbb{Q}$ dynamics of the risk factor: We will simulate r up to the risk horizon H using the $\mathbb{P}$ dynamics, hence $\bar{\theta}$. Then, to compute the price of the bond and equity call option at the different scenarios at time H we will use the $\mathbb{Q}$ dynamics, hence θ.
- SECOND ASSET of the PORTFOLIO: Call option on equity S with strike $K_c = 100$. Call option maturity: 2 years.
- RISK FACTOR S: The EQUITY CALL OPTION is driven by equity stock S_t. We choose a Black Scholes type model for the stock price S (but careful about the $\mathbb{Q}$ drift...)

Value at Risk and ES: An example III

- Equity under the physical measure: $dS_t = \mu S_t dt + \sigma_S S_t d\bar{W}_t$, $S_0 = 100$, $\mu = 0.09$, $\sigma_S = 0.2$.
- Equity under the pricing measure: $dS_t = r_t S_t dt + \sigma_S S_t dW_t$, with r_t the short term stochastic process given above. *Here the drift r_t is imposed by no-arbitrage!*
- We need both the $\mathbb{P}$ and $\mathbb{Q}$ dynamics of the risk factor: We will simulate S up to the risk horizon H using the $\mathbb{P}$ dynamics, hence μ. Then, to compute the price of the call option at the different scenarios at time H we will use the $\mathbb{Q}$ dynamics, hence r, where the drift in dr is θ.
- So

$$\Pi(t,T) = D(t,2y)N_B + D(t,2y)(S_{2y} - K_c)^+ =$$

$$= \exp\left(-\int_t^{2y} r_s ds\right) N_B + \exp\left(-\int_t^{2y} r_s\, ds\right)(S_{2y} - K_c)^+$$

Value at Risk and ES: An example IV

- TYPE OF RISK MEASURE
- VaR holding period: $H = 1y$.
- Confidence level: 99%
- ES holding period: $H = 1y$.
- Confidence level: 99%

Value at Risk and ES: An example V

- So, our loss is: $L_{1y} = \text{Portfolio}_0 - \text{Portfolio}_{1y}$, or

$$L_{1y} = \mathbb{E}_0^{\mathbb{Q}}\left[\exp\left(-\int_0^{2y} r_s ds\right) N_B + \exp\left(-\int_0^{2y} r_s\, ds\right)(S_{2y} - K_c)^+\right]$$

$$-\mathbb{E}_{1y}^{\mathbb{Q}}\left[\exp\left(-\int_{1y}^{2y} r_s ds\right) N_B + \exp\left(-\int_{1y}^{2y} r_s\, ds\right)(S_{2y} - K_c)^+\right]$$

- IMPORTANT: Notice that the risk factor r_t appears also in the DRIFT (local mean) of S under $\mathbb{Q}$, so that S and r need to be simulated consistently and jointly.

Value at Risk and ES: An example VI

- Correlation

$$\text{corr}(dr, dS) = \rho \qquad (dZ_t dW_t = \rho dt),$$

we try three cases:

- $\rho = 0$
- $\rho = -1$
- $\rho = 1$

Value at Risk and ES: An example VII

Results: One million paths in R

- $\rho = -1$: $\quad VaR = 13.07 \quad ES = 14.38$
- $\rho = 0$: $\quad VaR = 9.34 \quad ES = 10.85$
- $\rho = +1$: $\quad VaR = -0.99 \quad ES = -0.93$

Let's look at the three cases in detail

Value at Risk and ES: An example VIII

$$\rho = -1 : \quad VaR = 13.07 \quad ES = 14.38$$

Here there is totally negative correlation.

Remember that when interest rates go up in 1y, bonds go down: if r increases the zero coupon bond P decreases. This is also confirmed by the formula for the Vasicek zero coupon bond price $P(t,T) = A(t,T)\exp(-B(t,T)r_t)$ (recall $A > 0$ and $B > 0$): This is a decreasing function of r since it is an exponential with a negative exponent.

Totally negative correlation between r and S means that when r goes up (same as P goes down) S goes down and viceversa. Then we can write

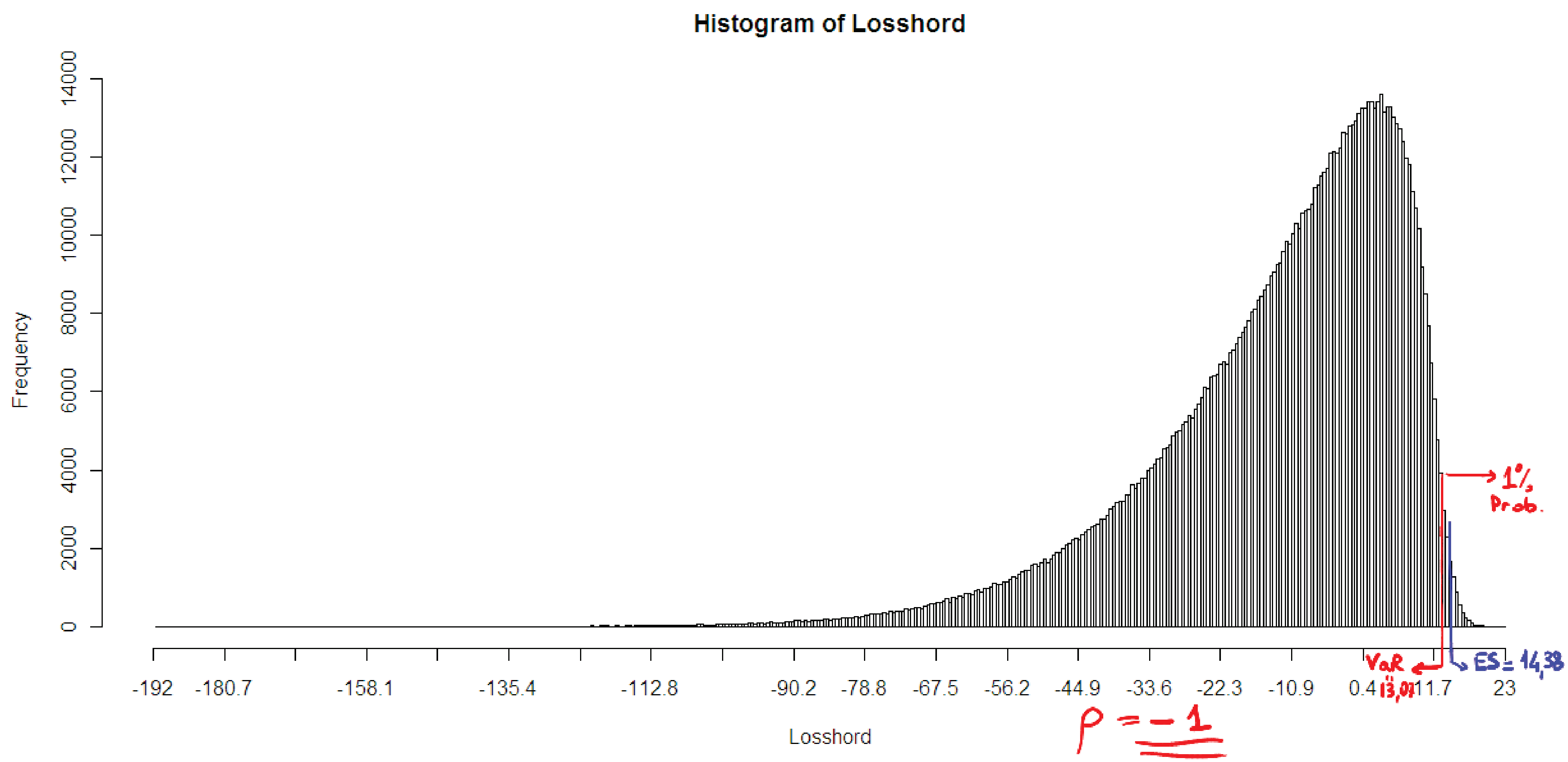
$$\boxed{r\uparrow \text{(equivalently } P\downarrow) \Rightarrow S\downarrow} \text{ and } \boxed{r\downarrow \text{(equivalently } P\uparrow) \Rightarrow S\uparrow}.$$

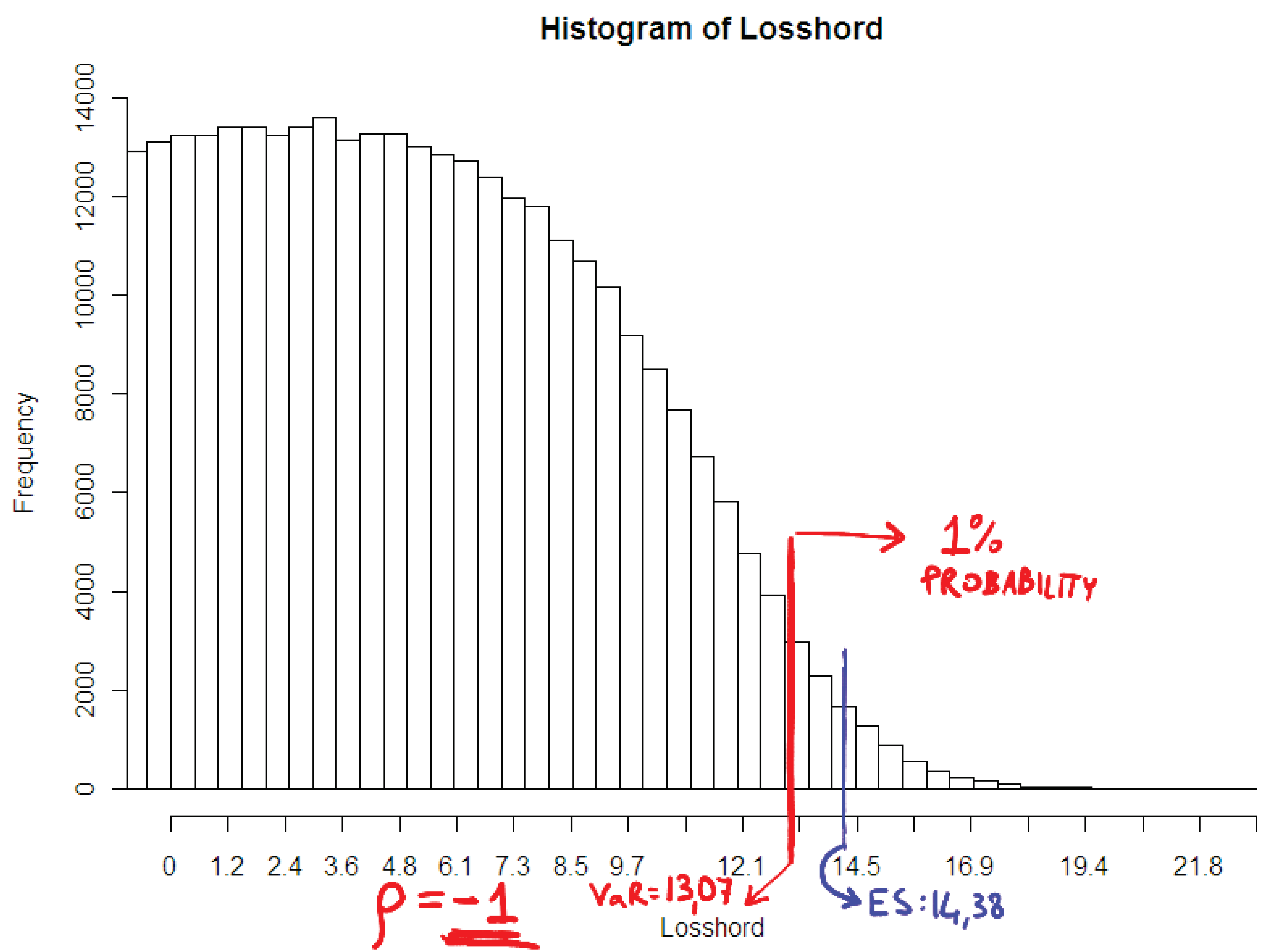
Value at Risk and ES: An example IX

$\boxed{r \uparrow \text{(equivalently } P \downarrow) \Rightarrow S \downarrow}$ and $\boxed{r \downarrow \text{(equivalently } P \uparrow) \Rightarrow S \uparrow}$.

Hence, in our portfolio, when the bond goes down a lot in one year, leading to an important loss, the stock goes into the same way due to the correlation and there is an important loss also in the equity call option, which is monotonically increasing in S. Same for the gain, in the other direction. Hence the correlation links losses (gains respectively) from the bond to losses (gains) from the stock and the effects compound by happening, statistically, in the same scenarios.

Then *the loss distribution will be more spread out and the percentiles will be larger*, as shown by our VaR, which is the largest in this case. We show the situation in a couple of plots:


Histogram of Losshord
Frequency
0
2000
4000
6000
8000
10000
12000
14000
-192
-180.7
-158.1
-135.4
-112.8
-90.2
-78.8
-67.5
-56.2
-44.9
-33.6
-22.3
-10.9
0.4
11.7
23
Losshord
1% Prob.
VaR 13,01
ES = 14,38
ρ = −1

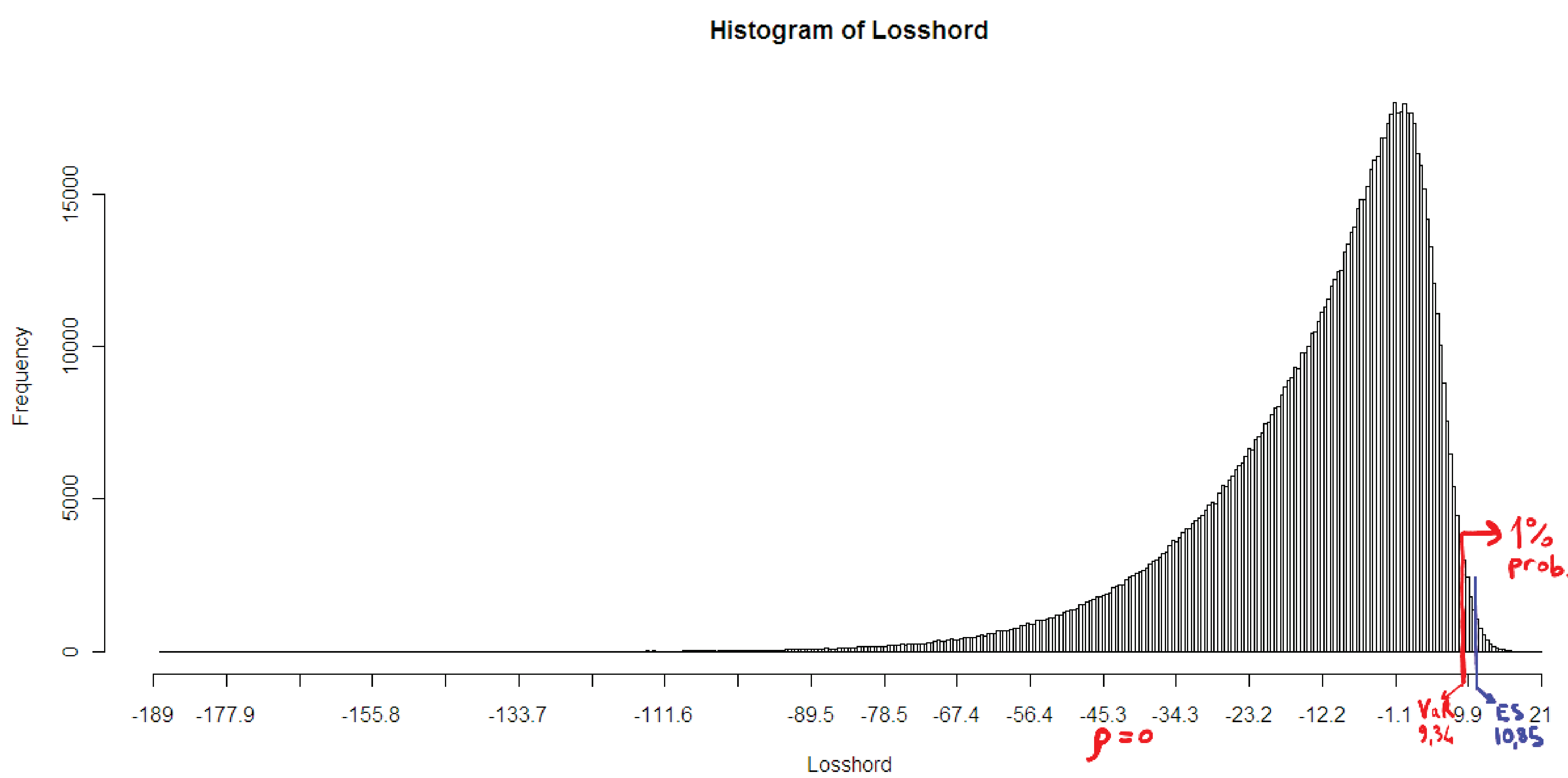

Histogram of Losshord
Frequency
0
2000
4000
6000
8000
10000
12000
14000
0
1.2
2.4
3.6
4.8
6.1
7.3
8.5
9.7
12.1
14.5
16.9
19.4
21.8
Losshord
1%
PROBABILITY
VaR=13,07
ES:14,38
ρ=-1

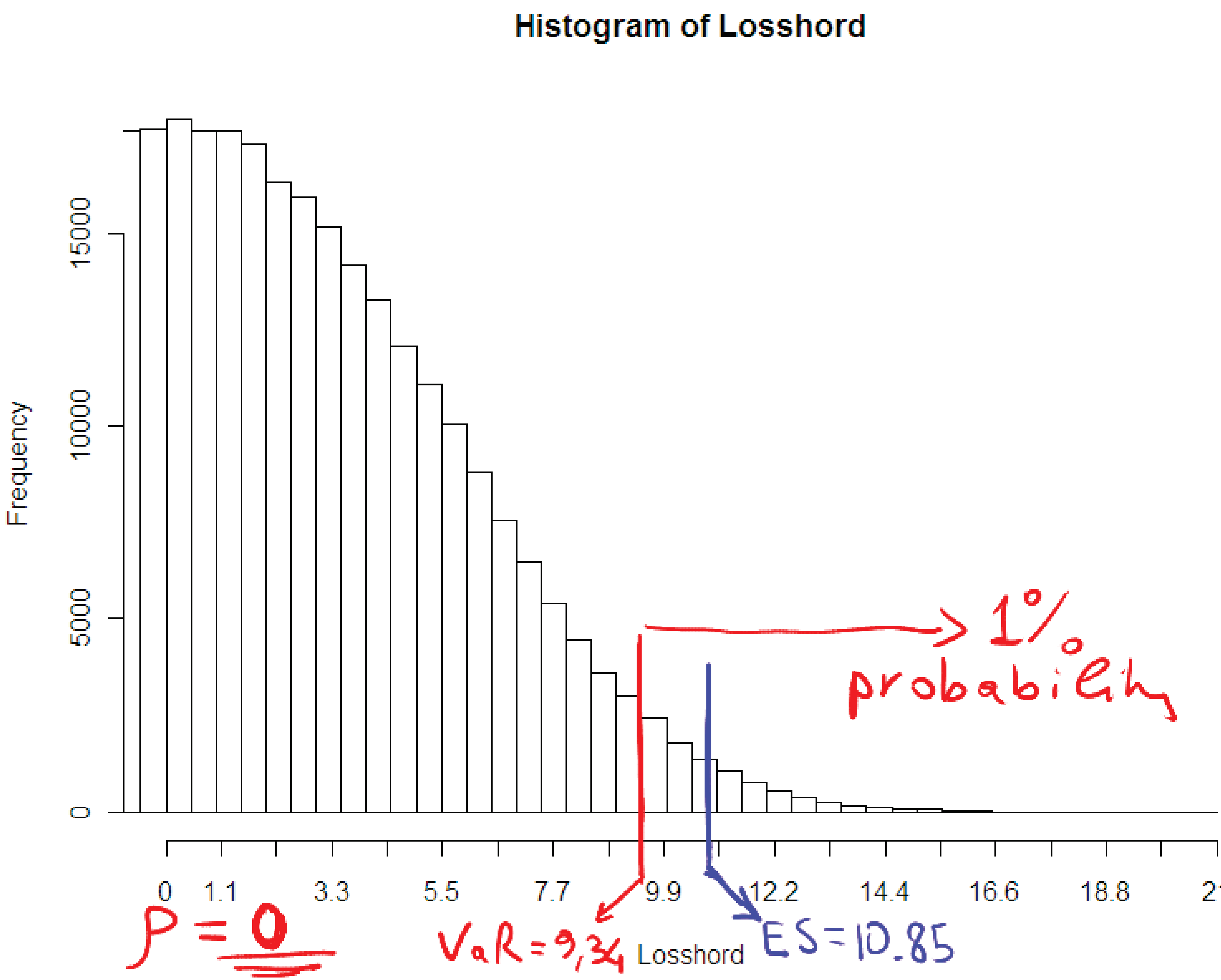
Value at Risk: An example I

Let's look at the second case:

$$\rho = 0 : \quad VaR = 9.34 \quad ES = 10.85$$

Here there is no correlation (and since the shocks are jointly gaussian, this means independence). Hence there is no link between the direction of P and the direction of S. As a consequence losses are less extreme than in the previous case because bad scenarios in r and S happen independently and don't combine.


Histogram of Losshord
Frequency
0
5000
10000
15000
-189
-177.9
-155.8
-133.7
-111.6
-89.5
-78.5
-67.4
-56.4
-45.3
-34.3
-23.2
-12.2
-1.1
9.9
21
Losshord
1% prob.
VaR 9,34
ES 10,85
ρ=0


Histogram of Losshord
Frequency
0
5000
10000
15000
0
1.1
3.3
5.5
7.7
9.9
12.2
14.4
16.6
18.8
21
Losshord
1% probability
P = 0
VaR = 9,34
ES = 10.85

Value at Risk: An example I

We may see that now that extreme losses or gains do not happen together, the distribution is less spread out. Compare to the plots for the case $\rho = -1$ and this is clear. The tail stops earlier and so does the VaR percentile. VaR is smaller here.

Value at Risk: An example I

Let's look at the third case:

$$\rho = 1: \quad VaR = -0.99 \quad ES = -0.93$$

Here there is total positive correlation. This means that when r goes up (equivalently P goes down), leading to a loss in the Bond portfolio, S goes up too, leading to a gain in the Call option.

In the opposite case, when S goes down, leading to a loss in the equity call option, then r goes down (equivalently P goes up) and we have a gain in the bond portfolio.

It is then obvious that we have here the less risky situation: when we lose on one of the two assets we gain from the other one, so that our losses will be always reduced compared to the other two cases. Indeed we may see that from the plot.

Value at Risk: An example II

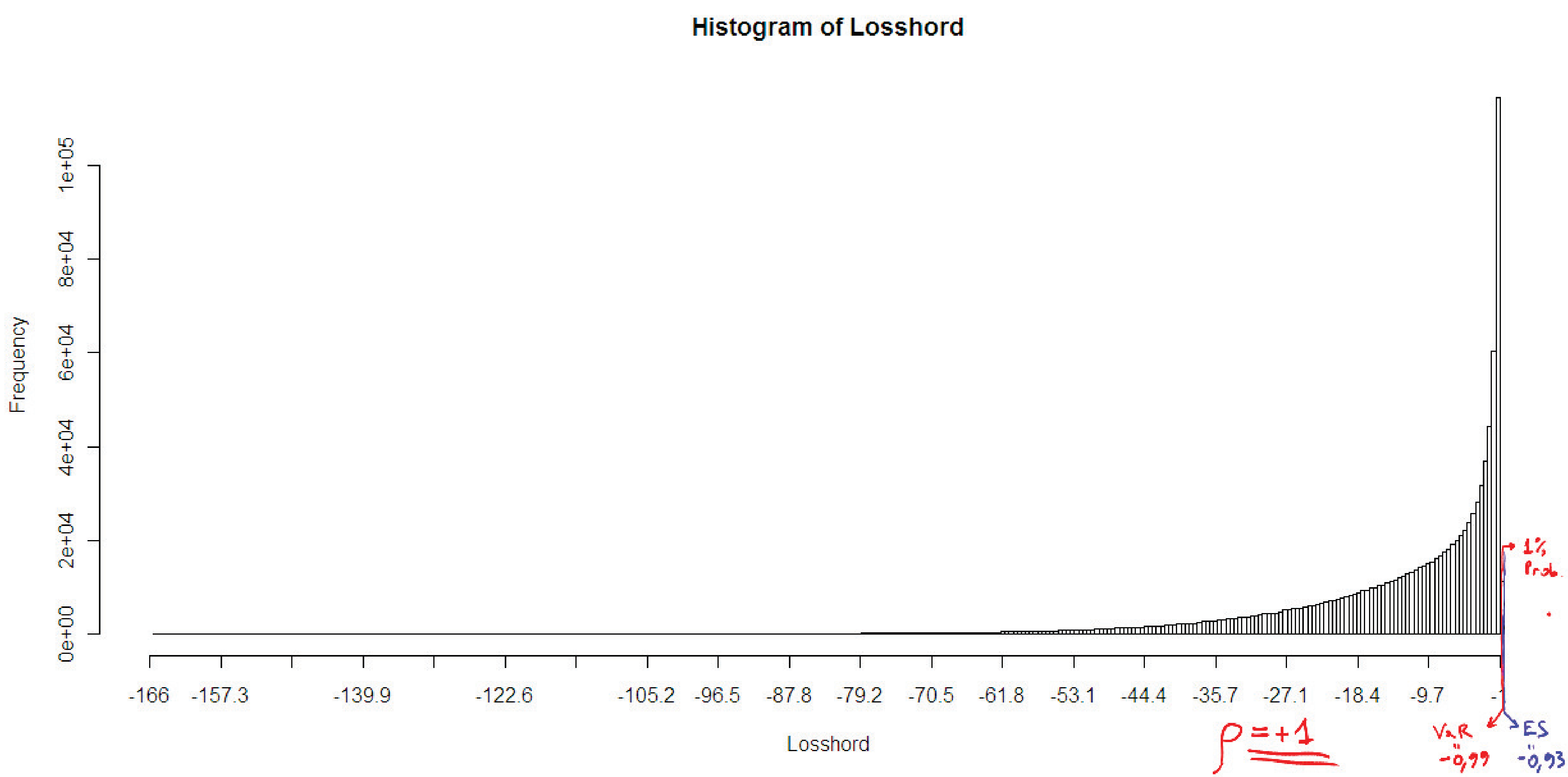
This holds to the extent that actually VaR and ES are negative and near zero, meaning that the worst we risk - according to these measures - is to make a small gain instead of a big one. But no actual losses.

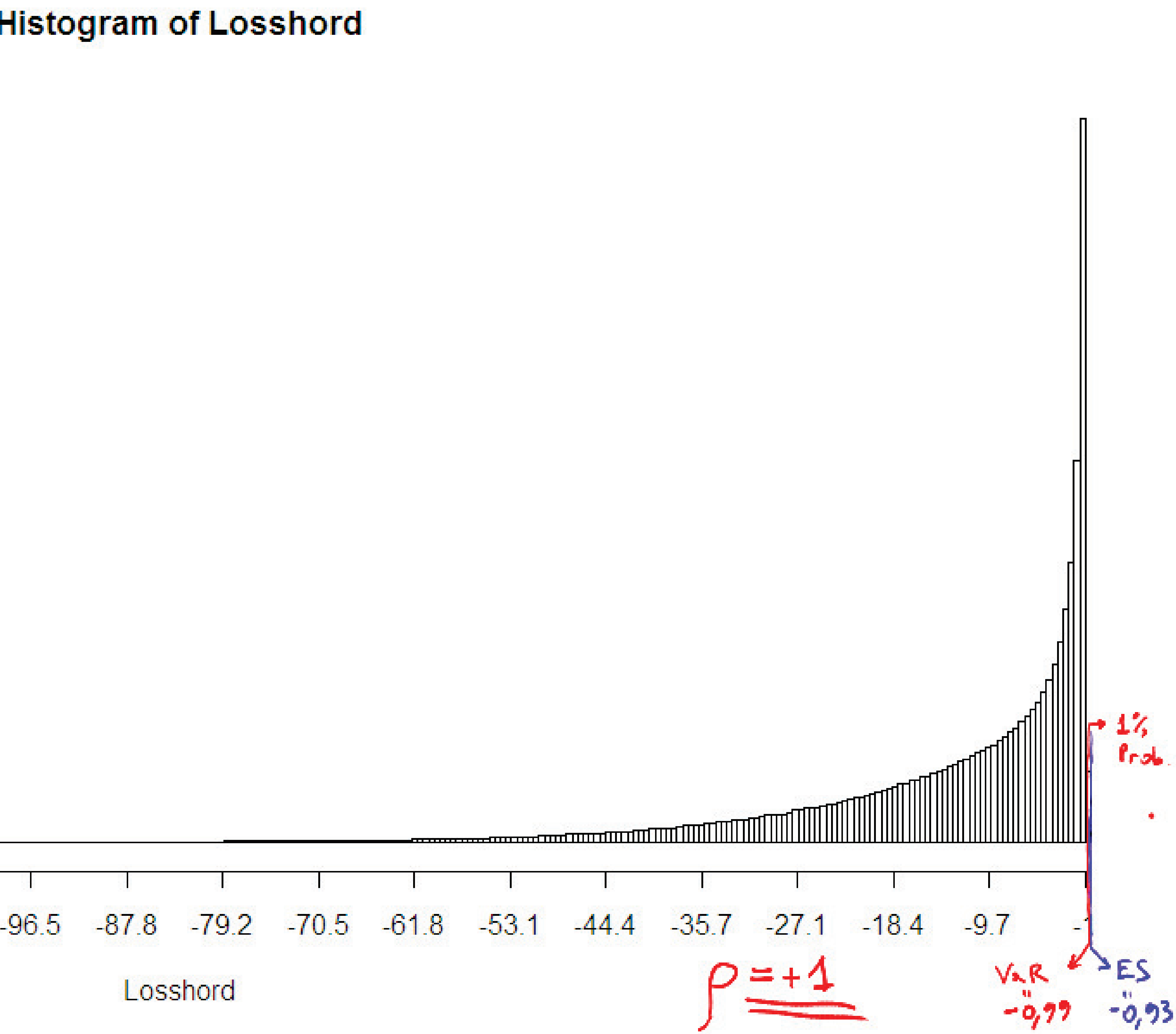
With a similar notation as before, we may now write

$$\boxed{r \uparrow \text{(equivalently } P \downarrow) \Rightarrow S \uparrow} \text{ and } \boxed{r \downarrow \text{(equivalently } P \uparrow) \Rightarrow S \downarrow}$$

showing clearly that P and S move into opposite directions.

Since the two assets move into opposite directions, the portfolio values will be much more concentrated near zero than in the previous cases. This is confirmed by the plot.


Histogram of Losshord
Frequency
0e+00
2e+04
4e+04
6e+04
8e+04
1e+05
-166
-157.3
-139.9
-122.6
-105.2
-96.5
-87.8
-79.2
-70.5
-61.8
-53.1
-44.4
-35.7
-27.1
-18.4
-9.7
Losshord
1% Prob.
$\rho = +1$
VaR = -0,99
ES = -0,93


Histogram of Losshord
-96.5
-87.8
-79.2
-70.5
-61.8
-53.1
-44.4
-35.7
-27.1
-18.4
-9.7
-1
Losshord
1% Prob.
VaR = -0,99
ES = -0,93
ρ = +1

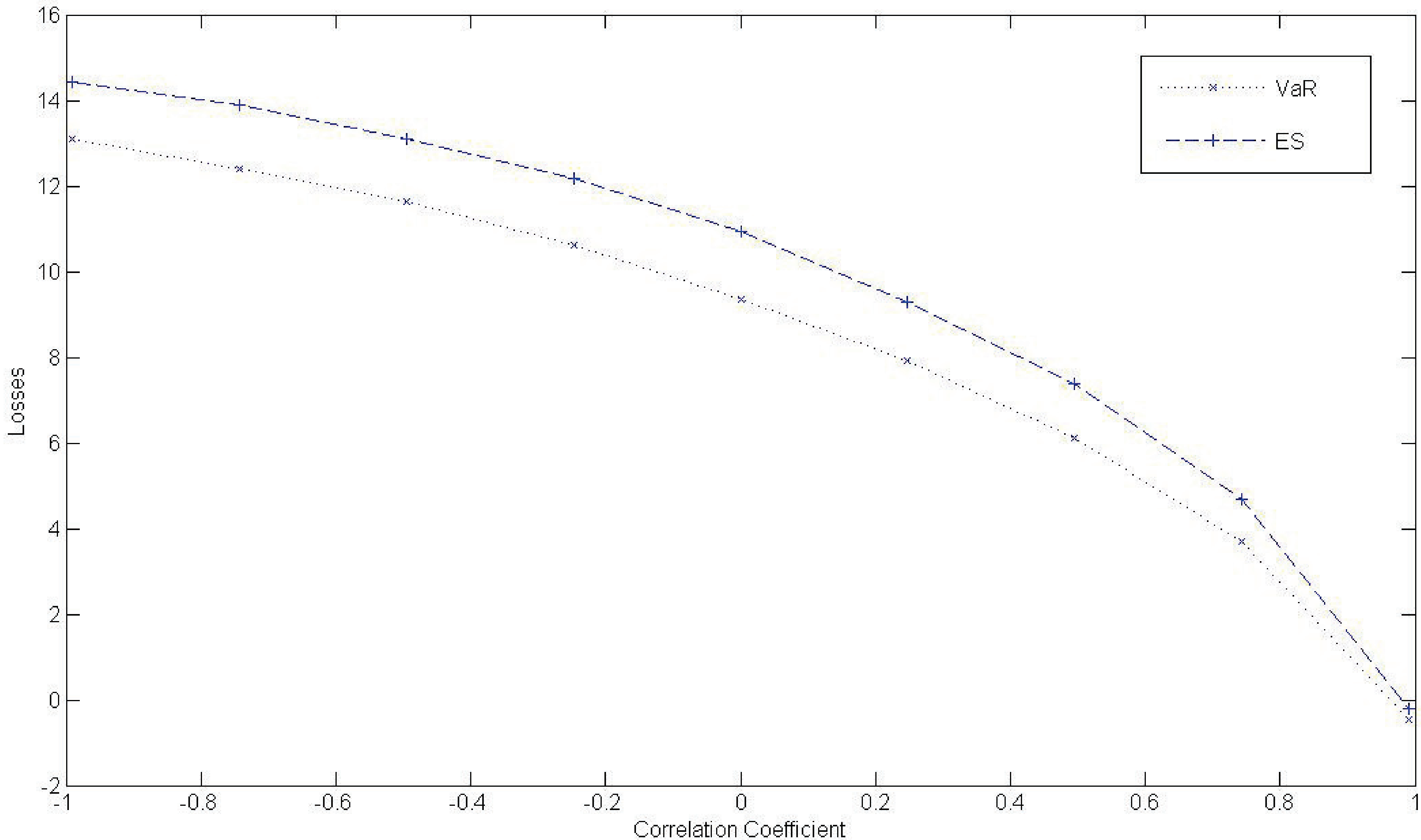
Value at Risk: An example I

So we have seen

- $\rho = -1: \quad VaR = 13.07 \quad ES = 14.38$
- $\rho = 0: \quad VaR = 9.34 \quad ES = 10.85$
- $\rho = +1: \quad VaR = -0.99 \quad ES = -0.93$

In this case the correlation between the two risk factors, interest rates and equity, r and S, plays a crucial role.
The next plot shows the impact of all possible values of correlation, ie correlation sensitivity for VaR and ES

Correlation sensitivity VaR and ES



Value at Risk: An example I

More generally, volatilities, correlation, dynamics and statistical dependencies have a very important impact on risk.

For very large portfolios it is difficult to obtain intuition on why some risk patterns are observed, as there are too many assets and parameters.

A rigorous quantitative analysis of risks is fundamental to have a safe result. However, the assumptions underlying the analysis need to be kept in mind and stress-tested

VaR type measures have also been applied to credit risk, leading to the Credit VaR measure we briefly discussed in the comparison with CVA in the credit part of this course.

Value at Risk: An example II

VaR of CVA itself is now one of the topical areas in the industry. This is not Credit VaR, but the VaR coming from the possible loss due to future adverse movements of CVA over a given risk horizon. Basel III is quite concerned with this.

Current research is focused on extending risk measures to properly include liquidity risk, see for example Brigo and Nordio ”Liquidity adjusted risk measures”.

Finale

"Essentially, all models are wrong, but some are useful".
Prof. G.E.P Box

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A final note on the exam. Not all the course material has been explained in detail in class. If in doubt on what is examinable, a good strategy is to look at the final exercise set. If a part of the course is not needed to solve the exercise set, then it is not examinable. However, in the final slide I provide a precise list.

Thank you for your attention.

Exam

For the exam, the exercise set should be studied very carefully.

The following slides of the course, in the present set, are examinable. In the following, ”RO” stands for ”Read Once”, and it is meant to be helpful for your general culture, job interviews etc, but not necessary for the exam.

RO 1-93; 94-203;
204-228; RO 229-250; 251-300; RO 301-306; 307-335;
342-361; RO 362-375; 376-404; 407- 436;
RO 437-444; 445-452; RO 453-471; 472-478; RO 479-495;
496-535; RO 536-548; 549-629;
(to be continued)