Interest Rate Models with Credit Risk, Collateral, Funding Liquidity Risk and Multiple Curves (M5MF30)

MSc Mathematics and Finance, IC, London, 2013-2014

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Office Hours During Term: Wednesdays 2.30pm-4.30pm

Imperial College London

Content I

Content II

Content III

Content IV

Content V

- Intensity models: Deterministic Intensity
- Intensity models: Stochastic Intensity

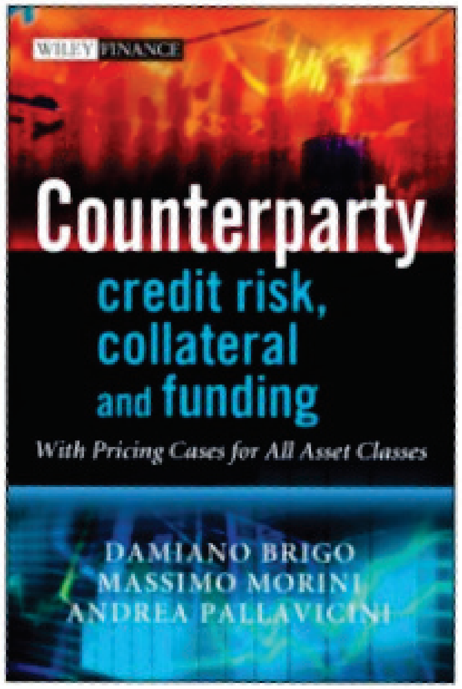
13. Intro to Counterparty Risk: Q & A
    - Counterparty Credit Risk
    - Credit VaR and CVA
    - CVA, Model Risk and WWR
    - Collateral
    - Netting
    - Credit and Debit Valuation Adjustments (CVA-DVA)

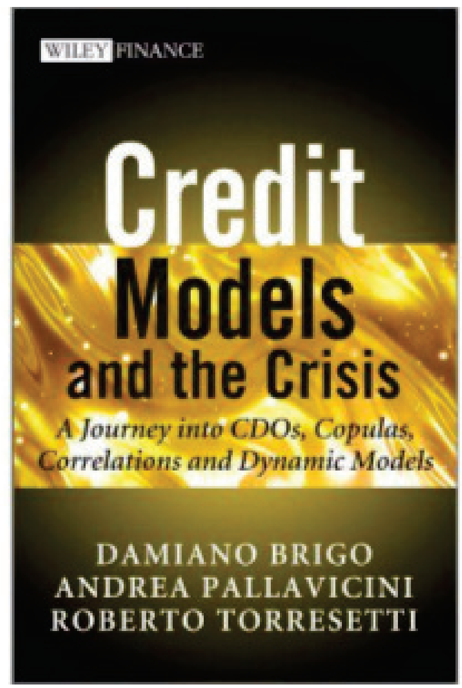
14. The mechanics of counterparty risk
    - General formula, Symmetry vs Asymmetry
    - Unilateral Credit Valuation Adjustment (UCVA)
    - Unilateral Debit Valuation Adjustment (UDVA)
    - Bilateral Risk and DVA
    - DVA Hedging?

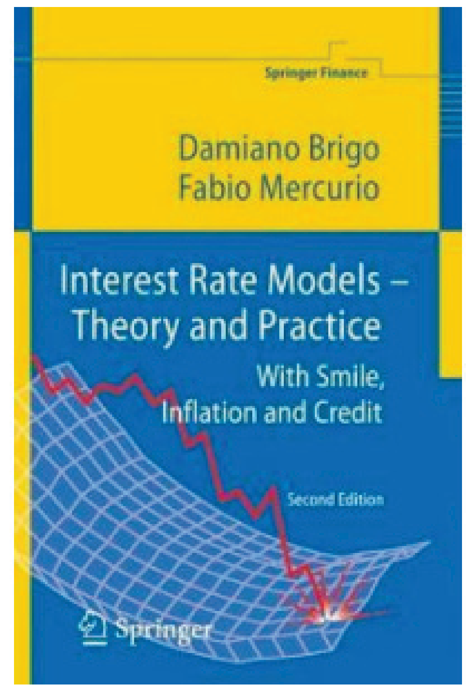
Content VI

Content VII

This course is mostly based on the books:







especially the first one (most recent) and the last one (least recent).

PART 0. OPTION PRICING AND ITS SIGNIFICANCE

In this introductory part we introduce the Black Scholes and Merton result, their precursors (Bachelier, DeFinetti...) and the refinements of their theory (Harrison, Kreps, Pliska....), pointing out its significance, successes and failures.

We also look at the derivatives markets and their significance

The Black Scholes and Merton Analysis

- Portfolio replication theory plus Ito's formula to derive the Black and Scholes PDE under certain assumptions on the dynamics of the stock price.
- The Feynman-Kac theorem to interpret the solution of the Black and Scholes PDE as an expected value of a function of the stock price with different dynamics.
- The Girsanov theorem to interpret the different dynamics of the stock price as a dynamics under a different (martingale) probability measure.

Description of the economy

We consider:

- A probability space with a r.c. filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t : 0 \leq t \leq T), P)$.
- In the given economy, two securities are traded continuously from time 0 until time T. The first one (a bond) is riskless and its (deterministic) price B_t evolves according to

$$dB_t = B_t r dt, \quad B_0 = 1, \tag{1}$$

which is equivalent to

$$B_t = e^{rt}, \tag{2}$$

where r is a nonnegative number. To state it differently, the short term interest rate is assumed to be constant and equal to r through time.

Description of the economy

- As for the second one, given the $(\mathcal{F}_t, P)$-Wiener process W_t, consider the following stochastic differential equation

$$dS_t = S_t[\mu dt + \sigma dW_t], \quad 0 \le t \le T, \tag{3}$$

with initial condition $S_0 > 0$, and where μ and σ are positive constants. Equation (49) has a unique (strong) solution which is given by

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\}, \quad 0 \le t \le T. \tag{4}$$

The risky asset, The B e S Assumptions, and Contingent Claims

$$dB_t = B_t r dt, \quad B_0 = 1,$$

$$dS_t = S_t[\mu dt + \sigma dW_t], \quad 0 \le t \le T,$$

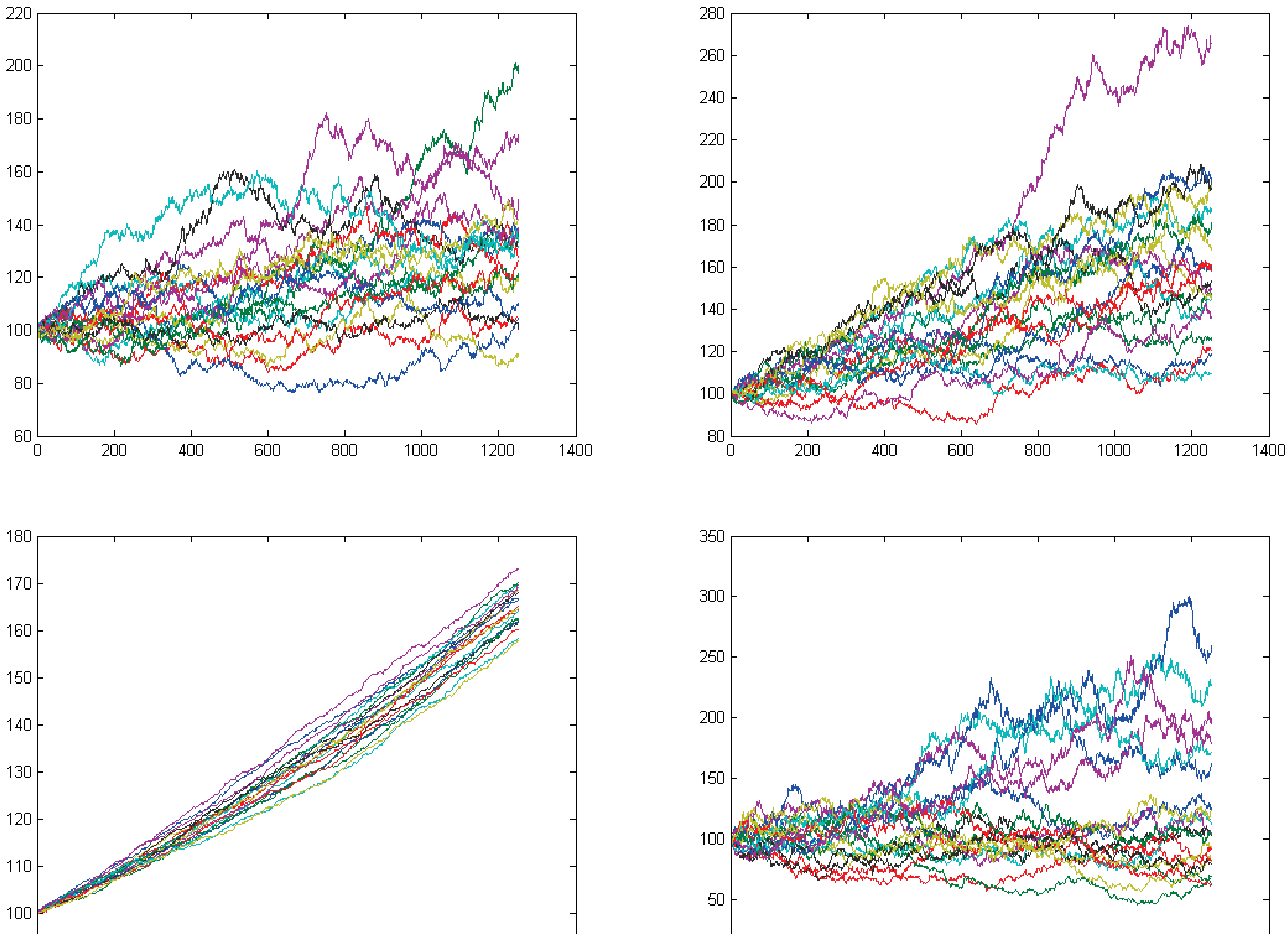
The second asset (a stock) is risky and its price is described by the process S_t. Furthermore, it is assumed that

- (i) there are no transaction costs in trading the stock;
- (ii) the stock pays no dividends or other distributions;
- (iii) shares are infinitely divisible;
- (iv) short selling is allowed without any restriction or penalty.

We refer to these assumptions as to Black and Scholes' *ideal conditions*.

Example of risky asset dynamics over 5 years:

$S_0 = 100, \quad (\mu, \sigma) = (5\%, 10\%), (10, 10), (10, 1), (1, 20)$



Contingent claim, Pricing problem, Complete Market

A **contingent claim** Y for the maturity T is any random variable which is $\mathcal{F}_T$–measurable.

We limit ourselves to *simple* contingent claims, i.e. claims of the form $Y = f(S_T)$.

The idea behind a claim is that it represents an amount that will be paid at maturity to the holder of the contract.

The **Pricing Problem** is giving a fair price to such a contract.

Loosely speaking, the market is said to be **complete** if every contingent claim has a price.

Trading strategies, Value process, gain process, self–financing

A **trading strategy** $\phi = (\phi^B, \phi^S)$ is a pair of functions $\mathcal{F}$–adapted. The pair (ϕ^B_t, ϕ^S_t) represents respectively amounts of bond and stock to be held at time t.

The **value process** is the process V describing the value of the portfolio constructed by following the strategy ϕ,

$$V_t(\phi) = \phi^B_t \, B_t + \phi^S_t \, S_t \ .$$

The **gain process** is defined as

$$G_t(\phi) = \int_0^t \phi^B_u \, dB_u + \int_0^t \phi^S_u \, dS_u \ .$$

and represents the income one obtains thanks to price movements in bond and stock when following the trading strategy ϕ.

Trading strategies, Value process, gain process, self–financing

The strategy is said to be *self–financing* if

$$\phi_t^B\ B_t + \phi_t^S\ S_t\ - (\phi_0^B\ B_0 + \phi_0^S\ S_0)\ = G_t(\phi)\ ,$$

or, in differential terms, $d\ V_t(\phi) = d\ G_t(\phi)$, i.e.

$$d(\phi_t^B\ B_t + \phi_t^S\ S_t)\ =\ \phi_t^B\ dB_t + \phi_t^S\ dS_t\ . \tag{5}$$

Intuitively, this means that the changes in value of the portfolio described by the strategy ϕ are only due to gains/losses coming from price movements, i.e. to changes in the prices B and S, without any cash inflow and outflow.

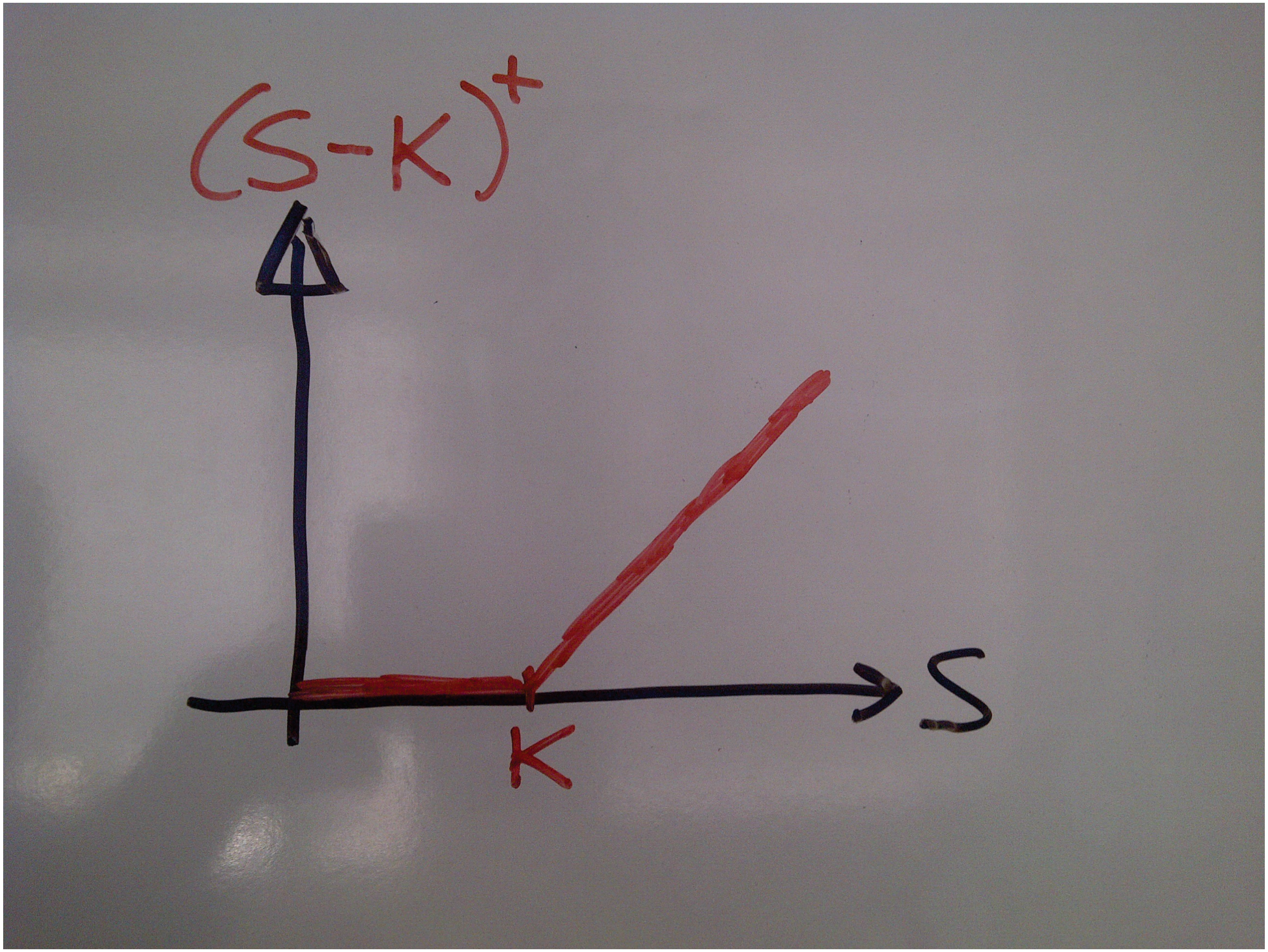
Arbitrage opportunity, arbitrage–free market

An **arbitrage opportunity** is a self–financing strategy ϕ such that

$$\phi_0^B B_0 + \phi_0^S S_0 = 0 , \quad \phi_T^B B_T + \phi_T^S S_T > 0 \text{ a.s.}$$

Basically, an arbitrage opportunity is a strategy which creates an almost surely positive cash inflow from nothing. It is sometimes called a **free lunch**.
The market is said to be **arbitrage–free** if there are no arbitrage opportunities.

Example of Claim: European Call Option



Example of Claim: European Call Option I

Suppose we have to price a simple claim $Y = f(S_T)$ at time t.

We focus on the case of a European call option: Let K be its strike price and T its maturity. The option payoff (to a long position) is represented by $Y = (S_T - K)^+ = \max(S_T - K, 0)$.

This is a contract which at maturity-time T pays nothing if the risky–asset price S_T is smaller than the strike price K, whereas it pays the difference between the two prices in the other case.

An investor who expects the risky–asset value to increase considerably can speculate by buying a call option.

Example of Claim: European Call Option II

An example of use of a call option is the following. Suppose now we are at time 0 and we plan to buy one share (unit) of a certain stock at time T. We wish to pay this stock the same price $K = S_0$ it has now, rather than the price it will have at time T, which could be much higher. What one can do in this situation is to buy a call option on the stock with maturity time T and strike price S_0.

He then buys the stock at time T paying S_T and receives $(S_T - S_0)^+$ from the option payoff. Clearly, the total amount he pays in T is then $S_T - (S_T - S_0)^+$ which equals S_T if $S_T \leq S_0$ and equals S_0 if $S_T \geq S_0$. Therefore, an European call option can be seen as a contract which locks the stock price at a desired value to be paid at maturity time T. This *locking* has of course a price, which we wish to determine.

The Black and Scholes PDE

Let $V_t = V(t, S_t)$ be the candidate claim (option) value at time t.
Assume the function $V(t, S_t)$ of time t and of the stock price S_t to have regularity $V \in C^{1,2}([0, T] \times R)$.
Apply Ito's Lemma to V so as to obtain

$$\begin{aligned} dV(t, S_t) &= \left(\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 \right) dt \\ &+ \frac{\partial V}{\partial S}(t, S_t)\sigma S_t dW_t. \end{aligned} \tag{6}$$

Set, for each $0 \leq t \leq T$,

$$\phi_t^S = \frac{\partial V}{\partial S}(t, S_t), \quad \phi_t^B = (V_t - \phi_t^S S_t)/B_t. \tag{7}$$

By construction, the value of this strategy at time t is V itself, since clearly $V(t, S_t) = \phi_t^B B_t + \phi_t^S S_t$.

The Black and Scholes PDE

Now assume ϕ to be self–financing. Since ϕ is self-financing

$$
\begin{aligned}
dV_t &= \phi_t^B dB_t + \phi_t^S dS_t \qquad (8)\\
&= \left[V(t, S_t) - \frac{\partial V}{\partial S}(t, S_t) S_t \right] r dt + \frac{\partial V}{\partial S}(t, S_t) S_t (\mu dt + \sigma dW_t).
\end{aligned}
$$

Then by equating (6) and (8) (ITO + SELF FINANCING), we obtain that V_t satisfies

$$
\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t) r S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t) \sigma^2 S_t^2 = r V(t, S_t), \qquad (9)
$$

which is the celebrated Black and Scholes partial differential equation with terminal condition $V_T = (S_T - K)^+$.

Black and Scholes' famous formula

The strategy (ϕ^B, ϕ^S) has final value equal to the claim Y we wish to price, and during its life the strategy does not involve cash inflows or outflows (self–financing condition). As a consequence, its initial value V_t at time t must be equal to the unique claim price to avoid arbitrage opportunities.
The solution of the above equation is given by

$$V_{BS}(t) = V_{BS}(t, S_t, K, T, \sigma, r) := S_t\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t)), \quad (10)$$

where

$$d_1(t) := \frac{\ln(S_t/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}},$$
$$d_2(t) := d_1(t) - \sigma\sqrt{T - t},$$

and $\Phi(\cdot)$ denotes the cumulative standard normal distribution function.

Black and Scholes' famous formula

Expression (10) is the celebrated Black and Scholes option pricing formula which provides the unique no-arbitrage price for the given European call option.

Notice that the coefficient μ does not appear in (10), indicating that investors, though having different risk preferences or predictions about the future stock price behaviour, must yet agree on this unique option price.

MORE ON THE SIGNIFICANCE OF THIS LATER.

Numerical example

Suppose the current stock value is $S_0 = 100$.
Suppose the risk free interest rate is $r = 2\% = 0.02$.
Suppose that the strike $K = 100$ (at the money option).
Assume the volatility $\sigma = 0.2 = 20\%$.
Take a maturity of $T = 5y$. CALL PRICE IS $V_{BS}(0) = 22.02$.

For example, in Matlab this is obtained through commands

```
S0=100; sig=0.2; r=0.02; K=100; T=5;
d1 = (r + 0.5*sig*sig)*T/(sig*sqrt(T));
d2 = (r - 0.5*sig*sig)*T/(sig*sqrt(T));
V0 = S0*normcdf(d1)-K*exp(-r*T)*normcdf(d2);
```

The same calculation with lower volatility $\sigma = 0.05 = 5\%$ would give

$$V_{BS}(0)|_{\sigma=0.05} = 10.5943, \quad V_{BS}(0)|_{\sigma=0.0001} = 9.52.$$

The last value is very close to the intrinsic value $S_0 - Ke^{-rT}$.

Numerical example

- Acme today is worth $S_0 = 100$.
- The more the value of acme goes up in 5 years, the more we gain as $S_{5y} - S_0$ grows. In a scenario where $S_{5y} = 200$, we gain 100.
- If however Acme goes down instead, $S_{5y} - S_0$ goes negative but the option $(S_{5y} - S_0)^+$ caps it at zero and we lose nothing. For example, in a scenario where Acme goes down to 60, we get $(60 - 100)^+ = (-40)^+ = 0$ ie we lose nothing
- With the original data, entering the gamble costs initially 22 USD out of 100 of stock notional. It is expensive. On the other hand, it is a gamble where we can only win and in principle have scenarios with unlimited profit.
- You will notice that:

$$\uparrow \sigma \Rightarrow V_{CallBS} \uparrow, \quad \uparrow S_0 \Rightarrow V_{CallBS} \uparrow, \quad \downarrow K \Rightarrow V_{CallBS} \uparrow \ldots.$$

Another numerical example

Take one more example where now the strike K is at the money forward and volatility very low, namely

```
S0=100; sig=0.0001; r=0.02; T=5; K=S0*exp(r*T);
```

Then

$$V_{BS}(0) = 0 \approx S_0 - Ke^{-rT} = S_0 - S_0 = 0.$$

Verifying the Self financing condition

Going back to the general Black Scholes result, we then prove that the strategy

$$\phi_t^S = \frac{\partial V_{BS}}{\partial S}(t, S_t), \quad \phi_t^B = (V_{BS}(t) - \phi_t^S S_t)/B_t$$

$$\Big(V_{BS}(t) = V_{BS}(t, S_t, K, T, \sigma, r) := S_t \Phi(d_1(t)) - K e^{-r(T-t)} \Phi(d_2(t)$$

is indeed self-financing. By Ito's Lemma, in fact, we have

$$dV_{BS}(t) = \frac{\partial}{\partial t} V_{BS}(t) dt + \frac{\partial}{\partial S} V_{BS}(t) dS_t + \frac{1}{2} \frac{\partial^2}{\partial S^2} V_{BS}(t) \sigma^2 S_t^2 dt. \quad (11)$$

Verifying the Self financing condition

Since straightforward differentiation of V_{BS} expression leads to

$$\frac{\partial}{\partial t} V_{BS}(t) = -\frac{S_t \Phi'(d_1(t))\sigma}{2\sqrt{T-t}} - rXe^{-r(T-t)}\Phi(d_2(t)),$$

$$\frac{\partial^2}{\partial S^2} V_{BS}(t) = \frac{\Phi'(d_1(t))}{S_t \sigma \sqrt{T-t}},$$

where $\Phi'(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, then it is enough to substitute ϕ^S and ϕ^B expressions given above to obtain from (11) that $dV_{BS}(t) = \phi_t^S dS_t + \phi_t^B dB_t$, which is the self–financing condition in differential form.

The Feynman Kac theorem for Risk Neutral Valuation

Different interpretation: the Feynman-Kac Theorem allows to interpret the solution of a parabolic PDE such as the Black and Scholes PDE in terms of expected values of a diffusion process. In general, given suitable regularity and integrability conditions, the solution of the PDE

$$\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)b(x)+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t,x)\sigma^2(x)=rV(t,x),\quad V(T,x)=f(x), \tag{12}$$

can be expressed as

$$V(t,x)=e^{-r(T-t)}\mathbb{E}^Q_{t,x}\{f(X_T)|\mathcal{F}_t\} \tag{13}$$

where the diffusion process X has dynamics starting from x at time t

$$dX_s=b(X_s)ds+\sigma(X_s)dW^Q_s\ , s\geq t,\ X_t=x \tag{14}$$

under the probability measure $\mathbb{Q}$ under which the expectation $\mathbb{E}^Q_{t,x}\{\cdot\}$ is taken. The process W^Q is a standard Brownian motion under $\mathbb{Q}$.

Risk Neutral interpretation of the B e S's formula

By applying this theorem to the Black and Scholes setup, with $b(x) = rx$, $\sigma(x) = \sigma\, x$ (so that the general PDE of the theorem coincides with the BeS PDE) we obtain:
The unique no-arbitrage price of the integrable contingent claim $Y = (S_T - K)^+$ (European call option) at time t, $0 \le t \le T$, is given by

$$V_{BS}(t) = \mathbb{E}^Q \left(e^{-r(T-t)} Y | \mathcal{F}_t \right). \tag{15}$$

The expectation is taken with respect to the so-called martingale measure $\mathbb{Q}$, i.e. a probability measure under which the risky–asset price $S_t/B_t = e^{-rt} S_t$ measured with respect to the risk-free asset price B_t is a martingale, i.e.

$$dS_t = S_t[rdt + \sigma dW_t^Q], \quad 0 \le t \le T, \tag{16}$$

An expression for $\mathbb{Q}$: Girsanov's theorem

Consider on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a stochastic differential equation

$$d\,X_t = b(X_t)\,dt + \,v(X_t)\,dW_t, \quad X_0.$$

Define the measure $\mathbb{Q}$ by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{b^Q(X_s) - b(X_s)}{v(X_s)}\right)^2 ds + \int_0^t \frac{b^Q(X_s) - b(X_s)}{v(X_s)}\,dW_s\right.$$

Then under $\mathbb{Q}$

$$dW_t^Q = -(b^Q(X_t) - b(X_t))/v(X_t)dt + dW_t$$

is a Brownian motion and

$$d\,X_t = b^Q(X_t)\,dt + \,v(X_t)\,dW_t^Q, \quad X_0.$$

The Risk Neutral measure via Girsanov's theorem

We apply Girsanov's theorem to move from

$$d\,S_t = \mu S_t\,dt + \sigma S_t\,dW_t$$

to

$$d\,S_t = rS_t\,dt + \sigma S_t\,dW_t^Q$$

We obtain

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \frac{\mu - r}{\sigma} W_T\right\}. \qquad (17)$$

main steps followed:

- Portfolio replication theory plus Ito's formula to derive the Black and Scholes PDE:

$$d\ S_t = \mu S_t\ dt + \sigma S_t\ dW_t$$

$$\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t) r S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 = rV(t, S_t),$$

$$V_T = \phi(S_T)$$

- The Feynman-Kac theorem to interpret the solution of the Black and Scholes PDE as an expected value of a function of the stock price with different dynamics

$$V(t, S_t) = \mathbb{E}^Q\{e^{-r(T-t)}\phi(S_T)|\mathcal{F}_t\}$$

$$d\ S_t = rS_t\ dt + \sigma S_t\ dW_t^Q$$

main steps followed:

- The Girsanov theorem to interpret the different dynamics of the stock price as a dynamics under a new (Risk neutral or martingale) probability measure P^*:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \frac{\mu - r}{\sigma} W_T\right\}.$$

The idea behind the martingale approach

Why martingales?

A martingale is a stochastic process representing a fair game. Loosely speaking, the above proposition states that in order to price under uncertainty one must price in a world where the probability measure is such that the risky asset evolves as a fair game when expressed in units of the risk–free asset.

Hence in our case S_t/B_t must be a fair game, ie a martingale.

martingales: local mean =0

For regular diffusion processes X_t martingale means ”zero-drift”, no up or down local direction: $dX_t = 0dt + \sigma(t, X_t)dW_t$.

Indeed, show that the drift of the SDE for $d(S_t/B_t)$ is zero under $\mathbb{Q}$.

The idea behind the martingale approach

Numeraire

When we consider S_t/B_t we may say that we are looking at S measured with respect to the numeraire B_t.
In general, as we shall see later on, it is possible to adopt any non-dividend paying asset price as numeraire, and price under the particular probability measure associated with that numeraire.
However, the canonical numeraire is the bank account B we have used now and the probability measure associated with the numeraire B is the risk neutral measure $\mathbb{Q}$.

The above analysis is easily generalized from a call option to any integrable claim $Y = f(S_T)$ different from a Call Option.

The idea behind the martingale approach

No need to know the real expected return

We noticed earlier that the coefficient μ does not appear in (10), indicating that investors, though having different risk preferences or predictions about the future stock price behaviour, must yet agree on this unique option price.

This property can also be inferred from (16), since, under $\mathbb{Q}$, the drift rate of the stock price process equals the risk-free interest rate while the variance rate is unchanged. For this reason the pricing rule (15) is often referred to as **risk-neutral valuation**, and the measure $\mathbb{Q}$ defines what is called **the risk-neutral world**.

Intuitively, in a risk-neutral world the expected rate of return on all securities is the risk-free interest rate, implying that investors do not require any risk premium for trading stocks.

Weak point of the derivation: Uniqueness of ϕ

The above derivation, however, is still not fully satisfactory, since we have implicitly assumed that (ϕ^B, ϕ^S) is the *unique* self-financing strategy replicating the claim with payoff $f(S_T)$. This uniqueness, anyway, can be obtained by applying the more general theory on complete markets, which is beyond the scope of this introduction.

Dynamic Hedging I

In the process of deriving the BS formula, we have also found a way to perfectly hedge the risk embedded in this contract.

Indeed look at the option pricing problem from the following point of view:

- You are the bank and you just sold a call option to the client.
- At the future time T you will have to pay $(S_T - K)^+$ to your client
- You client pays you V_0 for the option now, at time 0
- Clearly, if the equity goes up a lot in the future, $(S_T - K)^+$ could be very large
- You wish to avoid any risks and decide to hedge away the risk in this contract you sold.
- How should you do that?

Dynamic Hedging II

The answer to this question is in our derivation above.

- You cash in V_0 from the client and use it to buy, at time 0,

$$\frac{\partial V_0}{\partial S_0} = \Phi(d_1(0)) =: \phi_0^S =: \Delta_0 \quad \text{stock and}$$

$$\phi_0^B = (V_0 - \Delta_0 S_0)/B_0 \quad \text{bank account / bond (cash).}$$

- You then implement the self-financing trading strategy, **rebalancing continuously** (hence *dynamic hedging*) your ϕ_t^S, ϕ_t^B amounts of S and B according to

$$\phi_t^S = \frac{\partial V_t}{\partial S_t} = \Phi(d_1(t)) =: \Delta_t \quad \text{stock and}$$

$$\phi_t^B = (V_t - \Delta_t S_t)/B_t \quad \text{bank account / bond (cash).}$$

Dynamic Hedging III

- Because the strategy is self-financing, this rebalancing can be financed thanks to price movements of B and S and you need not add any cash or assets from outside.
- At final maturity we know that the final value will be $V_T = (S_T - K)^+$ as we posed this as boundary condition in our pricing problem.
- Hence by following the above strategy, set up with the initial V_0 and with no subsequent cost, we end up with the payout $(S_T - K)^+$ at maturity.
- We can then deliver this payout to our client and face no risk.
- Basically, our self financing trading strategy in the underlying S, set up with the initial payment V_0, completely replicated the claim we sold to our client.

Dynamic Hedging IV

- An obvious but often overlooked point it this: If we are perfectly hedged, all the money we received from the client (V_0) is spent to set up the hedge, and we as a bank make no gain.
- That's why in reality only partial hedges are often implemenented, in an attempt not to erode all potential profit.

The above framework is called ”**delta-hedging**”.

Basically one holds an amount of risky asset equal to the sensitivity of the contract price to the risky asset itself (delta).

This strategy is possible only in markets where all risks are directly linked to tradable assets and viceversa (roughly: ”complete markets”).

Dynamic Hedging V

Metatheorem/folklore: A market is complete if there are as many assets as independent sources of randomness.

In reality markets are incomplete, as there are some risks that are covered by no direct assets, and there are more risks than assets.

This can be partly addressed by including a few derivatives themselves among the basic assets, but it is hard to keep the market complete

For example, in credit risk with intensity models, where the default time is $\tau = \Lambda^{-1}(\xi)$, and Λ is the cumulated instantaneous credit spread and ξ is the jump to default exponential variable, we have that ξ cannot be hedged unless we introduce a credit derivative depending on ξ itself in the pool of our basic assets. And even then the hedge remains partial. We cannot hedge recovery rates, correlations...

Dynamic Hedging VI

A further problem is that continuous rebalancing does not happen. Real hedging happens in discrete time and this will imply an hedging error with respect to the idealized case

In the end hedging is more an art than a science, and it involves many pragmatic choices and rules of thumbs. However, a sound understanding of the idealized case is crucial to appreciate the subtleties in real market applications.

What does it all mean

So far we have tried to follow a technical path, but it is time to appreciate the significance of what we have done so far.

We now ask ourselves: What are the implications of what we have calculated on the big picture?

Mathematical Finance deals in large part with Derivatives. So, following our derivation above, why are derivatives so important, so popular and, often, unpopular?

What does it all mean? Call option and Gambling

Assume we wish to enter into a gamble (call option) against a bank, where:

- If the future price of the ACME stock in 1y is larger than the value of ACME today, we receive from the bank the difference between the two prices (on a given notional).
- If the future price of the ACME stock in 1y is smaller or equal than the value of ACME today, nothing happens.

The bank will charge us for entering this wage, since we can only win or get into a draw, whereas the bank can only lose or get to a draw.

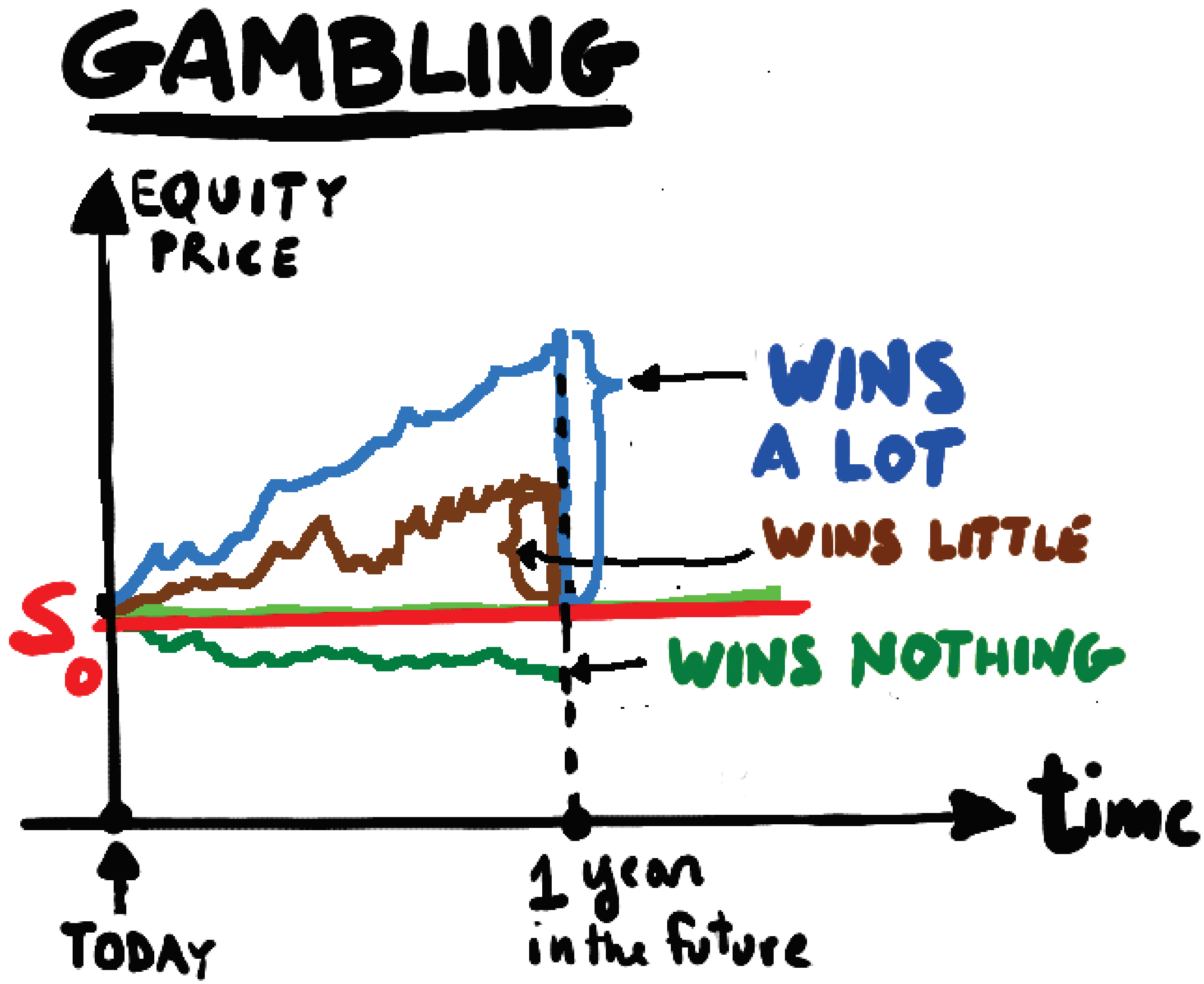


Figure: A one-year maturity Gamble on an equity stock. Call Option.

Call option and Gambling

We have an investor buying a call option on ACME with a 1y maturity.

The Bank's problem is finding the correct price of this option today. This price will be charged to the investor, who may also go to other banks.

This is an option pricing problem.

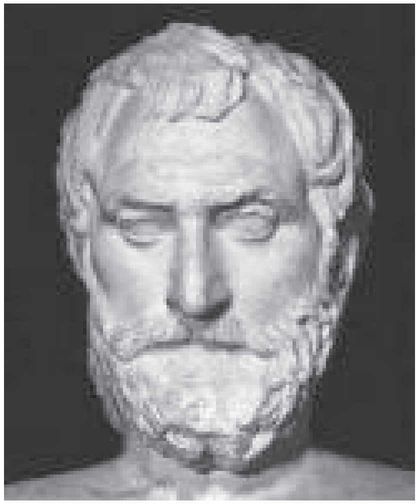
The market introduced options and more generally financial derivatives that may be much more complex than the above example. Such derivatives often work on different sectors: Foreign Exchange Rates, Interest Rates, Default Events, Metheorological events, Energy, etc.

Derivatives can be bought to protect or hedge some risk, but also for speculation or ”gambling”.

Options and Derivatives

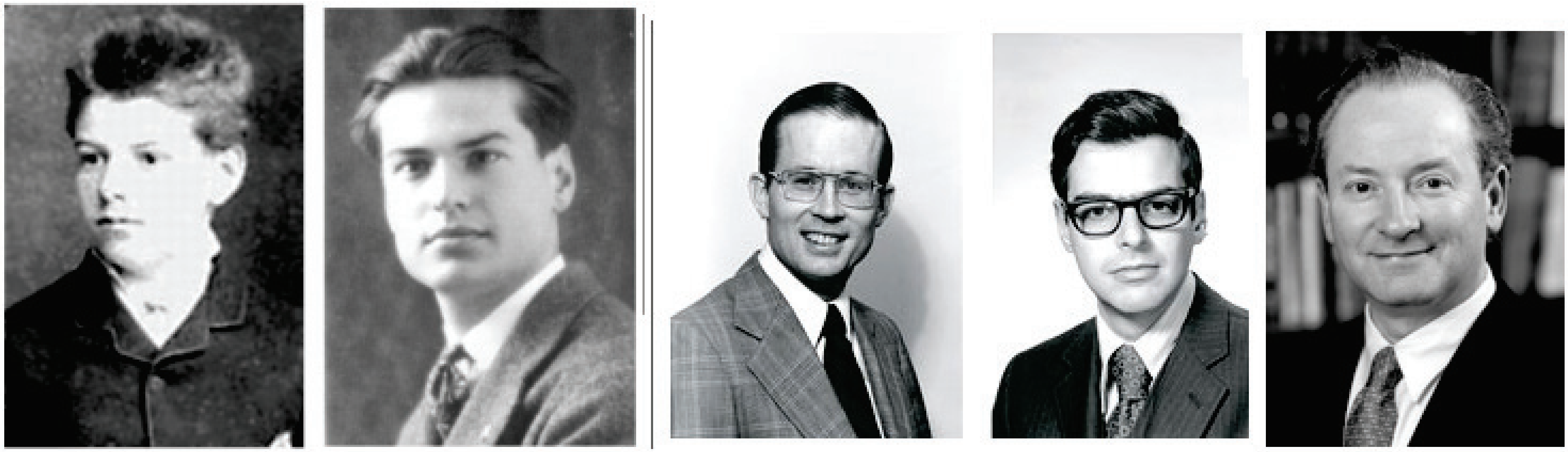
Derivatives outstanding notional as of June 2011 (BIS) is estimated at **708 trillions USD** (US GDP 2011: 15 Trillions; World GDP: 79 Trillions)

$$708000 \text{ billions}, \quad 708,000,000,000,000, \quad 7.08 \times 10^{14} \text{ USD}$$



How did it start? It has always been there. Around 580 B.C., Thales purchased options on the future use of olive presses and made a fortune when the olives crop was as abundant as he had predicted, and presses were in high demand. (Thales is also considered to be the father of the sciences and of western philosophy, as you know).

Options and Derivatives valuation: precursors



- **Louis Bachelier** (1870 – 1946) (First to introduce Bronwnian motion W_t in Finance, First in the modern study of Options);
- **Bruno de Finetti** (1906 – 1985) (Father of the subjective interpret of probability; defines the risk neutral measure in a way that is very similar to current theories: first to derive no arbitrage (ante-litteram!) through inequalities constraints, discrete setting, consistent betting quotients, see also Frank Ramsey (1903-1930).

Modern theory follows Nobel awarded **Black, Scholes and Merton** (and then Harrison and Kreps etc) on the correct pricing of options.

Black and Scholes: What does it mean?

We have derived the Black Scholes formula for a call option earlier. Let us recall the key points.

Let S_t be the equity price for ACME at time t.
For the value of the ACME stock S_t let us assume, as before, a SDE
$dS_t = \mu S_t dt + \sigma S_t dW_t$ or also

$$\underbrace{\frac{dS_t}{S_t}}_{\text{relative change in stock ACME between } t \text{ and } t+dt} = \underbrace{\mu}_{\text{instantaneous "mean" return of ACME between } t \text{ and } t+dt} dt + \underbrace{\sigma}_{\text{volatility for ACME}} \underbrace{dW_t}_{\text{New random shock}}$$

Black and Scholes: What does it mean?

Then we have seen there exists a formula (Black and Scholes') providing a unique fair price for the above gamble (option) on ACME in one year.

This Black Scholes formula **depends on the volatility** σ of ACME, and from the initial value S_0 of ACME today, but **does NOT depend on the expected return** μ of ACME.

This means that two investors with very different expectations on the future performance of ACME (for example one investor believes ACME will grow, the other one that ACME will go down) will be charged the same price from the bank to enter into the option.

The Gamble price does not depend on the investor perception of future markets. One would think that Red Investor should be willing to pay a higher price for the option with respect to Blue Investor. Instead, both will have to pay the gamble according to the green scenarios, where ACME grows with the same returns as a riskless asset

Derivatives prices independent of expected returns???

This seemingly counterintuitive result renders derivatives pricing independent of the expected returns of their underlying assets.

This makes derivatives valuations quite objective, and has contributed to derivatives growth worldwide.

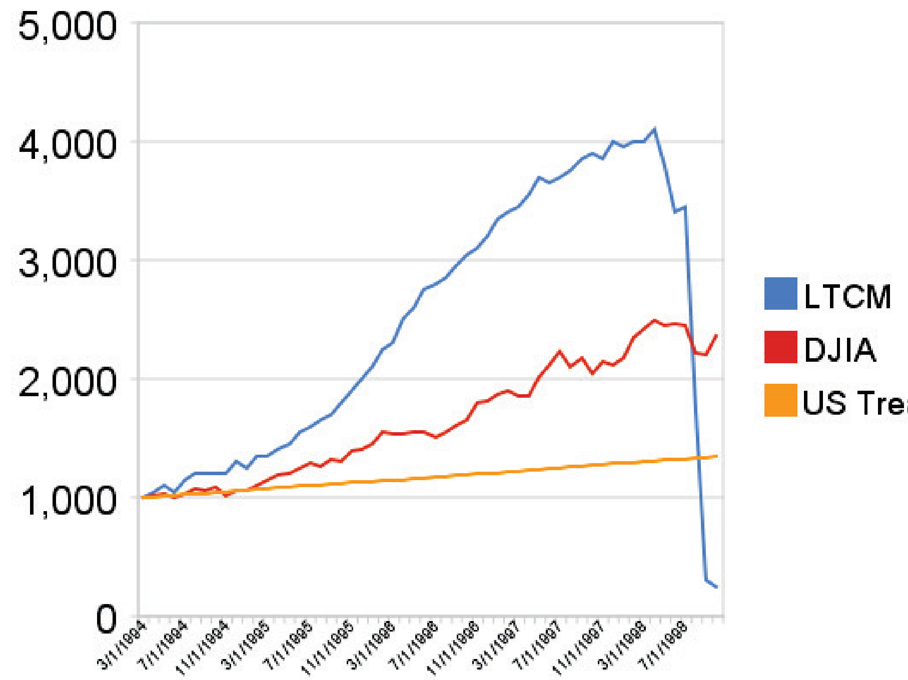
Today, derivatives are used for several purposes by banks and corporates all over the world

A mathematical result has contributed to create new markets that reached 708 trillions (US GDP: 15 Trillions)

But keep in mind that the derivation of the Black Scholes result holds only under the 4 ideal conditions and actually many more assumptions:

The Black Scholes Merton analysis assumptions

- Short selling is allowed without restrictions
- Infinitely divisible shares
- No transaction costs
- No dividends in the stock
- No default risk of the parties in the deal
- No funding costs: Cash can be borrowed or lent at the risk free rate r
- Continuous time and continuous trading/hedging
- Perfect market information
- ....



Sometimes the timing of the Nobel committee is funny, and we are not talking about the peace Nobel prize. Warning: anedoctal

1997: Nobel award.

1998: the US Long-Term Capital Management hedge fund has to be bailed out after a huge loss. The fund had Merton and Scholes in their board and made high use of leverage (derivatives). This leads us to...

The Credit Crisis: Is this Mathematics fault?

Quantitative Analysts (”quants”) and Academics guilty?

Over the past few years a number of articles has disputed the role of Mathematics in Finance, especially in relationship with Counterparty Credit Risk and Credit Derivatives (especially CDOs).
Quants have been accused to be unaware of models limitations and to have provided the market with a false sense of security.

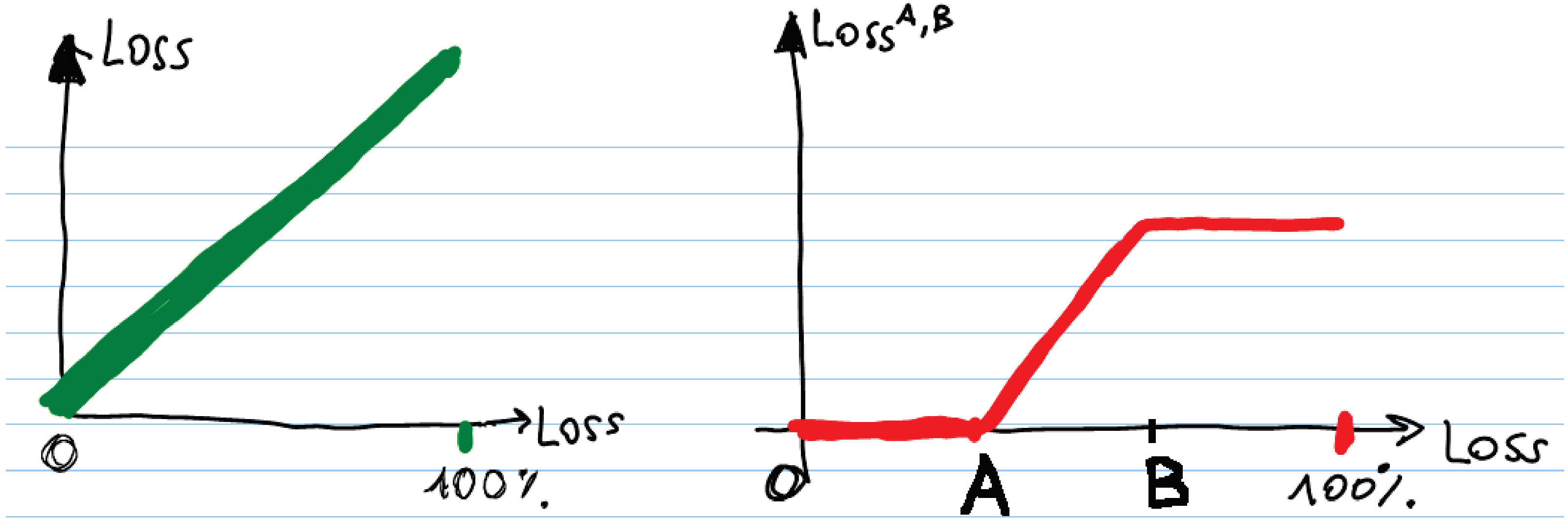
- “The formula that killed Wall Street”[1]
- “The formula that fell Wall Street”[2]
- “Wall Street Math Wizards forgot a few variables”[3]
- “Misplaced reliance on sophisticated (mathematical) models”[4]
- BUT WHAT IS THIS FORMULA PRECISELY?

[1] Recipe for disaster. Wired Magazine, 17.03.
[2] The Financial Times, Jones, S. (2009). April 24 2009.
[3] Lohr (2009), New York Times of September 12.
[4] Turner, J.A. (2009). The Turner Review. 03/2009. FSA, UK. www.fsa.gov.uk/pubs/other/turner_review.pdf.

CDOs: The standard synthetic case I

- Portfolio of names, say 125. Names may default, generating losses.
- A tranche is a portion of the loss between two percentages. The $3\% - 6\%$ tranche focuses on the losses between 3% (attachment point) and 6% (detachment point).
- The CDO protection seller agrees to pay to the buyer all notional default losses (minus the recoveries) in the portfolio whenever they occur due to one or more defaults, within 3% and 6% of the total pool loss.
- In exchange for this, the buyer pays the seller a periodic fee on the notional given by the portion of the tranche that is still "alive" in each relevant period.
- Valuation problem: What is the fair price of this "insurance"?

CDOs: The standard synthetic case II



- Pricing (marking to market) a tranche: taking expectation of the future tranche losses under the pricing measure.
- From nonlinearity, the tranche expectation will depend on the loss distribution: marginal distributions of the single names defaults **and** dependency among different names' defaults. Dependency is commonly called "correlation".
- Abuse of language: correlation is a complete description of dependence for jointly Gaussians, but more generally it is not.

Copulas

The complete description is either the whole multivariate distribution or the so-called “copula function” (marginal distributions have been standardized to uniform distributions).

CDO Valuation: The culprit.

One-factor Gaussian copula

$$\int_{-\infty}^{+\infty} \prod_{i=1}^{125} \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Lambda_i(T))) - \sqrt{\rho_i}m}{\sqrt{1-\rho_i}}\right) \varphi(m)dm.$$

“MEA COPULA!” From Nobel award to universal scapegoat

Introduced in Credit Risk modeling by David X. Li. Commentators went from suggesting a Nobel award to blaming Li for the whole Crisis.

The scapegoat

David Li, 2005, Wall Street Journal

[...] "The most dangerous part," Mr. Li himself says of the model, "is when people believe everything coming out of it." Investors who put too much trust in it or don't understand all its subtleties may think they've eliminated their risks when they haven't.

Indeed, these models are static. they ignore Credit Spread Volatilities, that in Credit can be 100%; this has further paradoxical consequences in copula models for wrong way risk, as we will see later on.

Tranches and Correlations

The dependence of the tranche on "correlation" is crucial. The market assumes a Gaussian Copula connecting the defaults of the 125 names, parametrized by a correlation matrix with 125*124/2 = 7750 entries. However, when looking at a tranche:

$$7750 \ \text{parameters} \longrightarrow \ 1 \ \text{parameter}.$$

The unique parameter is reverse-engineered to reproduce the price of the liquid tranche under examination. "Implied correlation". Once obtained it is used to value related products.

Problem with this implied "compound correlation"

If at a given time the $3\% - 6\%$ tranche for a five year maturity has a given implied correlation, the $6\% - 9\%$ tranche for the same maturity will have a different one. The two tranches on the *same pool* are priced (and hedged!!!) with two inconsistent loss distributions

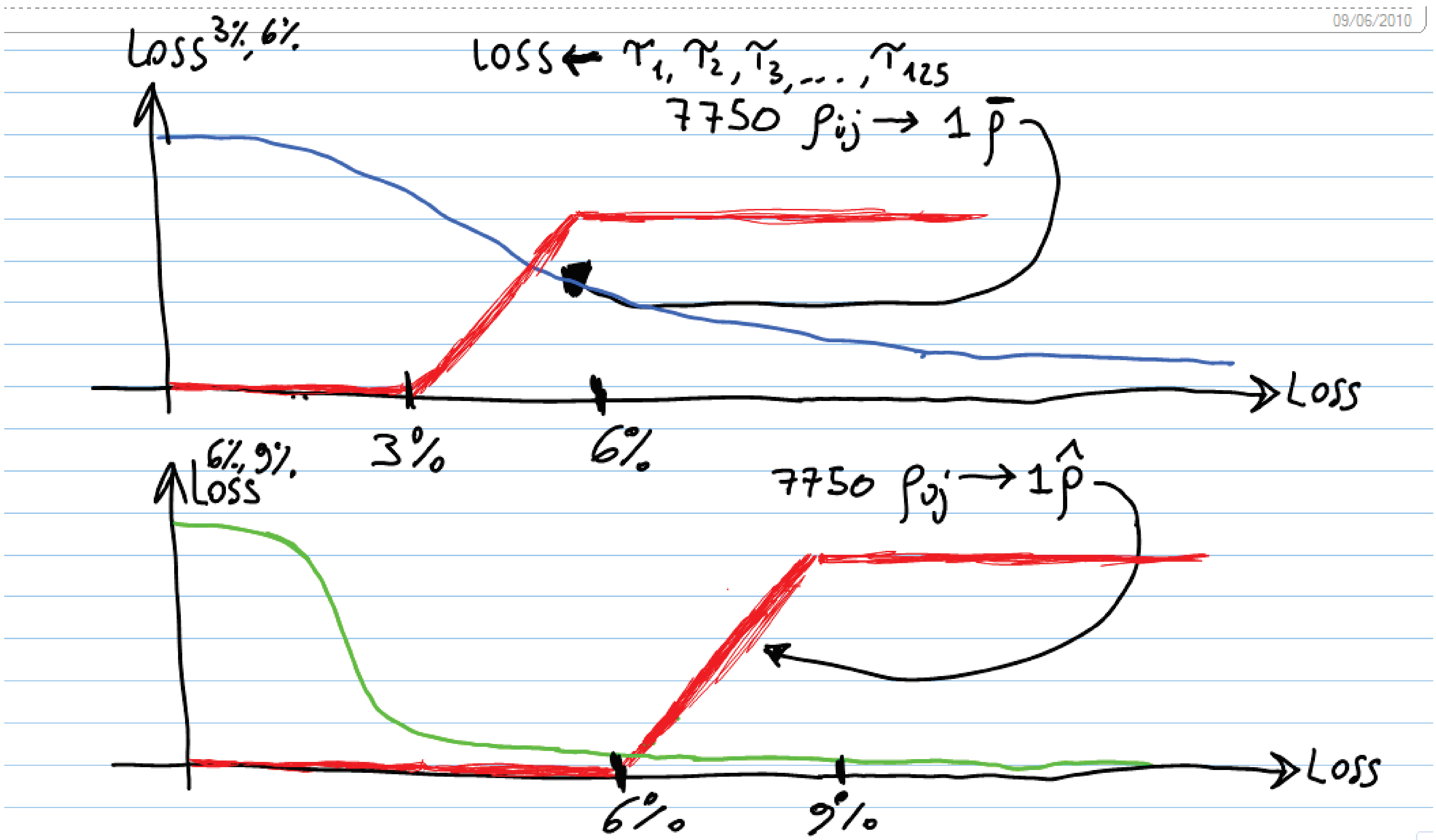


Figure: Compound correlation inconsistency

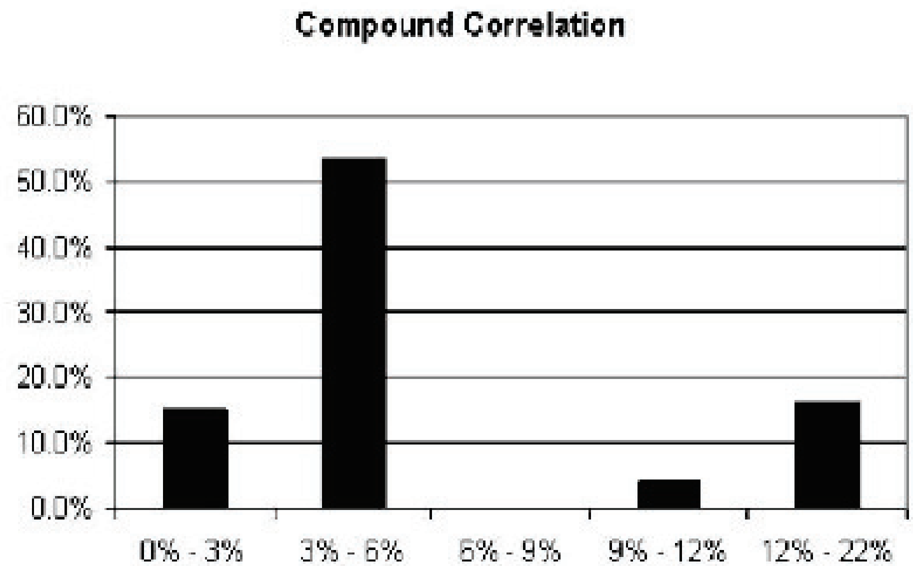


Figure: (After Edvard Munch's The Scream; Compound correlation DJ-iTraxx S5, 10y on 3 Aug 2005)

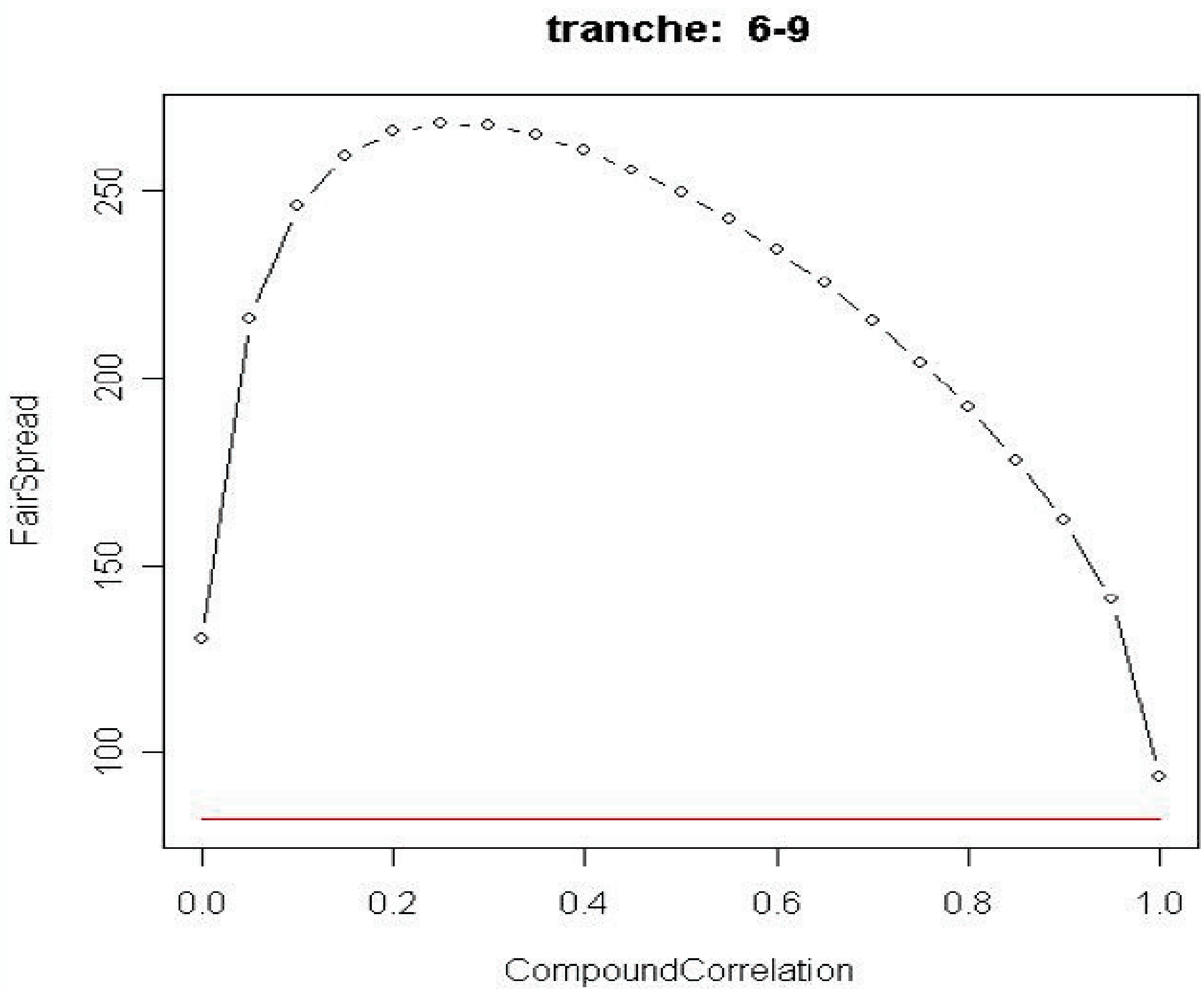


Figure: Non-invertibility compound correl DJ-iTraxx S5, 10y on 3 Aug 2005

Base correlation I

As a possible remedy for non-invertibility of compound correlation and other matters, the market introduced Base Correlation, which is still prevailing in the market.

Problems with base correlation

Base correlation is easier to interpolate but is inconsistent even at single tranche level, in that it prices the $3\% - 6\%$ tranche by decomposing it into the $0\% - 3\%$ tranche and $0\% - 6\%$ tranche and using two different correlations (and hence distributions) for those. This inconsistency shows up occasionally in negative losses (i.e. in defaulted names resurrecting).

[in the graph we use put-call parity to simplify]

Base correlation II

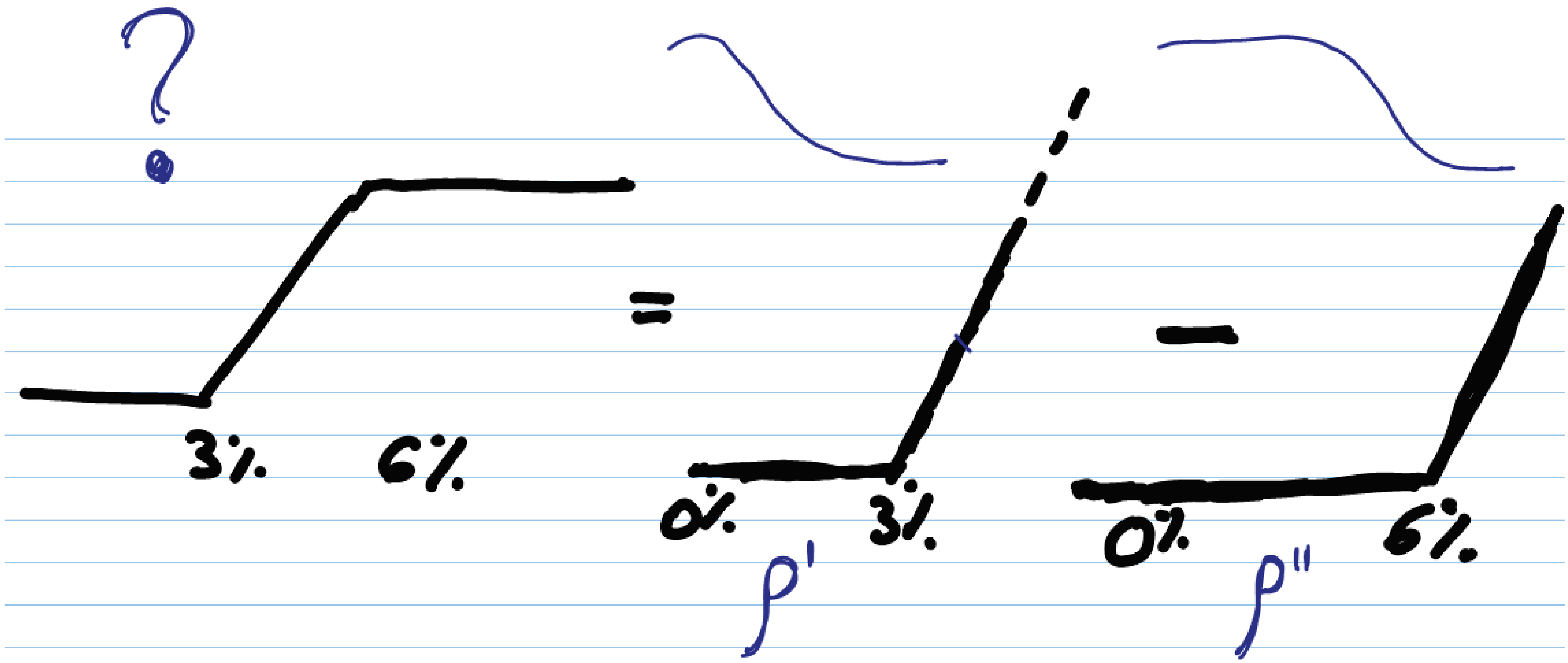


Figure: Base correlation inconsistency

Base correlation III

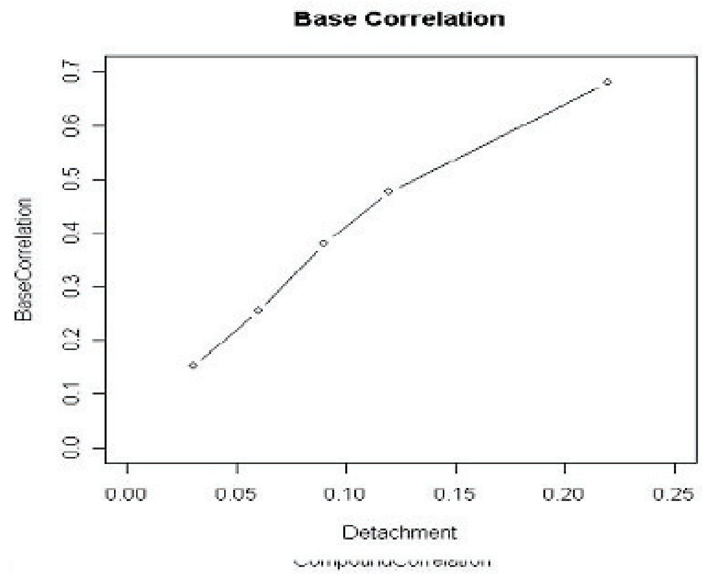


Figure: (Base correl DJ-iTraxx S5, 10y on 3 Aug 2005)

Base correlation

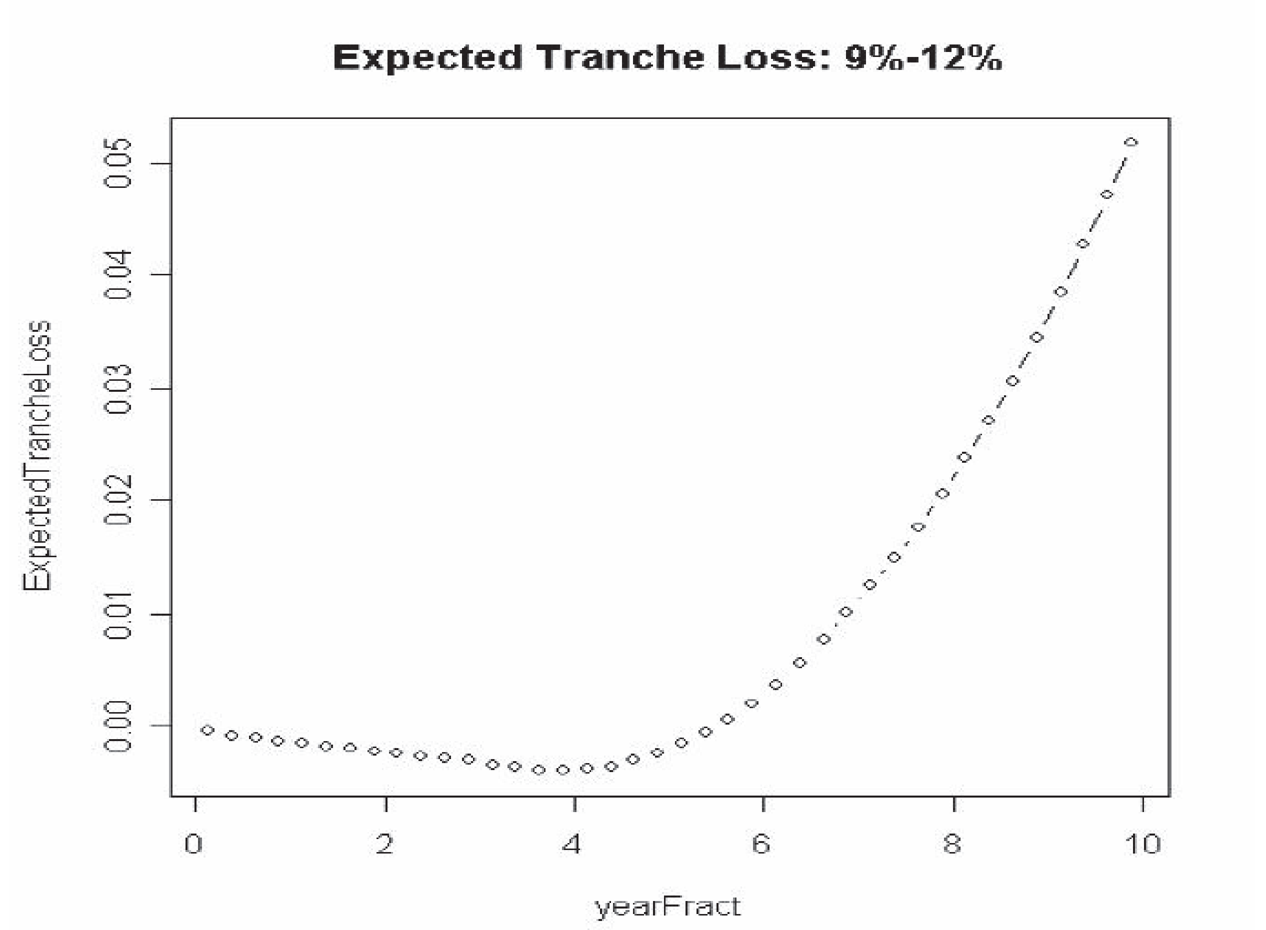
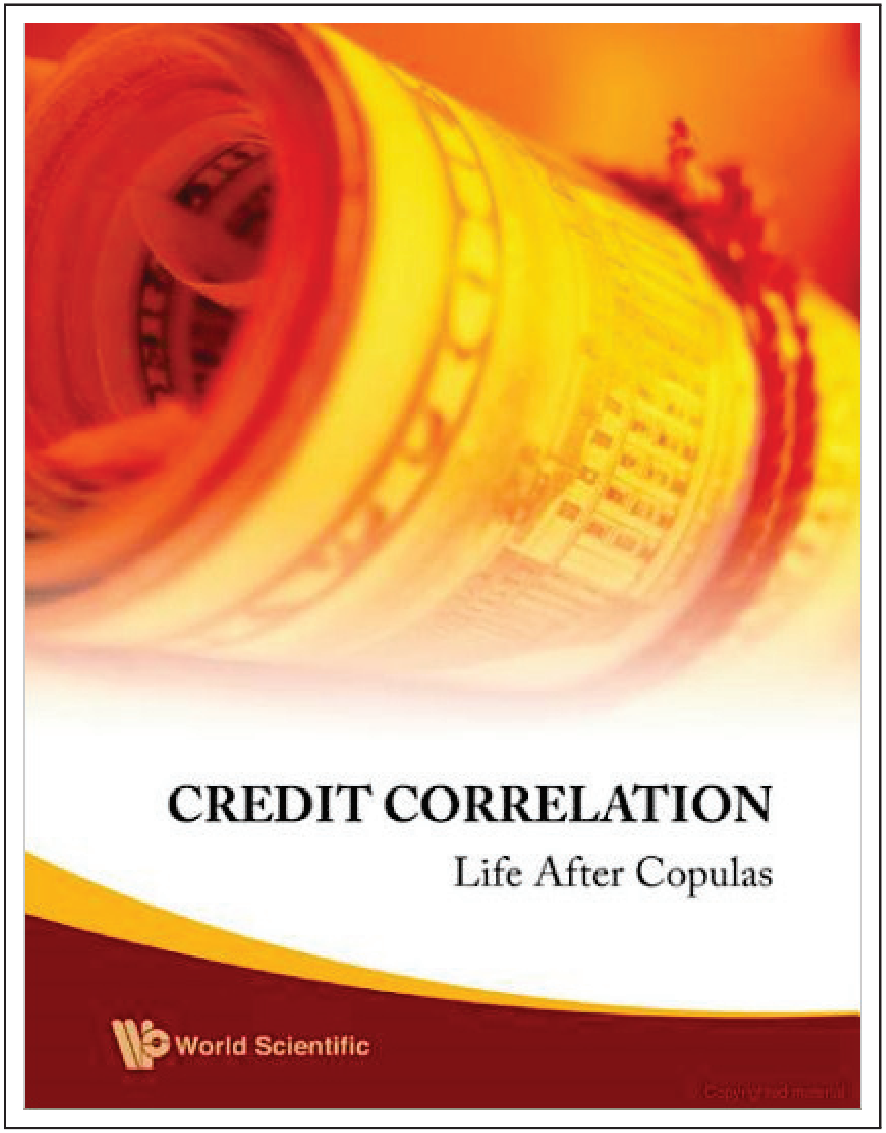


Figure: Expected tranche loss coming from Base correlation calibration, 3d August 2005, First published in 2006. The locally negative loss distribution means there are defaulted names RESURRECTING with positive probability

Some facts

Proceedings of a Conference held in London in 2006 by Merrill Lynch.
A number of proposals to improve the static copula models used (and abused) for credit derivatives have been presented. I was there.
Quants and Academics were well aware (and had been for years) of the models limitations and were trying to overcome them.



A few journalist have very short memory...

12 Sept 2005. Wall Street Journal

How a Formula [Base correlation + Gaussian Copula] Ignited Market That Burned Some Big Investors.

There are many other publications preceeding the crisis started in 2007. Such publications questioned the use of the Gaussian copula and the notion of implied and base correlation. For example, see our 2006 article

Implied Correlation: A paradigm to be handled with care, 2006, SSRN, `http://ssrn.com/abstract=946755`

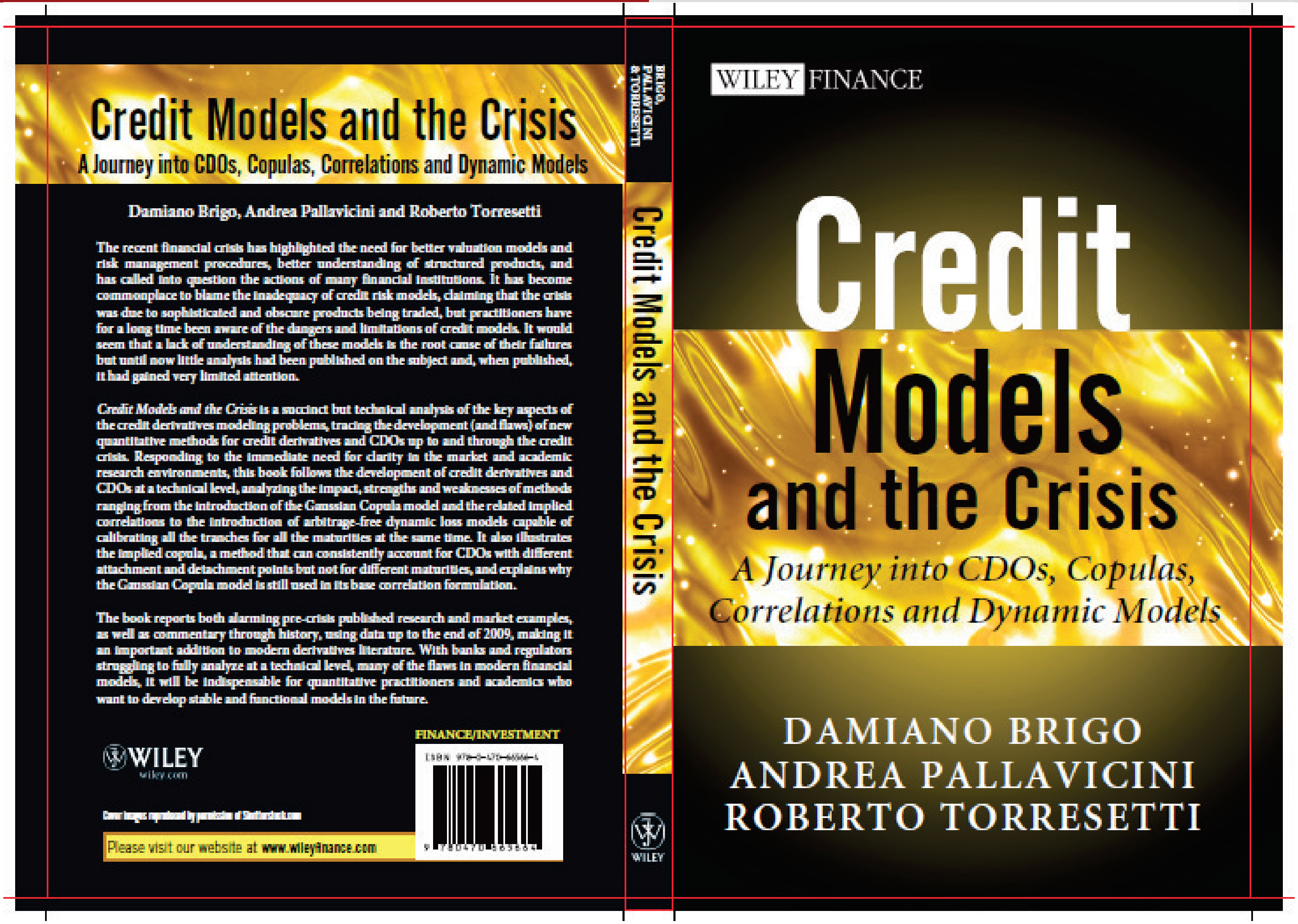


Figure: This book collects research published originally in 2006, warning against the flaws of the industry credit derivatives models. Related papers in the journals *Mathematical Finance, Risk Magazine, IJTAF*

Beyond copulas: GPL and GPCL Models (2006-on)

We model the total number of defaults in the pool by t as

$$Z_t := \sum_{j=1}^{n} \delta_j Z_j(t)$$

(for integers δ_j) where Z_j are independent Poissons. This is consistent with the Common Poisson Shock framework, where defaults are linked by a Marshall Olkin copula (Lindskog and McNeil).

$$\text{Example :} \quad n = 125, \;\; Z_t = 1\, Z_1(t) + 2\, Z_2(t) + \ldots + 125\, Z_{125}(t).$$

If Z_1 jumps there is just one default (idiosyncratic), if Z_{125} jumps there are 125 ones and the whole pool defaults one shot (total systemic risk), otherwise for other Z_i's we have intermediate situations (sectors).

The GPL and GPCL Models: Default clusters?

- Thrifts in the early 90s at the height of the loan and deposit crisis.
- Airliners after 2001.
- Autos and financials more recently. From the September, 7 2008 to the October, 8 2008, we witnessed seven credit events: Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir, Kaupthing.

S&P ratings and default clusters

Moreover, S&P issued a request for comments related to changes in the rating criteria of corporate CDO. Tranches rated 'AAA' should be able to withstand the default of the largest single industry in the pool with zero recoveries. Stressed but plausible scenario that a cluster of defaults in the objective measure exists.

The GPL and GPCL Models

Problem: infinite defaults. Solution 1: **GPL:** Modify the aggregated pool default counting process so that this does not exceed the number of names, by simply capping Z_t to n, regardless of cluster structures:

$$C_t := \min(Z_t, n)$$

Solution 2: **GPCL**. Force clusters to jump only once and deduce single names defaults consistently.
The first choice is ok at top level but it does not really go down towards single names. The second choice is a real top down model, but combinatorially more complex.

Calibration

The GPL model is calibrated to the market quotes observed on March 1 and 6, 2006. Deterministic discount rates are listed in Brigo, Pallavicini and Torresetti (2006). Tranche data and DJi-TRAXX fixings, along with bid-ask spreads, are (I=index,T=Tranche,Tl=Tranchelet)

	Att-Det	**March, 1 2006**		**March, 6 2006**		
		5y	7y	3y	5y	7y
I		35(1)	48(1)	20(1)	35(1)	48(1)
T	0-3	2600(50)	4788(50)	500(20)	2655(25)	4825(25)
	3-6	71.00(2.00)	210.00(5.00)	7.50(2.50)	67.50(1.00)	225.50(2.50)
	6-9	22.00(2.00)	49.00(2.00)	1.25(0.75)	22.00(1.00)	51.00(1.00)
	9-12	10.00(2.00)	29.00(2.00)	0.50(0.25)	10.50(1.00)	28.50(1.00)
	12-22	4.25(1.00)	11.00(1.00)	0.15(0.05)	4.50(0.50)	10.25(0.50)
Tl	0-1	6100(200)	7400(300)			
	1-2	1085(70)	5025(300)			
	2-3	393(45)	850(60)			

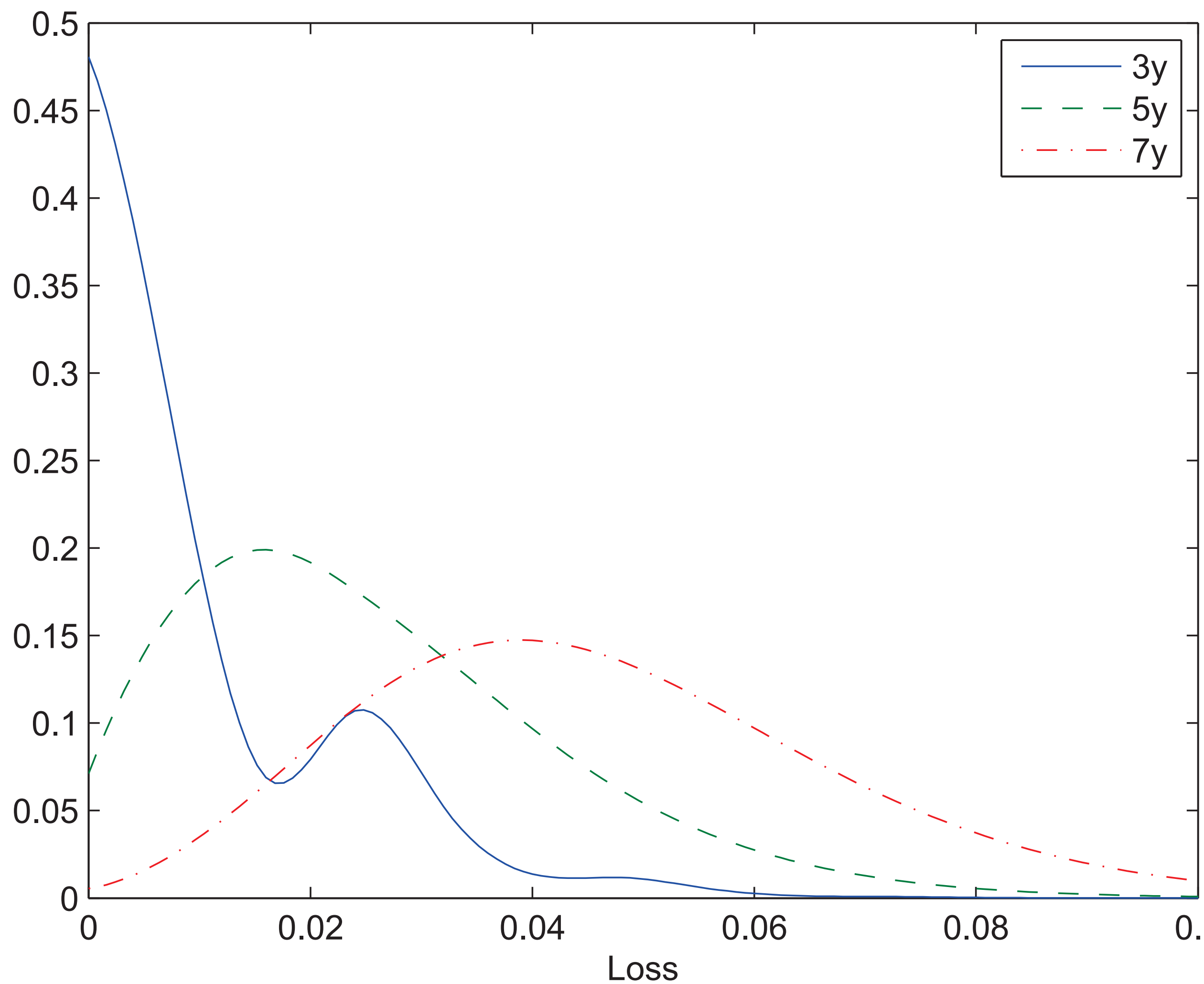
Calibration: All standard tranches up to seven years

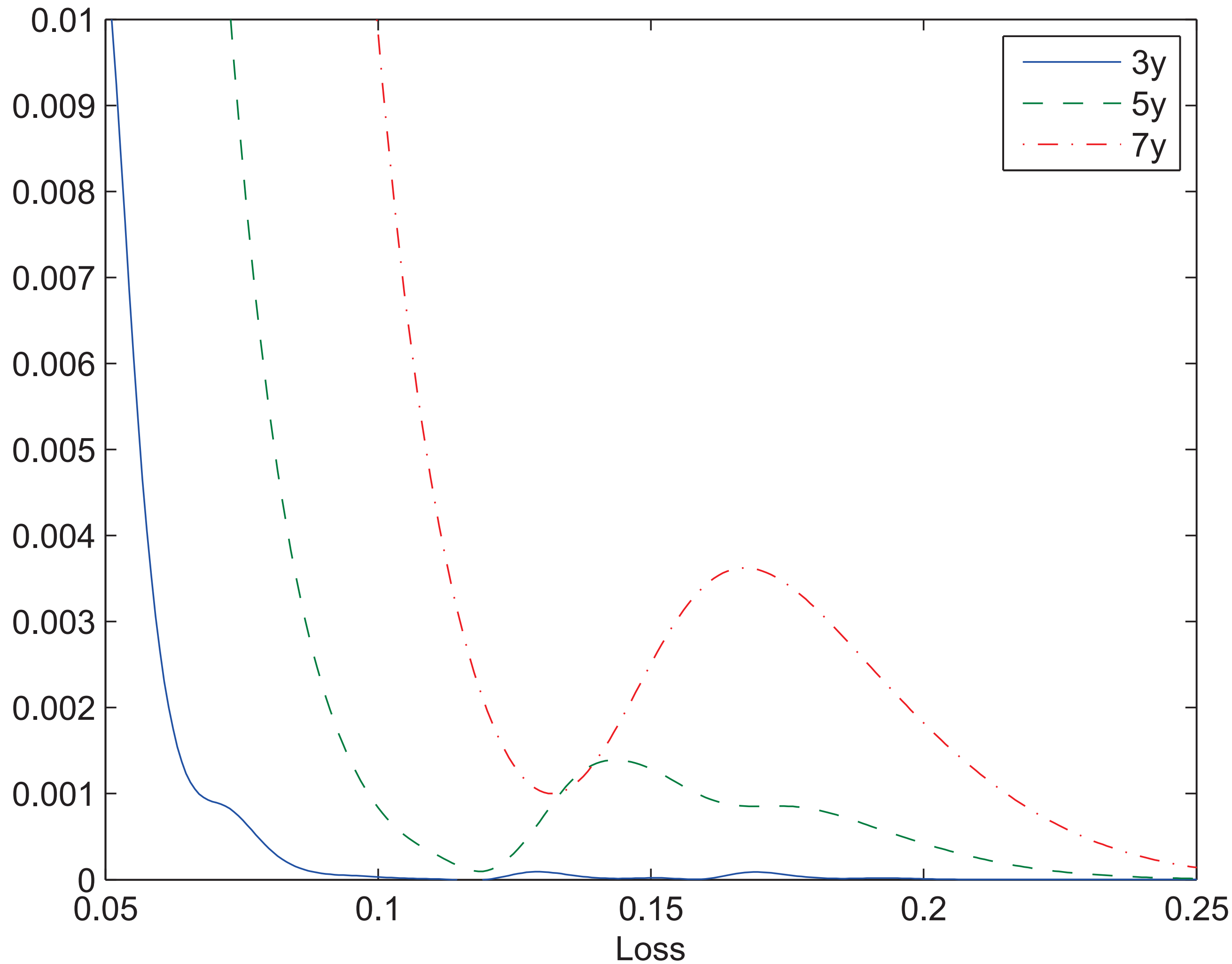
As a first calibration example we consider standard DJi-TRAXX tranches up to a maturity of 7y with constant recovery rate of 40%. The calibration procedure selects five Poisson processes. The 18 market quotes used by the calibration procedure are almost perfectly recovered. In particular all instruments are calibrated within the bid-ask spread (we show the ratio calibration error / bid ask spread).

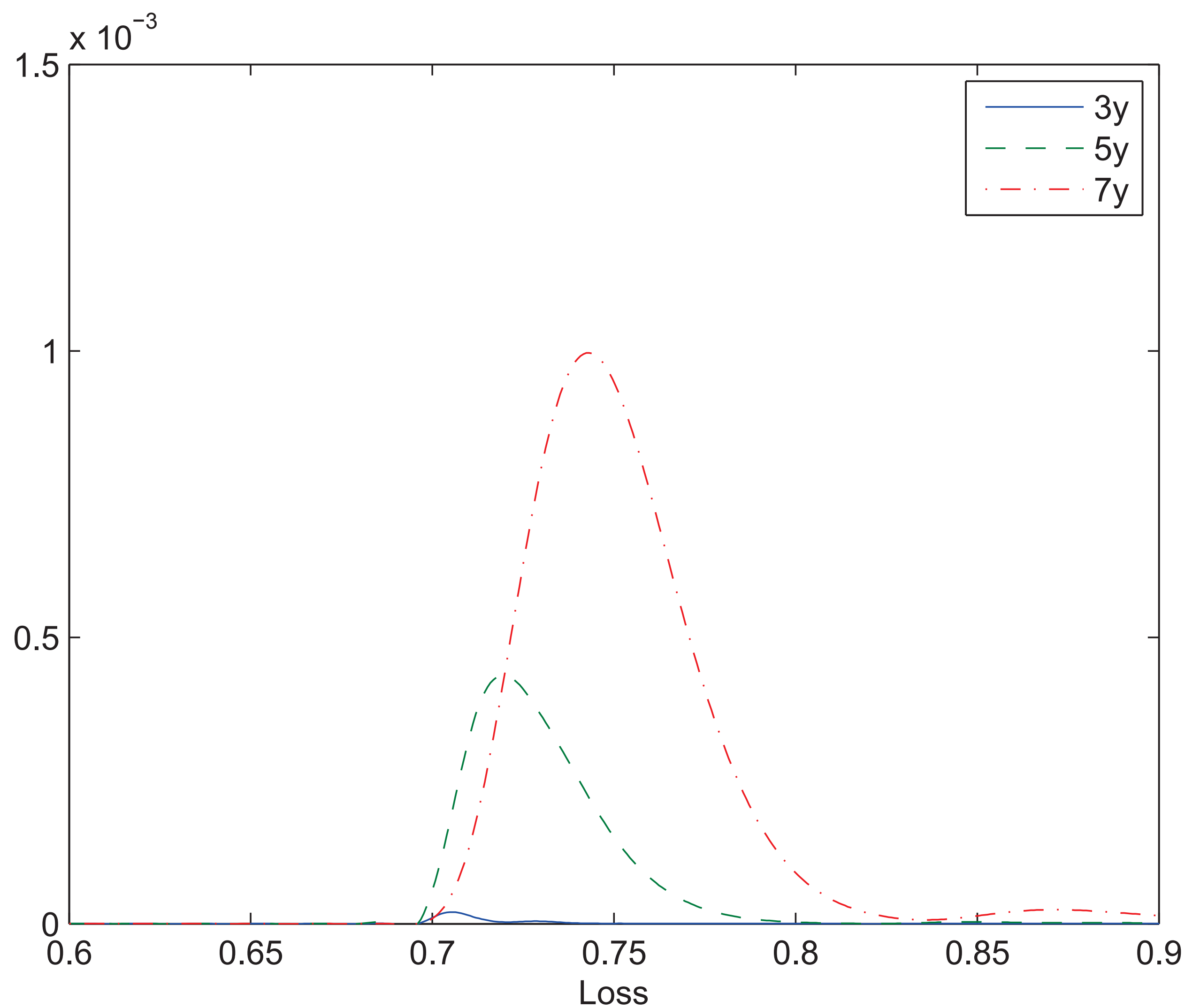
	Att-Det	**Maturities**		
		3y	5y	7y
Index		-0.4	-0.2	-0.9
Tranche	0-3	0.1	0.0	-0.7
	3-6	0.0	0.0	0.7
	6-9	0.0	0.0	-0.2
	9-12	0.0	0.0	0.0
	12-22	0.0	0.0	0.2

δ	$\Lambda(T)$		
	3y	5y	7y
1	0.535	2.366	4.930
3	0.197	0.266	0.267
16	0.000	0.007	0.024
21	0.000	0.003	0.003
88	0.000	0.002	0.007

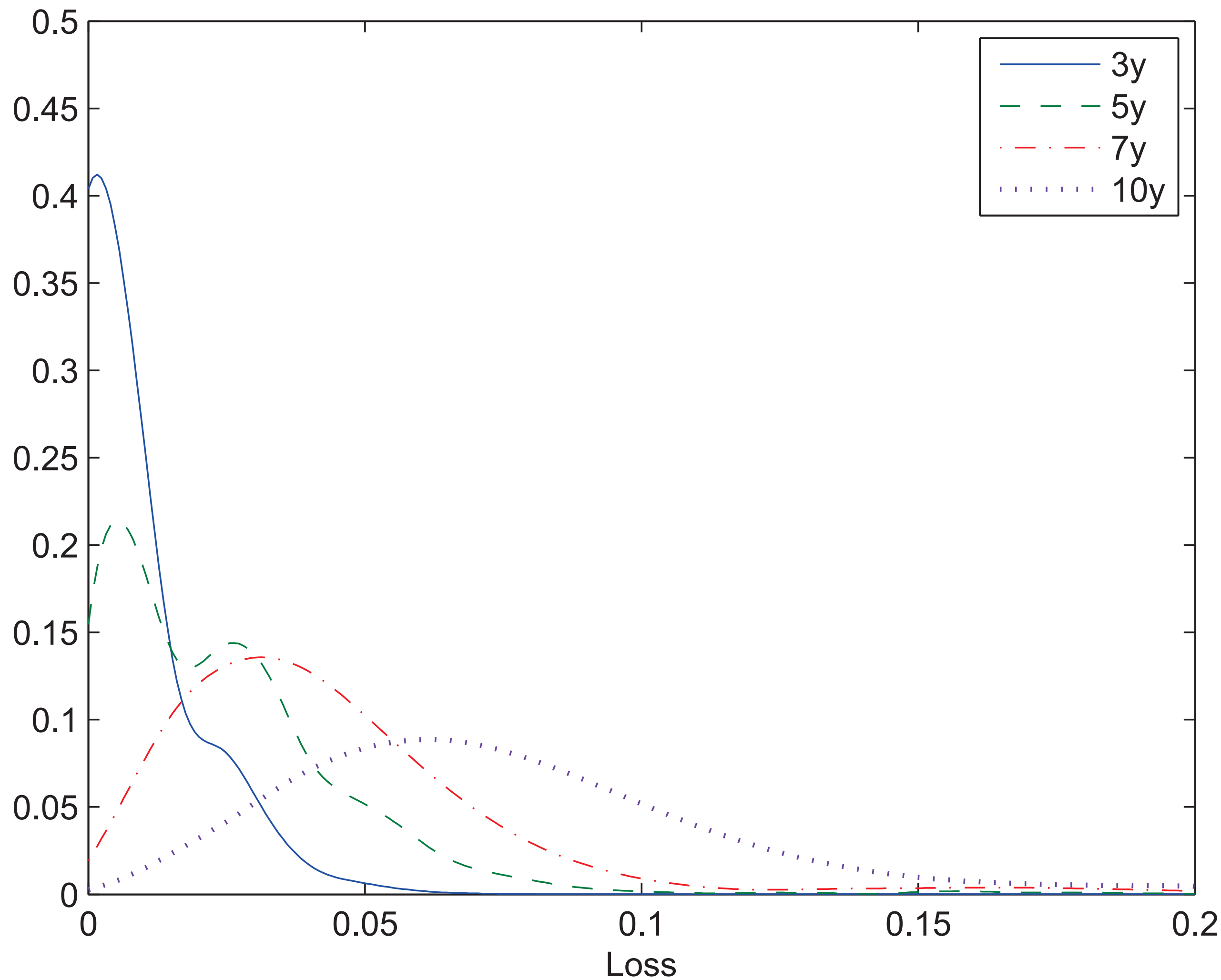
Loss distribution of the calibrated GPL model at different times



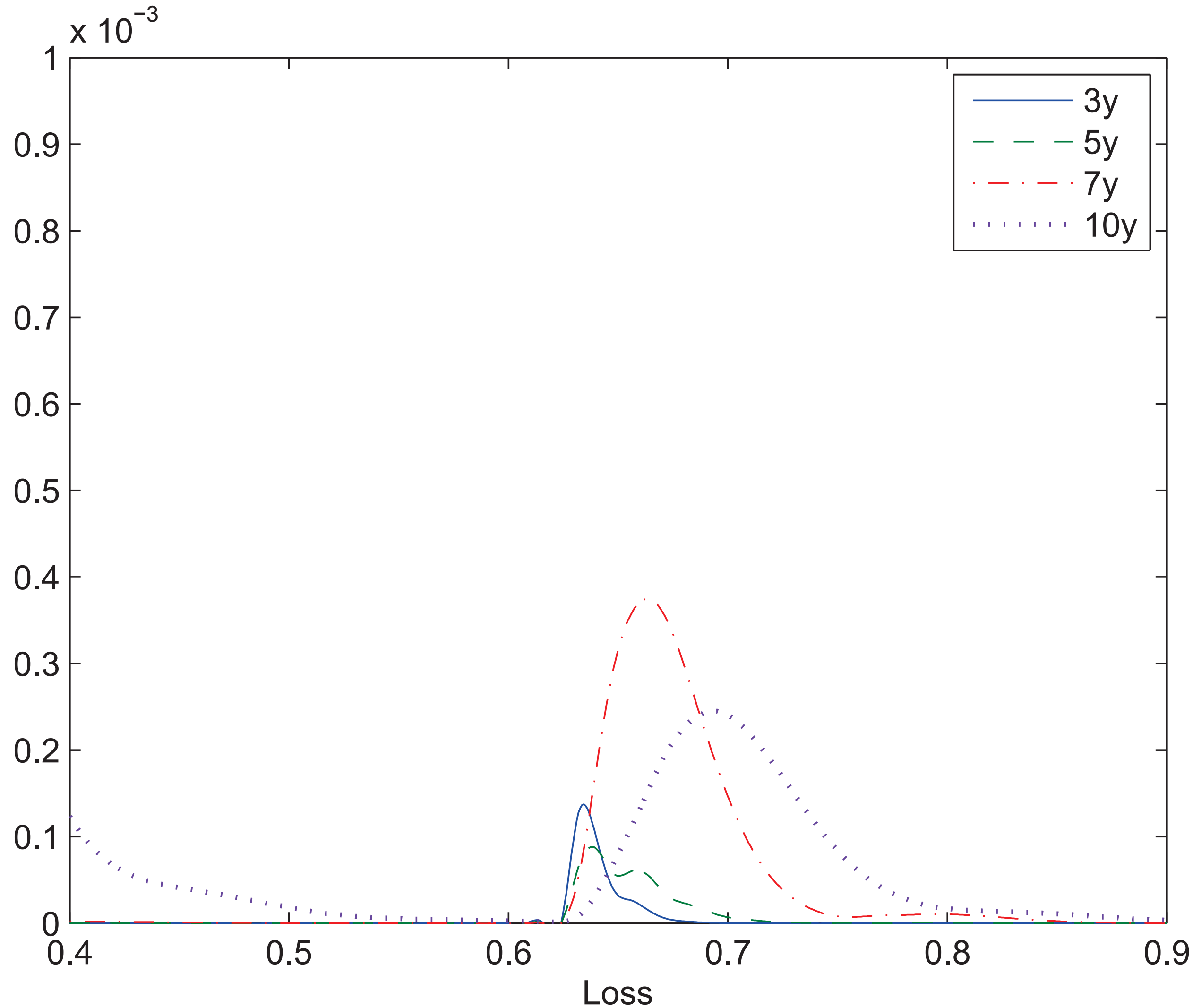

0.01
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
0.05
0.1
0.15
0.2
0.25
Loss
3y
5y
7y


x 10^-3
1.5
1
0.5
0
0.6
0.65
0.7
0.75
0.8
0.85
0.9
Loss
3y
5y
7y

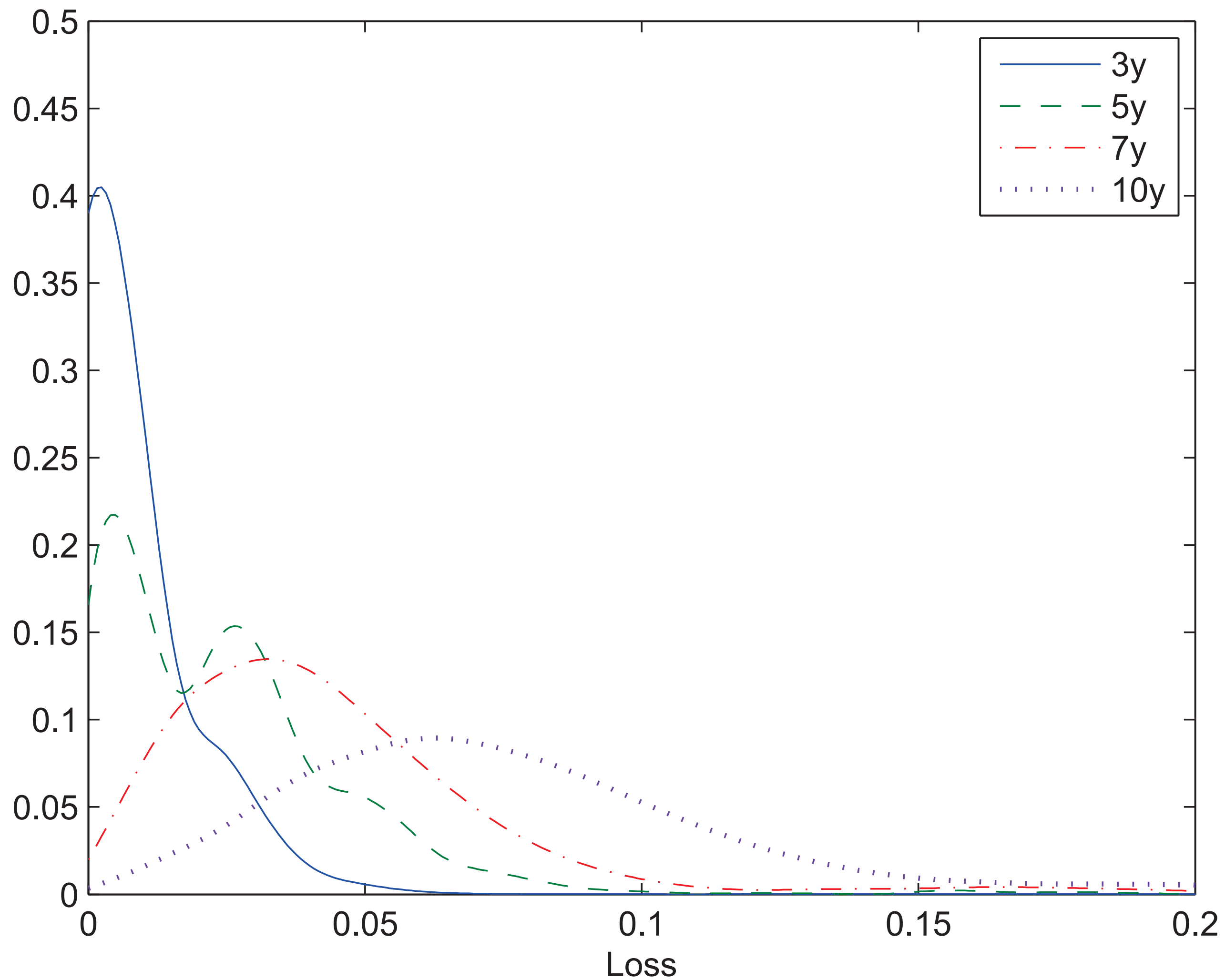
October 2 2006, GPL, Calibration up to 10y



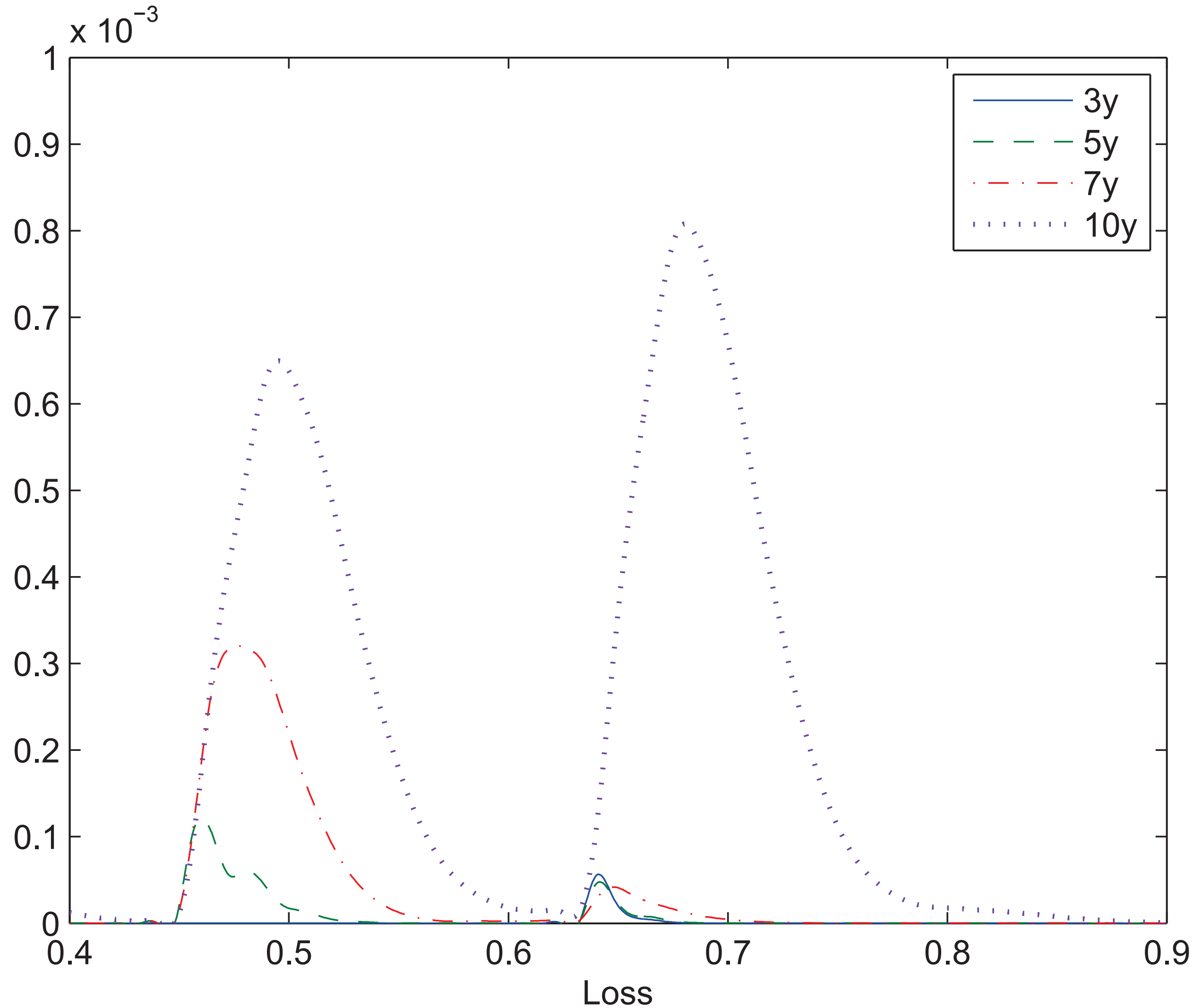
October 2 2006, GPL tail



October 2 2006, GPCL, Calibration up to 10y



October 2 2006, GPCL tail



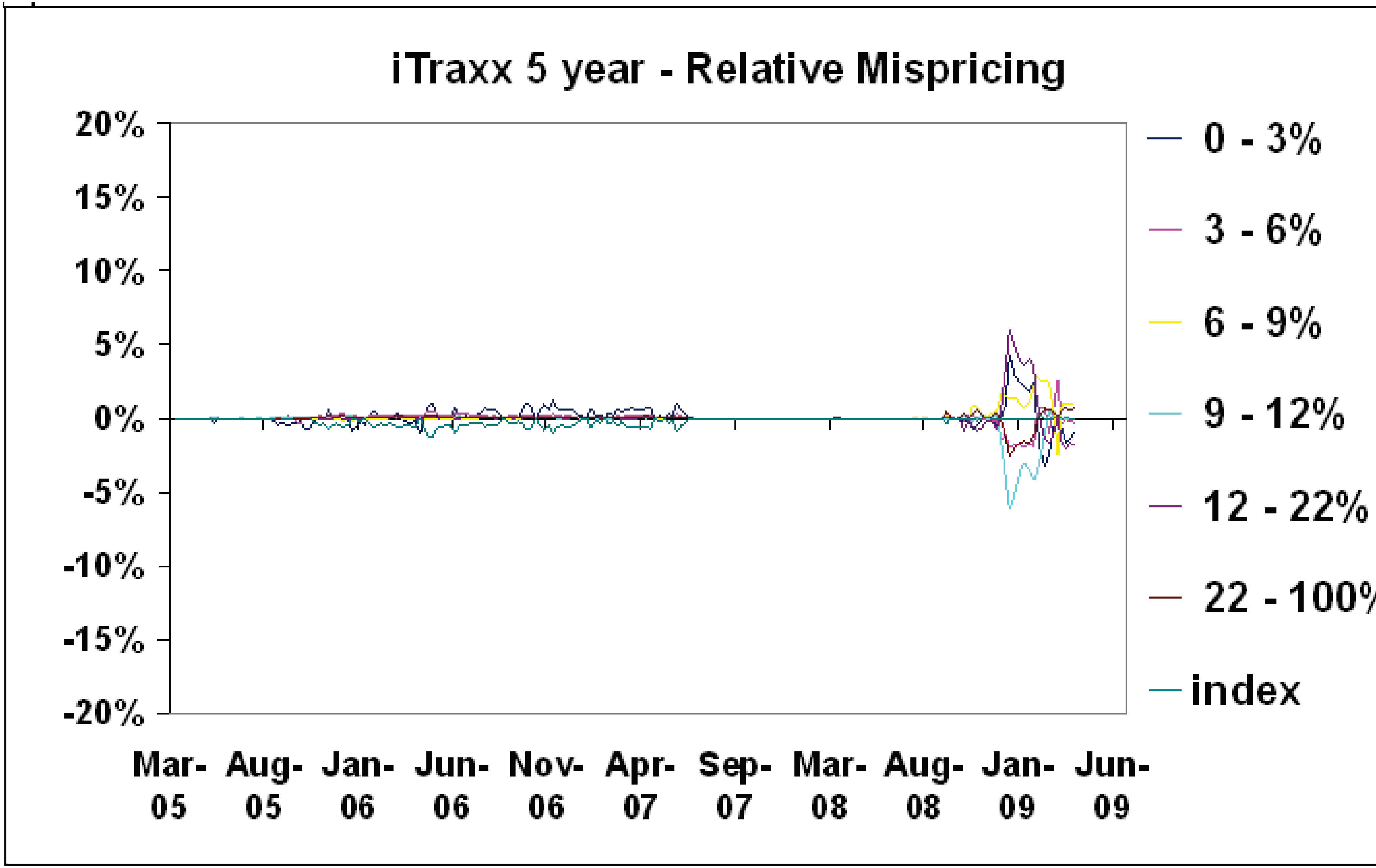
Calibration comments I

Sector / systemic calibration:

Notice the large portion of mass concentrated near the origin, the subsequent modes (default clusters) when moving along the loss distribution for increasing values, and the bumps in the far tail. Modes in the tail represent risk of default for large sectors. This is systemic risk as perceived by the dynamical model from the CDO quotes. With the crisis these probabilities have become larger, but they were already observable pre-crisis. Difficult to get this with parametric copula models.

History of calibration in-crisis with a different parametrization (α's fixed a priori):

Calibration comments II



Calibration in-crisis

A full treatment of the calibration in crisis and a model extension is given in the book ”Credit Models and the Crisis” by Brigo, Pallavicini and Torresetti (2010), Wiley.

The synthetic CDO case?

- We have illustrated how a complex situation in CDO markets has been trivialized by media and even regulators
- Models (such as base correlation) were indeed inadequate, but the industry and researchers had been looking for much more powerful and consistent alternatives
- We have seen the example of the GPL model, a fully consistent arbitrage free dynamic model for CDOs
- So why didn't the media pick this up? Why didn't the media realize the glitches they were signalling were the same the Wall Street Journal had reported years earlier in 2005?
- We hope the CDO case study illustrates the lack of rigour in a broad part of investigative journalism, especially in connection with complex and technical subjects.
- We cannot blame (even poor) modeling for policy, regulation, incentives, banking model, governance, lack of culture...
- We have a duty to make our research visible and heard to society

Is Maths Guilty and Wrong?

- Mathematics is not wrong. We have to be careful in understanding what is meant when saying that one uses *mathematical models*.
- Mathematical models are a simplification of reality, and as such, are always ”wrong”, even if they try to capture the salient features of the problem at hand.
- **”All models are wrong, but some models are useful”** (Prof. George E.P. Box)
- The core mathematical theory behind derivatives valuation is correct, but the assumptions on which the theory is based may not reflect the real world when the market evolves over the years.

Is Mathematics guilty?

- Although the models used in Credit Derivatives and counterparty risk have limits that have been highlighted before the crisis by several researchers, the ongoing crisis is due to factors that go well beyond any methodological inadequacy: the killer formula

$$\int_{-\infty}^{+\infty} \prod_{i=1}^{125} \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Lambda_i(T))) - \sqrt{\rho_i}m}{\sqrt{1-\rho_i}}\right) \varphi(m)dm.$$

Versus

The Crisis:
US real estate policy, Originate to Distribute (to Hold?) system fragility, volatile monetary policies,
myopic compensation and incentives system, lack of homogeneity in regulation, underestimation of liquidity risk, lack of data, fraud corrupted data...(Szegö 2009, The crash sonata in D major, JRMFI).

And what about the data?

Data and Inputs quality

For many financial products, and especially RMBS (Residential Mortgage Backed Securities), quite related to the asset class that triggered the crisis, the problem is in the data rather than in the models.

Risk of fraud

At times data for valuation in mortgages CDOs (RMBS and CDO of RMBS) can be distorted by fraud (see for example the FBI Mortgage fraud report, 2007, www.fbi.gov/publications/fraud/mortgage_fraud07.htm.

Pricing a CDO on this underlying:

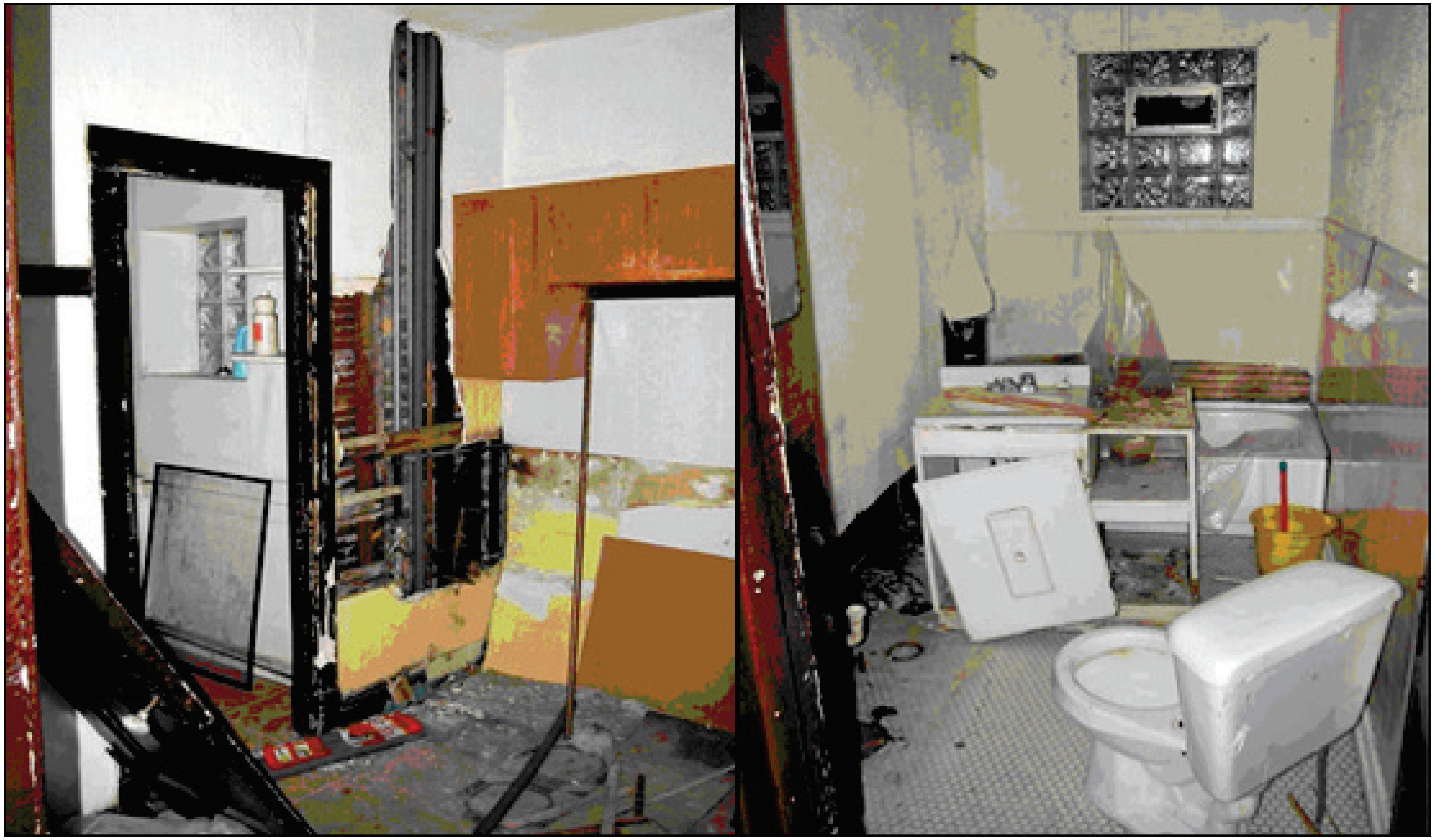


Figure: The above photos are from condos that were involved in a mortgage fraud. The appraisal described ”recently renovated condominiums” to include Brazilian hardwood, granite countertops, and a value of 275,000 USD

And what about the data?

At times it is not even clear what is in the portfolio: From the offering circular of a huge RMBS (more than 300.000 mortgages)

Type of property	% of Total
Detached Bungalow	2.65%
Detached House	16.16%
Flat	13.25%
Maisonette	1.53%
Not Known	2.49%
New Property	0.02%
Other	0.21%
Semi Detached Bungalow	1.45%
Semi Detached House	27.46%
Terraced House	34.78%
Total	100.00%

Mathematics or Magic?

All this is before modeling. Models obey a simple rule that is popularly summarized by the acronym GIGO (Garbage In $\rightarrow$ Garbage Out). As Charles Babbage (1791–1871) famously put it:

On two occasions I have been asked,
"Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?"
I am not able rightly to apprehend
the kind of confusion of ideas
that could provoke such a question.

So, in the end, how can the crisis be mostly due to models inadequacy, and to quantitative analysts and academics pride and unawareness of models limitations?

Interesting times...

We are indeed going through very interesting times. New derivatives are appearing, eg Longevity swaps, but there's much more beyond derivatives: *We need better models, not no models.*

We need to model risks that were absent/neglected in classical theory: Counterparty credit risk, liquidity risk, funding risk... Nonlinearities!

We need to understand systemic risk, contagion, the dynamics of dependence, and how to deal with scarcity of data and data proxying...

We need to enhance consistency of models in different areas

Optimal execution, algo trading, high freq trading, risk optimization...

All these areas, and many more, require quantitative input and good quantitative finance.

Interesting times...

This is not a good idea:

Rather then accusing mathematical finance for failures that are more managerial, political and behavioural in nature, we should derive better models that may account for the types of risks that had been neglected earlier.

But before doing that, we need to learn the classical theory pretty well.

... we need to learn the classical theory pretty well...

So let's get started

PART ONE: TERM STRUCTURE MODELS

In this part of the course we look at the classical theory of term structure models. No credit risk. No liquidity risk. No multiple curves. Just the classical theory. We'll look at the more modern aspects later.

Risk Neutral Valuation

Bank account $dB(t) = r_t B(t)dt$, $B(t) = B_0 \exp\left(\int_0^t r_s ds\right)$.

Risk neutral measure Q associated with numeraire B, $Q = Q^B$.

Recall shortly the risk-neutral valuation paradigm of Harrison et al (1983), generalizing the result of Black and Scholes we have seen above, characterizing no-arbitrage theory:

A future stochastic payoff H_T, built on an underlying fundamental asset, paid at a future time T and satisfying some technical conditions, has as unique price at current time t the *risk neutral world* expectation

$$E_t^B\left[\frac{B(t)}{B(T)}H(\text{Asset})_T\right] = E_t^Q\left[\exp\left(-\int_t^T r_s\, ds\right)H(\text{Asset})_T\right]$$

Risk neutral valuation I

$$E_t^B\left[\frac{B(t)}{B(T)}H(\text{Asset})_T\right] = E_t^Q\left[\exp\left(-\int_t^T r_s\, ds\right)H(\text{Asset})_T\right]$$

As we have seen above. "Risk neutral world" means that all fundamental underlying assets must have as locally deterministic drift rate the risk-free interest rate r:

$$d\ \text{Asset}_t = \boxed{r_t}\ \text{Asset}_t\ dt +$$

$$+\text{Asset-Volatility}_t\ (d\ \text{Brownian-motion-under-Q})_t$$

Nothing strange at first sight. To value **future unknown** quantities now, we discount at the relevant interest rate and then take **expectation**. The mean is a reasonable estimate of unknown quantities with known distributions.

Risk neutral valuation I

But what is surprising is that we do not take the mean in the **real world**, where statistics and econometrics based on the observed data are used. Indeed, in the real world probability measure P, we have

$$d\ \text{Asset}_t = \boxed{\mu_t}\ \text{Asset}_t\ dt +$$

$$+\text{Asset-Volatility}_t\ (d\ \text{Brownian-motion-under-P})_t.$$

But when we consider risk-neutral valuation, or no-arbitrage pricing, we do not use the real-world P-dynamics with μ but rather the risk-neutral world Q-dynamics with r.

We have a feeling for why this happens, since we derived the Black Scholes formula, a special case of the above framework, earlier. Basically we can avoid μ thanks to a replicating self-financing strategy in the underlying asset whose value does not depend on μ.

Risk neutral valuation II

From the risk neutral valuation formula we see that one fundamental quantity is r_t, the instantaneous interest rate.

As a very important special case of the general valuation formula, if we take $H_T = 1$, we obtain the Zero-Coupon Bond

Zero-coupon Bond, LIBOR rate I

A T–**maturity zero–coupon bond** is a contract which guarantees the payment of one unit of currency at time T. The contract value at time $t < T$ is denoted by $P(t, T)$:

$$P(T, T) = 1,$$

$$P(t, T) = E_t^Q \left[\frac{B(t)}{B(T)} 1 \right] = E_t^Q \exp\left(- \int_t^T r_s \, ds \right) = E_t^Q D(t, T)$$

All kind of rates can be expressed in terms of zero–coupon bonds and vice-versa. ZCB's can be used as fundamental quantities.
The **spot–Libor rate** at time t for the maturity T is the constant rate at which an investment has to be made to produce an amount of one unit

Zero-coupon Bond, LIBOR rate II

of currency at maturity, starting from $P(t, T)$ units of currency at time t, when accruing occurs **proportionally** to the investment time.

$$P(t,T)(1 + \ (T-t)\ L(t,T)) = 1, \quad L(t,T) = \frac{1 - P(t,T)}{(T-t)\ P(t,T)} \quad .$$

Notice:

$$r(t) = \lim_{T \to t^+} L(t,T) \approx L(t, t+\epsilon),$$

ϵ small.

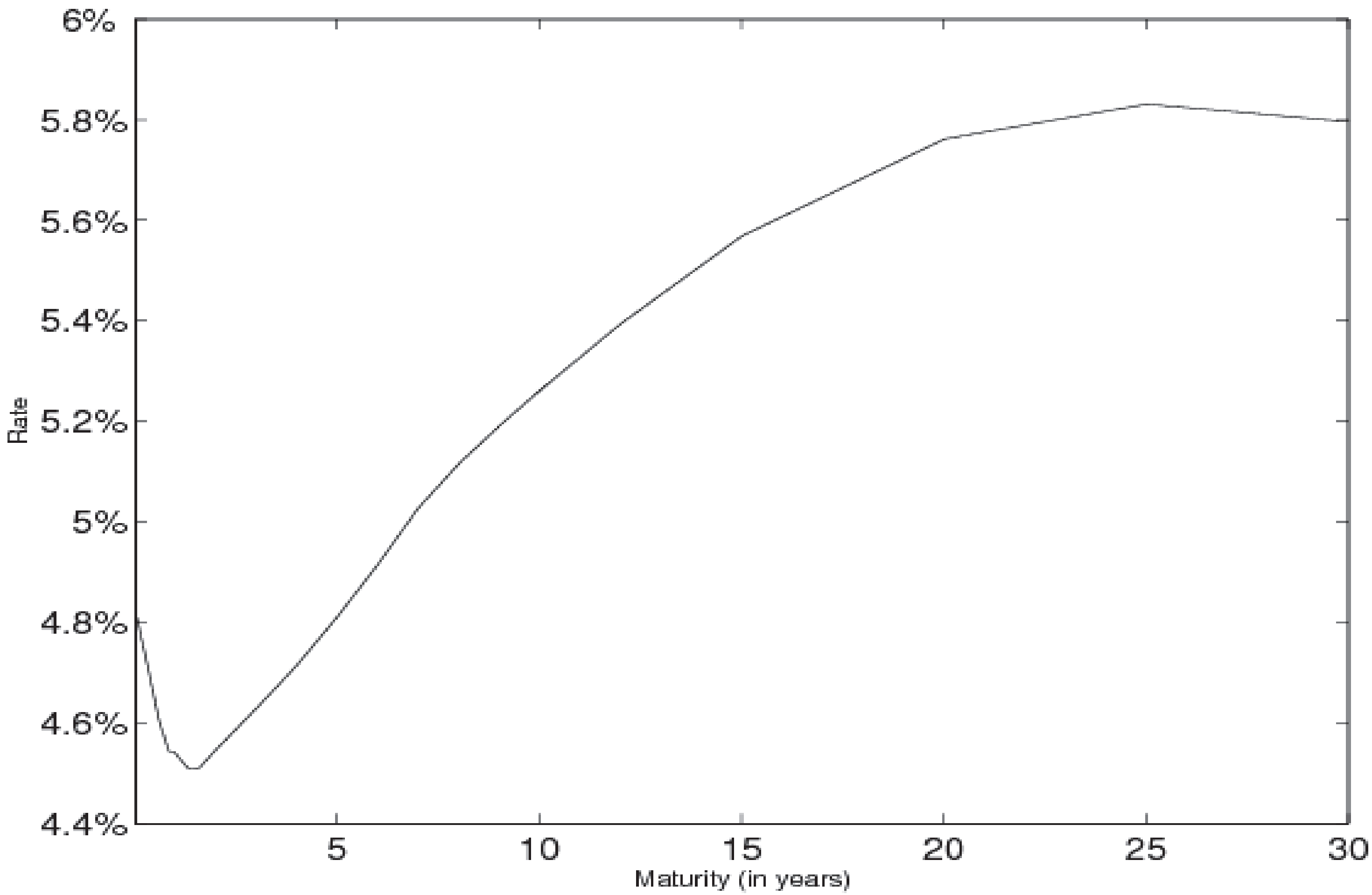
LIBOR, zero coupon curve (term structure) I

The **zero–coupon curve** (often referred to as "yield curve" or "term structure") at time t is the graph of the function

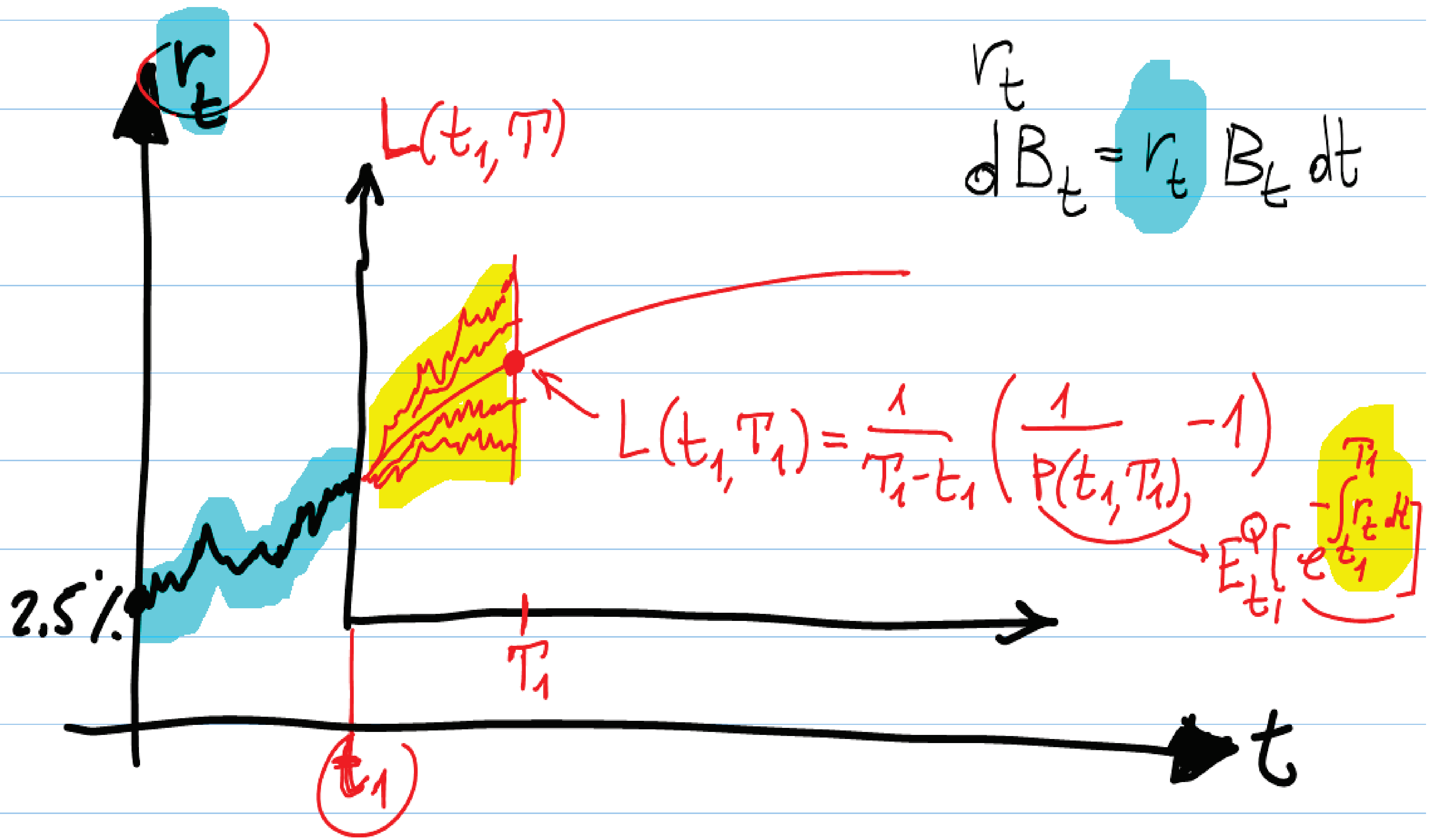
$$T \mapsto L(t, T), \quad \text{initial point} \quad r_t \approx L(t, t + \epsilon).$$

This function is called *term structure of interest rates* at time t.

Zero-coupon curve $T \mapsto L(t, t+T)$ stripped from market EURO rates on 13 Feb 2001 I



Zero-coupon curve $T \mapsto L(t, t+T)$ stripped from market EURO rates on 13 Feb 2001 II



LIBOR, zero coupon curve (term structure) I

This figure illustrates the different variables at play:

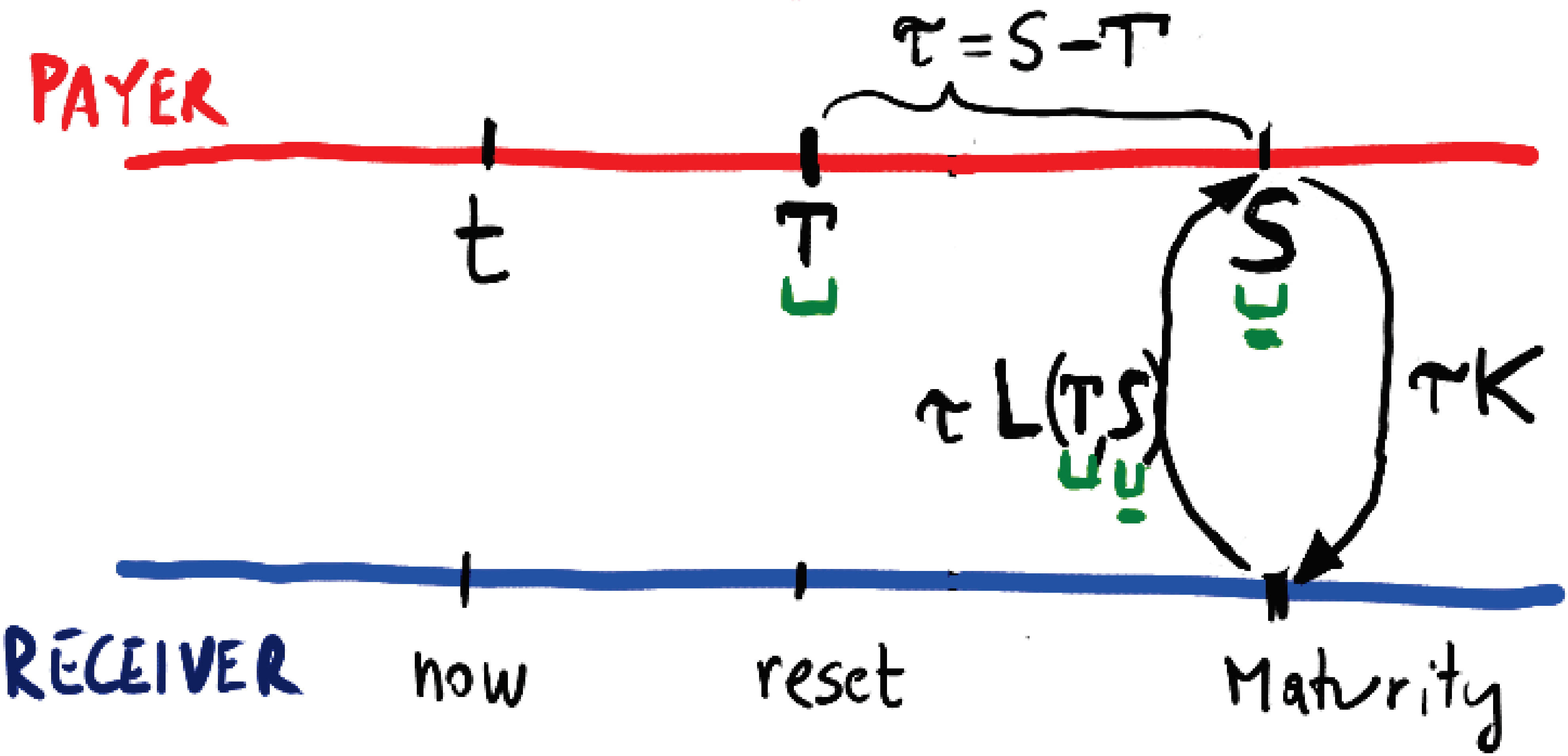
- the fundamental process is the short rate $t \mapsto r_t$. We show one path (in black, with a cyan contour) of the short rate r from time 0 (starting from $r_0 = 2.5\% = 0.025$) to t_1.
- Then at t_1 we show the term structure of interest rates $T \mapsto L(t_1, T)$ (in red), highlighting a point $L(t_1, T_1)$.
- As we have seen before, $L(t_1, T_1)$ is a function of $P(t_1, T_1)$ which, in turn, is $E_{t_1}[\exp(-\int_{t_1}^{T_1} r_t dt)]$.
- This means that the point $L(t_1, T_1)$ of the term structure is obtained through an expectation of an integral of every path of r from t_1 to T_1.
- Some of these paths are shown as zig-zagging lines in red from t_1 to T_1 in the picture.

Products not depending on the curve dynamics: FRA's and IRS's I

	At time S, with reset time T $(S > T)$	
Fixed payment	$\longrightarrow$ $(S-T)K$ $\longrightarrow$	
	$\longleftarrow$ $(S-T)\ L(T,S)$ $\longleftarrow$	Float. payment

A **forward rate agreement** FRA is a contract involving three time instants: The current time t, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest rate payment for the period $T \mapsto S$ with fixed rate K at maturity S against an interest rate payment over the same period with rate $L(T,S)$.
Basically, this contract allows one to lock–in the interest rate between T and S at a desired value K.

Products not depending on the curve dynamics: FRA



Products not depending on the curve dynamics: FRA I

The FRA is said to be a Receiver FRA if we pay floating $L(T,S)$ and receive Fixed K. It is a Payer FRA if we pay K and receive floating $L(T,S)$.
By easy static no-arbitrage arguments, the price of a receiver FRA is:

$$\mathrm{FRA}(t,T,S,K) = P(t,S)(S-T)K - P(t,T) + P(t,S)\ .$$

$(S-T)$ may be replaced by a year fraction τ. The price of a payer FRA is exactly the opposite, since cash flows go into the opposite direction.
The Proof is as follows.
The Receiver Fra Price is obtained by taking the risk neutral expectation of the FRA Discounted Cash Flows. As payments happen in S, we need to discount them back to t through $D(t,S)$.

$$\mathrm{FRA}(t,T,S,K) = E_t[D(t,S)\tau K - D(t,S)\tau L(T,S)] =$$

FRA Pricing: I

$$E_t[D(t,S)\tau K - D(t,S)\tau L(T,S)] =$$
$$= \tau K E_t[D(t,S)] - E_t[D(t,S)\tau L(T,S)] =$$
$$= \tau K P(t,S) - E_t[D(t,S)\tau L(T,S)] =$$

now use $D(t,S) = D(t,T)D(T,S)$ (ok for D, not for P)

$$= \tau K P(t,S) - E_t[\tau D(t,T)D(T,S)L(T,S)] =$$
$$= \tau K P(t,S) - E_t[E_T\{\tau D(t,T)D(T,S)L(T,S)\}] =$$
$$= \tau K P(t,S) - E_t[\tau D(t,T)L(T,S)E_T\{D(T,S)\}] =$$
$$= \tau K P(t,S) - E_t[\tau D(t,T)L(T,S)P(T,S)] =$$
$$= \tau K P(t,S) - E_t[D(t,T)P(T,S)(1/P(T,S) - 1)] =$$
$$= \tau K P(t,S) - E_t[D(t,T)] + E_t[D(t,T)P(T,S)] =$$
$$= \tau K P(t,S) - E_t[D(t,T)] + E_t[D(t,T)E_T[D(T,S)]] =$$
$$= \tau K P(t,S) - E_t[D(t,T)] + E_t[E_T[D(t,T)D(T,S)]] =$$
$$= \tau K P(t,S) - E_t[D(t,T)] + E_t[E_T[D(t,S)]] =$$
$$= \tau K P(t,S) - E_t[D(t,T)] + E_t[D(t,S)] =$$
$$= \tau K P(t,S) - P(t,T) + P(t,S)$$

FRA Pricing I

Note that this derivation did not require any modeling assumptions. We have made no assumption on the dynamics of interest rates. We have only used very general no-arbitrage principles to derive this formula.

The value of K which makes the contract fair (=0) is the **forward LIBOR interest rate** prevailing at time t for the expiry T and maturity S: $K = F(t; T, S)$. This is derived by solving in K

$$\tau K P(t, S) - P(t, T) + P(t, S) = 0.$$

$$K = F(t; T, S) := \frac{1}{S - T}\left(\frac{P(t, T)}{P(t, S)} - 1\right) .$$

Notice that incidentally we have found, with the above derivation, that

$$E_t[D(t, S)L(T, S)] = P(t, S)F(t, T, S).$$

Are Forward rates expectations of future interest rates? I

It is important to notice that while

$$\mathbb{E}_t^Q[D(t,S)L(T,S)] = P(t,S)F(t,T,S),$$

we also have

$$\mathbb{E}_t^Q[L(T,S)] \neq F(t,T,S).$$

The second one would follow from the first one only if D and L were independent. Clearly this is not the case. We will be able to write

$$\mathbb{E}_t^{Q^S}[L(T,S)] = F(t,T,S)$$

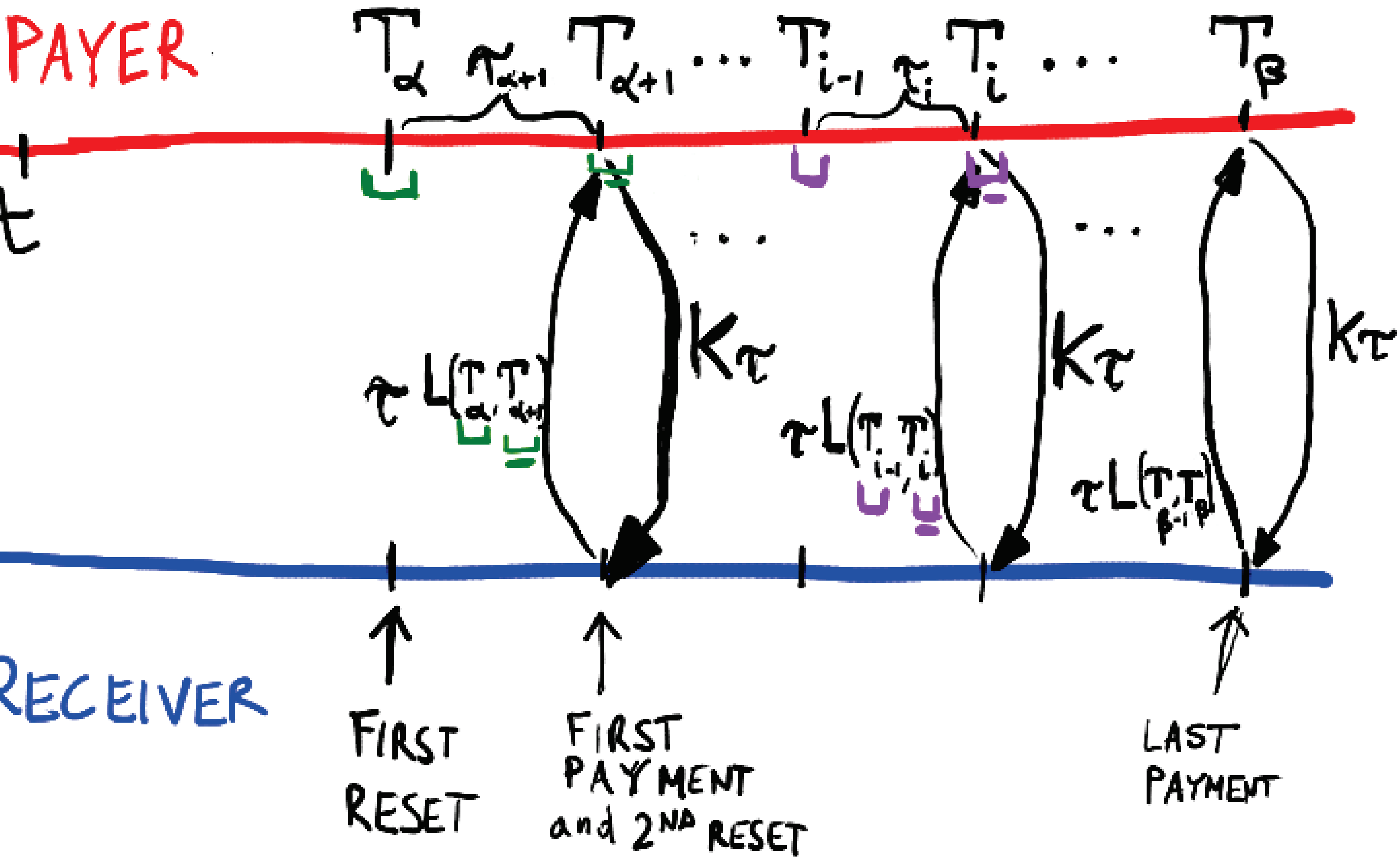
only under a different probability measure Q^S, called S forward measure. We'll deal with this later in the course.

Products not depending on the curve dynamics: IRS I

An Interest Rate Swap (PFS) is a contract that exchanges payments between two differently indexed legs, starting from a future time–instant. At future dates $T_{\alpha+1}, ..., T_\beta$,

$$
\begin{array}{ccc}
 & \longrightarrow \;\; \tau_j K \;\; \longrightarrow & \\
\text{at } T_j: \quad \text{Fixed Leg} & & \text{Float. Leg} \\
 & \longleftarrow \;\; \tau_j \;\; L(T_{j-1}, T_j) \;\; \longleftarrow & \\
 & \text{or taking } \; E_{T_\alpha}[\cdot]: & \\
 & \tau_j \;\; F(T_\alpha; T_{j-1}, T_j) &
\end{array}
$$

where $\tau_i = T_i - T_{i-1}$. The IRS is called “payer IRS” from the company paying K and “receiver IRS” from the company receiving K.


PAYER
t
RECEIVER
FIRST RESET
FIRST PAYMENT and 2ND RESET
LAST PAYMENT
Kτ
Kτ
Kτ

Products not depending on the curve dynamics: IRS I

The *discounted* payoff at a time $t < T_\alpha$ of a receiver IRS is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)\, \tau_i (K - L(T_{i-1}, T_i)), \quad \text{or alternatively}$$

we may proceed as follows. (i) value the swap at the future first reset T_α. (ii) Take the T_α IRS price, which is a random payoff when seen from t, and dicount it back at t. This will help later with swaptions and this is why we do this. We obtain

$$D(t, T_\alpha) E_{T_\alpha} [\sum_{i=\alpha+1}^{\beta} D(T_\alpha, T_i)\, \tau_i (K - L(T_{i-1}, T_i))] =$$

$$= D(t, T_\alpha) \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\, \tau_i (K - F(T_\alpha; T_{i-1}, T_i)).$$

Products not depending on the curve dynamics: FRA's and IRS's I

Now rather than taking risk neutral expectations and going through the calculations, we simply note that IRS can be valued as a collection of FRAs. In particular, a receiver IRS can be valued as a collection of (receiver) FRAs.

$$\text{ReceiverIRS}(t, [T_\alpha, \ldots, T_\beta], K) = \sum_{i=\alpha+1}^{\beta} \text{FRA}(t, T_{i-1}, T_i, K) =$$

$$= \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) - P(t, T_\alpha) + P(t, T_\beta), \quad \text{or alternatively}$$

$$= \sum_{i=\alpha+1}^{\beta} P(t, T_i)\, \tau_i (K - F(t; T_{i-1}, T_i)).$$

Products not depending on the curve dynamics: FRA's and IRS's II

Analogously,

$$\text{PayerIRS}(t, [T_\alpha, \ldots, T_\beta], K) =$$

$$= \sum_{i=\alpha+1}^{\beta} P(t, T_i)\, \tau_i (F(t; T_{i-1}, T_i) - K), \quad \text{or alternatively}$$

$$= -\sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) + P(t, T_\alpha) - P(t, T_\beta).$$

The value $K = S_{\alpha,\beta}(t)$ which makes
$\text{IRS}(t, [T_\alpha, \ldots, T_\beta], K) = 0$
is the **forward swap rate.**
Denote $F_i(t) := F(t; T_{i-1}, T_i)$.

Products not depending on the curve dynamics: FRA's and IRS's I

Three possible formulas for the forward swap rate:

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)}$$

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t), \quad w_i(t) = \frac{\tau_i P(t,T_i)}{\sum_{j=\alpha+1}^{\beta} \tau_j P(t,T_j)}$$

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}} .$$

The second expression is a “weighted” average: $0 \le w_i \le 1$, $\sum_{j=\alpha+1}^{\beta} w_j = 1$. The weights are functions of the F's and thus random at future times.

Products not depending on the curve dynamics: FRA's and IRS's I

Recall the Receiver IRS Formula

$$\text{ReceiverIRS}(t, [T_\alpha, \dots, T_\beta], K) =$$
$$= \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) - P(t, T_\alpha) + P(t, T_\beta)$$

and combine it with

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}$$

Products not depending on the curve dynamics: FRA's and IRS's II

to obtain

$$\text{ReceiverIRS}(t, [T_\alpha, \ldots, T_\beta], K) =$$

$$= (K - S_{\alpha,\beta}(t)) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$$

Analogously,

$$\text{PayerIRS}(t, [T_\alpha, \ldots, T_\beta], K) =$$

$$= (S_{\alpha,\beta}(t) - K) \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$$

Products depending on the curve dynamics: Caplets and CAPS I

A **cap** can be seen as a payer IRS where each exchange payment is executed only if it has positive value.

$$\text{Cap discounted payoff:} \quad \sum_{i=\alpha+1}^{\beta} D(t, T_i)\, \tau_i (L(T_{i-1}, T_i) - K)^+ \; .$$

$$= \sum_{i=\alpha+1}^{\beta} D(t, T_i)\, \tau_i (F_i(T_{i-1}) - K)^+ \; .$$

Suppose a company is Libor–indebted and has to pay at $T_{\alpha+1}, \ldots, T_\beta$ the Libor rates resetting at $T_\alpha, \ldots, T_{\beta-1}$.

Products depending on the curve dynamics: Caplets and CAPS II

The company has a view that libor rates will increase in the future, and wishes to protect itself

$$\text{buy a cap: } (L-K)^+ \longrightarrow^{CAP} \text{Company} \longrightarrow^{DEBT} L$$

$$\text{or Company} \longrightarrow^{NET} L-(L-K)^+ = \min(L,K)$$

The company pays at most K at each payment date.
A cap contract can be decomposed additively: Indeed, the discounted payoff is a sum of terms (**caplets**)

$$D(t,T_i)\,\tau_i(L(T_{i-1},T_i)-K)^+$$

$$= D(t,T_i)\,\tau_i(F_i(T_{i-1})-K)^+ .$$

Products depending on the curve dynamics: Caplets and CAPS III

Each caplet can be evaluated separately, and the corresponding values can be added to obtain the cap price (notice the "call option" structure!).
However, even if separable, the payoff is not linear in the rates. This implies that, roughly speaking, we need the whole distribution of future rates, and not just their means, to value caplets. This means that the dynamics of interest rates is needed to value caplets: We cannot value caplets at time t based only on the current zero curve $T \mapsto L(t, T)$, but we need to specify how this infinite-dimensional object moves, in order to have its distribution at future times. This can be done for example by specifying how r moves.

Products depending on the curve dynamics: Floors

A **floor** can be seen as a receiver IRS where each exchange payment is executed only if it has positive value.

$$\text{Floor discounted payoff:} \quad \sum_{i=\alpha+1}^{\beta} D(t,T_i)\,\tau_i(K - L(T_{i-1},T_i))^+ \,.$$

$$= \sum_{i=\alpha+1}^{\beta} D(t,T_i)\,\tau_i(K - F_i(T_{i-1}))^+ \,.$$

The floor price is the risk neutral expectation E of the above discounted payoff.

Products depending on the curve dynamics: SWAPTIONS I

Finally, we introduce options on IRS's **(swaptions)**.
A (payer) swaption is a contract giving the right to enter at a future time a (payer) IRS.
The time of possible entrance is the maturity.
Usually maturity is first reset of underlying IRS.
IRS value at its first reset date T_α, i.e. at maturity, is, by our above formulas,

$$\text{PayerIRS}(T_\alpha, [T_\alpha, \ldots, T_\beta], K) =$$

$$= \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\, \tau_i (F(T_\alpha; T_{i-1}, T_i) - K) =$$

$$= (S_{\alpha,\beta}(T_\alpha) - K) \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)$$

Products depending on the curve dynamics: SWAPTIONS II

Call $C_{\alpha,\beta}(T_\alpha)$ the summation on the right hand side.
The option will be excercised only if this IRS value is positive. There results the payer–swaption discounted–payoff at time t:

$$D(t, T_\alpha)C_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha) - K)^+ =$$

$$D(t, T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\, \tau_i(F(T_\alpha; T_{i-1}, T_i) - K)\right)^+ .$$

Unlike Caps, this payoff **cannot be decomposed** additively.
Caps can be decomposed in caplets, each with a single fwd rate.
Caps: Deal with each caplet **separately**, and put results together.
Only **marginal** distributions of different fwd rates are involved.
Not so with swaptions: The summation is *inside* the positive part operator $()^+$, and not outside.

Products depending on the curve dynamics: SWAPTIONS III

With swaptions we will need to consider the *joint* action of the rates involved in the contract.
The **correlation** between rates is fundamental in handling swaptions, contrary to the cap case.

Which variables do we model? I

For some products (Forward Rate Agreements, Interest Rate Swaps) the **dynamics** of interest rates is not necessary for valuation, the current curve being enough.
For caps, swaptions and more complex derivatives a dynamics is necessary.
Specifying a stochastic dynamics for interest rates amounts to choosing an **interest-rate model**.

- Which quantities do we model? Short rate r_t? LIBOR rates $L(t, T)$? Forward LIBOR rates $F_i(t) = F(t; T_{i-1}, T_i)$? Forward Swap rates $S_{\alpha,\beta}(t)$? Bond Prices $P(t, T)$?
- How is randomness modeled? i.e: What kind of stochastic process or stochastic differential equation do we select for our model? (Markov diffusions)

Which variables do we model? II

- What are the consequences of our choice in terms of valuation of market products, ease of implementation, goodness of calibration to real data, pricing complicated products with the calibrated model, possibilities for diagnostics on the model outputs and implications, stability, robustness, etc?

First Choice: short rate r I

This approach is based on the fact that the zero coupon curve at any instant, or the (informationally equivalent) zero bond curve

$$T \mapsto P(t,T) = E_t^Q \exp\left(-\int_t^T \boxed{r_s}\, ds\right)$$

is completely characterized by the probabilistic/dynamical properties of r.

So we write a model for r, the initial point of the curve $T \mapsto L(t,T)$ for $T = t$ at every instant t.

Typically a stochastic differential equation for r is chosen.

$$d\ r_t = \text{local\_mean}(t, r_t)dt +$$

$$+\text{local\_standard\_deviation}(t, r_t) \times \boxed{\text{stochastic\_change}_t}$$

First Choice: short rate r II

which we write

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)\boxed{dW_t}$$

The local mean b is called the “drift” and the local standard deviation σ is the “diffusion coefficient”

First Choice: short rate r I

Dynamics of $r_t = x_t$ under the risk–neutral-world probability measure

1. **Vasicek (1977):**

$$dx_t = k(\theta - x_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma).$$

2. **Cox-Ingersoll-Ross (CIR, 1985):**

$$dx_t = k(\theta - x_t)dt + \sigma \sqrt{x_t} dW_t, \quad \alpha = (k, \theta, \sigma), \quad 2k\theta > \sigma^2 \,.$$

3. **Dothan / Rendleman and Bartter:**

$$dx_t = a x_t dt + \sigma x_t dW_t, \; (x_t = x_0 \; e^{(a - \frac{1}{2}\sigma^2)t + \sigma W_t}, \; \alpha = (a, \sigma)).$$

4. **Exponential Vasicek:**

$$x_t = \exp(z_t), \quad dz_t = k(\theta - z_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma).$$

Every different choice has important consequences.

First Choice: short rate r. Example: Vasicek I

$$dx_t = k(\theta - x_t)dt + \sigma dW_t, \quad r_t = x_t.$$

The Vasicek model has some peculiarities that make it attractive.
The equation is linear and can be solved explicitly.
Joint distributions of many important quantities are Gaussian. Many formula for prices (i.e. expectations)
The model is mean reverting: The expected value of the short rate tends to a constant value θ with velocity depending on k as time grows towards infinity, while its variance does not explode.
However, this model features also some drawbacks.
Rates can assume negative values with positive probability.
Gaussian distributions for the rates are not compatible with the market implied distributions.

First Choice: short rate r. Example: Vasicek II

The choice of a particular dynamics has several important consequences, which must be kept in mind when designing or choosing a particular short-rate model. A typical comparison is for example with the Cox Ingersoll Ross (CIR) model.

First Choice: short rate r. Example: CIR I

$$dy(t) = \kappa[\mu - y(t)]dt + \nu \boxed{\sqrt{y(t)}} dW(t), \quad r_t = y_t$$

For the parameters κ, μ and ν ranging in a reasonable region, this model implies **positive** interest rates, but the instantaneous rate is characterized by a **noncentral chi-squared distribution**.

The model is mean reverting: The expected value of the short rate tends to a constant value μ with velocity depending on κ as time grows towards infinity, while its variance does not explode.

This model maintains a certain degree of analytical tractability, but is **less tractable** than Vasicek, especially as far as the extension to the multifactor case with correlation is concerned

CIR is usually closer to market implied distributions of rates than Vasicek.

First Choice: short rate *r*. Example: CIR II

Therefore, the CIR dynamics has both some advantages and disadvantages with respect to the Vasicek model.

CIR and Vasicek models: some intuition I

The parameters of the CIR model are similar to those of the Vasicek model in terms of interpretation.

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW_t$$

κ: speed of mean reversion
μ: long term mean reversion level
ν: volatility.

CIR model I

$$E[y_t] = y_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$$

$$\text{VAR}(y_t) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa t})^2$$

After a long time the process reaches (asymptotically) a stationary distribution around the mean μ and with a corridor of variance $\mu\nu^2/2\kappa$. The largest κ, the fastest the process converges to the stationary state. So, ceteris paribus, increasing κ kills the volatility of the interest rate. The largest μ, the highest the long term mean, so the model will tend to higher rates in the future in average.
The largest ν, the largest the volatility. Notice however that κ and ν fight each other as far as the influence on volatility is concerned. We see some plots of scenarios now

CIR model II

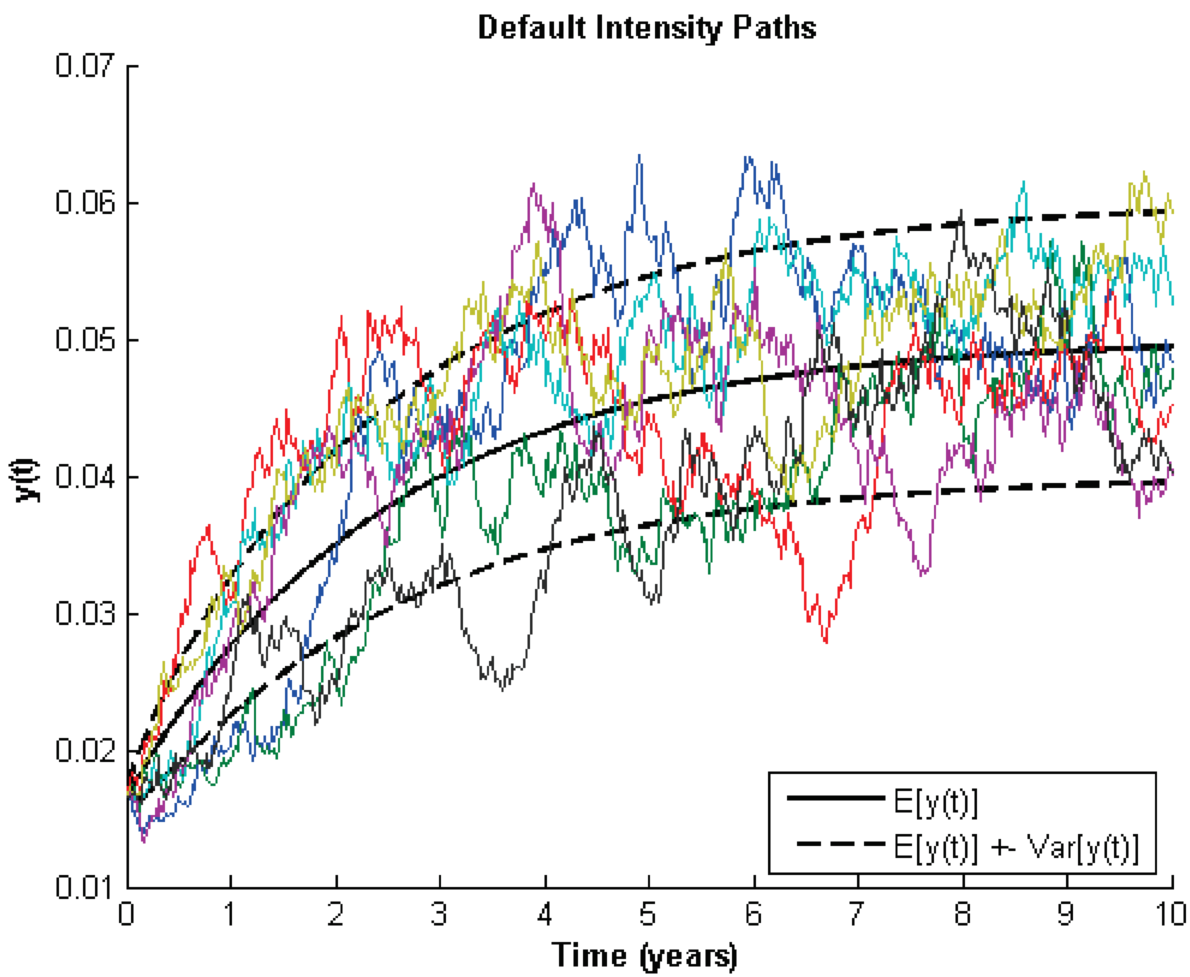


Figure: $y0 = 0.0165, \kappa = 0.4, \mu = 0.05, \nu = 0.04$

Case Study: Vasicek I

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \quad \alpha = (k, \theta, \sigma).$$

Compute

$$d[e^{kt} r_t] = ke^{kt} r_t dt + e^{kt} dr_t = \ldots = e^{kt}[k\theta \, dt + \sigma dW_t]$$

Integrating both sides between s and t we obtain, for each $s \leq t$,

$$e^{kt} r_t - e^{ks} r_s = \int_s^t e^{ku} k\theta \, du + \int_s^t e^{ku} \sigma dW_u$$

Now, multiplying both sides by e^{-kt} we get

$$r(t) = r(s)e^{-k(t-s)} + \theta \left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)} dW(u), \qquad (18)$$

Case Study: Vasicek II

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma\int_s^t e^{-k(t-u)}dW(u), \qquad (19)$$

so that $r(t)$ conditional on r_s is normally distributed with mean and variance given respectively by

$$E\{r(t)|r_s\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\mathrm{VAR}\{r(t)|r_s\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right].$$

(Ito isometry: for deterministic $v(t)$ we have
$\mathrm{VAR}(\int v(u)dW_u) = E[(\int v(u)dW_u)^2] = \int v(u)^2 du)$
This implies that, for each time t, the rate $r(t)$ can be negative with positive probability.

Case Study: Vasicek III

$$E\{r(t)|r_s\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\mathrm{VAR}\{r(t)|r_s\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right],$$

and r is normally distributed. The possibility of negative rates is indeed a major drawback of the Vasicek model. However, the analytical tractability that is implied by a Gaussian density is hardly achieved when assuming other distributions for r.

The short rate r is mean reverting, since the expected rate tends, for t going to infinity, to the value θ.

The price of a pure-discount bond can be derived by computing the expectation $P(t,T) = E_t \exp(-\int_t^T r_u du)$.

Notice: The integral in the exponent is Gaussian since r is Gaussian. Its mean and variance can be computed from

Case Study: Vasicek IV

$$E_t \int_t^T r_u du = \int_t^T E_t[r_u] du =$$

$$\int_t^T [r(t) e^{-k(u-t)} + \theta \left(1 - e^{-k(u-t)}\right)] du = \dots$$

$$E_t \left[\left(\int_t^T r_u du \right)^2 \right] = E_t \left[\int_t^T \int_t^T r_u r_v du dv \right] = \int_t^T \int_t^T E_t[r_u r_v] du dv =$$

$$= \int_t^T \int_t^T E_t \left\{ \left[r_t e^{-k(u-t)} + \theta(1 - e^{-k(u-t)}) + \sigma \int_t^u e^{-k(u-z)} dW_z \right] \right.$$

$$\left. \left[r_t e^{-k(v-t)} + \theta(1 - e^{-k(v-t)}) + \sigma \int_t^v e^{-k(v-z)} dW_z \right] \right\} du dv$$

...

Case Study: Vasicek V

This can be computed by using the isometry

$$E\left[\int_t^u f(z)dW_z \int_t^v g(z)dW_z\right] = \int_t^{\min(u,v)} f(z)g(z)dz.$$

One obtains (moment generating function of a Gaussian)

$$X := -\int_t^T r_u du \sim \mathcal{N}(M, V^2),$$

$$P(t,T) = E[e^X] = \exp(M + V^2/2)$$

By completing the (now trivial) computations we have

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

$$A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\right\}$$

Case Study: Vasicek VI

$$B(t,T) = \frac{1}{k}\left[1 - e^{-k(T-t)}\right].$$

Put Option on a S-maturity Zero coupon bond. Payoff at T (discounted back at t)

$$\exp\left(-\int_t^T r_u du\right)(X - P(T,S))^+$$

The price at time t of a European option with strike X, maturity T and written on a pure discount bond maturing at time S is the risk neutral expectation of the above quantity, and is denoted by ZBP(t,T,S,X). Here is how one can compute it.

Case Study: Vasicek. Bond Option I

$$E_t \left[\exp\left(-\int_t^T r_u du \right) (X - P(T,S))^+ \right]$$

Recall:

$$r(T) = r(t)e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right) + \sigma \int_t^T e^{-k(T-u)} dW(u),$$

and $P(T,S) = A(T,S)e^{-B(T,S)r(T)}$. Moreover, integrating both sides of $dr = k(\theta - r)dt + \sigma dW$ we get

$$-\int_t^T r_u du = (r_T - r_t)/k - \theta(T-t) - (\sigma/k)\int_t^T dW_u.$$

Case Study: Vasicek. Bond Option II

The above expectation depends only on the random vector

$$\left[\int_t^T dW(u), \int_t^T e^{-k(T-u)} dW(u)\right]$$

which is normally distributed (isometry)

$$\mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} T-t & (1-e^{-k(T-t)})/k \\ . & (1-e^{-2k(T-t)})/(2k) \end{bmatrix}\right),$$

$$E_t\left[\exp\left(-\int_t^T r_u du\right)(X - P(T,S))^+\right]$$

$$= E_t\left[e^{aY_2+bY_1+c}\left(X - \alpha e^{\gamma Y_2}\right)^+\right]$$

Case Study: Vasicek. Bond Option III

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} T-t & (1-e^{-k(T-t)})/k \\ . & (1-e^{-2k(T-t)})/(2k) \end{bmatrix} \right),$$

so that we know how to compute the expectation explicitly. One obtains, after a lot of computations (but there are easier ways)

$$\text{ZBP}(t,T,S,X) = \left[XP(t,T)\Phi(\sigma_p - h) - P(t,S)\Phi(-h) \right],$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function, and

$$\sigma_p = \sigma \sqrt{\frac{1-e^{-2k(T-t)}}{2k}} B(T,S), \quad h = \frac{1}{\sigma_p} \ln \frac{P(t,S)}{P(t,T)X} + \frac{\sigma_p}{2}.$$

Case Study: Vasicek. Caplet I

A caplet can be seen as a put option on a zero bond.
If N is the notional amount, and $\tau = S - T$, we have

$$\mathrm{Cpl}(t, T, S, X, N) = E\left(e^{-\int_t^S r_s ds} N\tau(L(T,S) - X)^+ | \mathcal{F}_t\right)$$

$$= E\left(E\left[e^{-\int_t^S r_s ds} N\tau(L(T,S) - X)^+ | \mathcal{F}_T\right] | \mathcal{F}_t\right)$$

$$= E\left(E\left[e^{-\int_t^T r_s ds} e^{-\int_T^S r_s ds} N\tau(L(T,S) - X)^+ | \mathcal{F}_T\right] | \mathcal{F}_t\right)$$

$$= E\left(e^{-\int_t^T r_s ds} E\left[e^{-\int_T^S r_s ds} | \mathcal{F}_T\right] N\tau(L(T,S) - X)^+ | \mathcal{F}_t\right)$$

$$= NE\left(e^{-\int_t^T r_s ds} P(T,S)\tau(L(T,S) - X)^+ | \mathcal{F}_t\right),$$

Case Study: Vasicek. Caplet II

where we used iterated conditioning. Using the definition of the LIBOR rate $L(T,S)$, we obtain

$$= NE\left(e^{-\int_t^T r_s ds} P(T,S)\left[\frac{1}{P(T,S)} - 1 - X\tau\right]^+ |\mathcal{F}_t\right)$$

$$= N(1+X\tau)E\left(e^{-\int_t^T r_s ds}[1/(1+X\tau) - P(T,S)]^+|\mathcal{F}_t\right),$$

We have thus seen that a caplet can be expressed as a put option on a bond, for which we derived a formula earlier.

$$\text{Cpl}(t,T,S,X,N) = N(1+X\tau)\text{ZBP}(t,T,S,1/(1+X\tau))$$

Case Study: Vasicek. Summary I

In Vasicek's model we can:

- Solve explicitly the SDE for r

$$dr_t = k(\theta - r_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma)$$

  because it is **linear**, and find the **normal** distribution of r;
- Find the price of a bond $P(t,T) = P(t,T;\alpha;r_t)$ thanks to the fact that adding up jointly normal variables one obtains a **normal** random variable, so that $\int r_s ds$ is normal;
- Find the price of a put option on a zero coupon bond

$$\mathrm{ZBP}(t,T,S,X) = \mathrm{ZBP}(t,T,S,X;\alpha,r_t)$$

  by means of the expectation of a certain random variable based on a **bivariate normal** distribution coming from properties of Brownian motions;

Case Study: Vasicek. Summary II

- Find the price of a caplet

$$\mathrm{Cpl}(t, T, S, X, N) = N(1 + X\tau)\mathrm{ZBP}((t, T, S, 1/(1 + \tau X); \alpha, r_t)$$

as a price of a zero-bond put option thanks to **iterated conditioning** (property of conditional expectations).

Even this simple example shows that in order to price financial products one needs to master probability and statistics. Also, **analytical tractability** is often related to **linearity** and **normality**.

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation I

We can consider the objective measure Q_0-dynamics of the Vasicek model as a process of the form

$$dr(t) = [k\,\theta - (k + \lambda\,\sigma)r(t)]dt + \sigma dW^0(t), \quad r(0) = r_0 \ ,$$

where λ is a new parameter, contributing to the market price of risk. Compare this Q_0 dynamics to the risk-neutral Q-dynamics

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t), \quad r(0) = r_0 \ .$$

Notice that for $\lambda = 0$ the two dynamics coincide. More generally, the above Q_0-dynamics is expressed again as a linear Gaussian stochastic differential equation, although it depends on the new parameter λ.

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation II

Requiring that the dynamics be of the same nature under the two measures (linear-Gaussian), imposes a Girsanov change of measure:

$$\left.\frac{dQ}{dQ_0}\right|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \lambda^2\, r(s)^2 ds + \int_0^t \lambda\, r(s) dW^0(s)\right)$$

although λ has to be assumed to be constant and not depending on r, which is not true in general. However, under this choice we obtain a short rate process that is tractable under both measures.

Important: In traditional finance, one first postulates a dynamics under the objective measure Q^0, and then writes the risk neutral dynamics by adding one or more parameters. For example, one would write

$$dr(t) = k(\theta - r(t))dt + \sigma dW^0(t), \quad r(0) = r_0 \ .$$

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation III

under the objective measure Q^0 and then

$$dr(t) = [k\,\theta - (k - \lambda\,\sigma)r(t)]dt + \sigma dW(t), \quad r(0) = r_0$$

under the risk neutral measure.
We did the contrary because in pricing practice one starts from the risk neutral dynamics first.

$$dr(t) = [k\,\theta - (k + \lambda\,\sigma)r(t)]dt + \sigma dW^0(t)$$

(Statistics, historical estimation, econometrics).

$$dr(t) = k(\theta - r(t))dt + \sigma dW(t)$$

(Pricing, risk neutral valuation).

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation IV

It is clear why tractability under the risk-neutral measure is a desirable property: claims are priced under that measure, so that the possibility to compute expectations in a tractable way with the Q-dynamics is important. Yet, why do we find it desirable to have a tractable dynamics under Q_0 too?

If we are provided with a series $r_0, r_1, r_2, \ldots, r_n$ of daily observations of a proxy of $r(t)$ (say a monthly rate, $r(t) \approx L(t, t + 1m)$), and we wish to incorporate information from this series in our model, we can estimate the model parameters on the basis of this daily series of data.

However, data are collected in the real world, and their statistical properties characterize the distribution of our interest-rate process $r(t)$ under the objective measure Q_0. Therefore, what is to be estimated from historical observations is the Q_0 dynamics. The estimation

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation V

technique can provide us with estimates for the objective parameters k, λ, θ and σ, or more precisely for combinations thereof.
If we are provided with a series $r_0, r_1, r_2, \ldots, r_n$ of daily observations of a proxy of $r(t)$, their statistical properties characterize the distribution of our interest-rate process $r(t)$ under the objective measure Q_0.
Therefore, what is to be estimated from historical observations is the Q_0 dynamics, with the objective parameters k, λ, θ and σ.
On the other hand, prices are computed through expectations under the risk-neutral measure. When we observe prices, we observe expectations under the measure Q. Therefore, when we calibrate the model to derivative prices we need to use the Q dynamics, thus finding the parameters k, θ and σ involved in the Q-dynamics and reflecting current market prices of derivatives.

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation VI

We could then combine the two approaches. For example, since the diffusion coefficient is the same under the two measures, we might estimate σ from historical data through a maximum-likelihood estimator, while finding k and θ through calibration to market prices. However, this procedure may be necessary when very few prices are available. Otherwise, it might be used to deduce historically a σ which can be used as initial guess when trying to find the three parameters that match the market prices of a given set of instruments.
Maximum-likelihood estimator for the Vasicek model. Write

$$dr(t) = [b - ar(t)]dt + \sigma dW^0(t),$$

with b and a suitable constants.

$$r(t) = r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-u)} dW^0(u).$$

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation VII

Given $\mathcal{F}_s$ the variable $r(t)$ is normally distributed with mean $r(s)e^{-a(t-s)} + \frac{b}{a}(1 - e^{-a(t-s)})$ and variance $\frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})$.
It is natural to estimate the following functions of the parameters: $\beta := b/a$, $\alpha := e^{-a\delta}$ and $V^2 = \frac{\sigma^2}{2a}(1 - e^{-2a\delta})$, where δ denotes the time-step of the observed proxies. The maximum likelihood estimators for α, β and V^2 are easily derived as

$$\widehat{\alpha} = \frac{n\sum_{i=1}^n r_i r_{i-1} - \sum_{i=1}^n r_i \sum_{i=1}^n r_{i-1}}{n\sum_{i=1}^n r_i^2 - \left(\sum_{i=1}^n r_{i-1}\right)^2}, \quad \widehat{\beta} = \frac{\sum_{i=1}^n [r_i - \widehat{\alpha} r_{i-1}]}{n(1-\widehat{\alpha})},$$

$$\widehat{V^2} = \frac{1}{n}\sum_{i=1}^n \left[r_i - \widehat{\alpha} r_{i-1} - \widehat{\beta}(1-\widehat{\alpha})\right]^2.$$

Case Study: Vasicek. Objective measure, econometrics, statistics, historical estimation VIII

The estimated quantities give complete information on the δ-transition probability for the process r under Q_0, thus allowing for example simulations at one-day spaced future discrete time instants.

First Choice: short rate r. Questions to ask. I

Back to short rate models in general. When choosing a model, one should ask:

- Does the dynamics imply positive rates, i.e., $r(t) > 0$ a.s. for each t?
- What distribution does the dynamics imply for the short rate r? Is it, for instance, a fat-tailed distribution?
- Are bond prices $P(t, T) = E_t \left\{ e^{-\int_t^T r(s)ds} \right\}$ (and therefore spot rates, forward rates and swap rates) explicitly computable from the dynamics?
- Are bond-option (and cap, floor, swaption) prices explicitly computable from the dynamics?
- Is the model mean reverting, in the sense that the expected value of the short rate tends to a constant value as time grows towards infinity, while its variance does not explode?

First Choice: short rate r. Questions to ask. II

- How do the volatility structures implied by the model look like?
- Does the model allow for explicit short-rate dynamics under the forward measures?
- How suited is the model for Monte Carlo simulation?
- How suited is the model for building recombining lattices (trees)?
- Does the chosen dynamics allow for historical estimation techniques to be used for parameter estimation purposes?

First Choice: Modeling r. Endogenous models. I

Model	**Dist**	**Analytic** $P(t,T)$	**Analytic Options**	**Multif**	**M-R**	$r > 0$?
Vasicek	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
CIR	n.c. χ^2	Yes	Yes	Yes	Yes	Yes
Dothan	$e^{\mathcal{N}}$	"Yes"	No	No	"Yes"	Yes
Exp. Vasicek	$e^{\mathcal{N}}$	No	No	No	Yes	Yes

These models are **endogenous**. $P(t,T) = E_t(e^{-\int_t^T r(s)ds})$ can be computed as an expression (or numerically in the last two) depending on the model parameters.

For example, in Vasicek and CIR, given k, θ, σ and $r(t)$, once the function $T \mapsto P(t,T; k, \theta, \sigma, r(t))$ is known, we know the whole interest-rate curve at time t. At $t = 0$ (initial time), the interest rate curve is an **output** of the model, rather than an input, depending on k, θ, σ, r_0 in the dynamics.

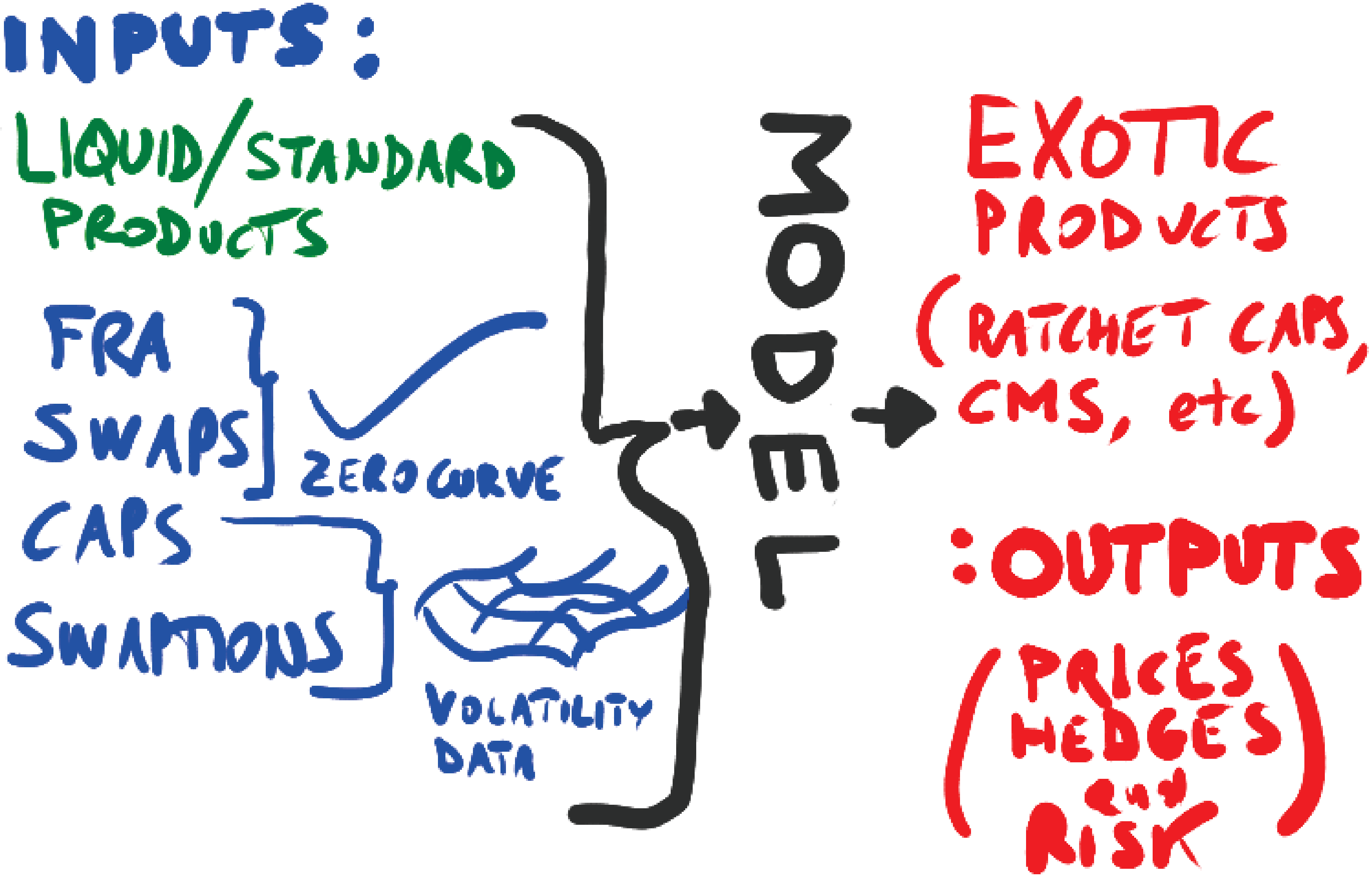
First Choice: Modeling r. Endogenous models. II

If we have the initial curve $T \mapsto P^M(0,T)$ from the market, and we wish our model to incorporate this curve, we need forcing the model parameters to produce a curve as close as possible to the market curve. This is the **calibration of the model to market data**. In the Vasicek case, run an optimization to have

$$\text{Fit } \; T \mapsto P(0,T;k,\theta,\sigma,r_0) \; \text{ to } \; T \mapsto P^M(0,T) \; \text{ through } k,\theta,\sigma,r_0.$$

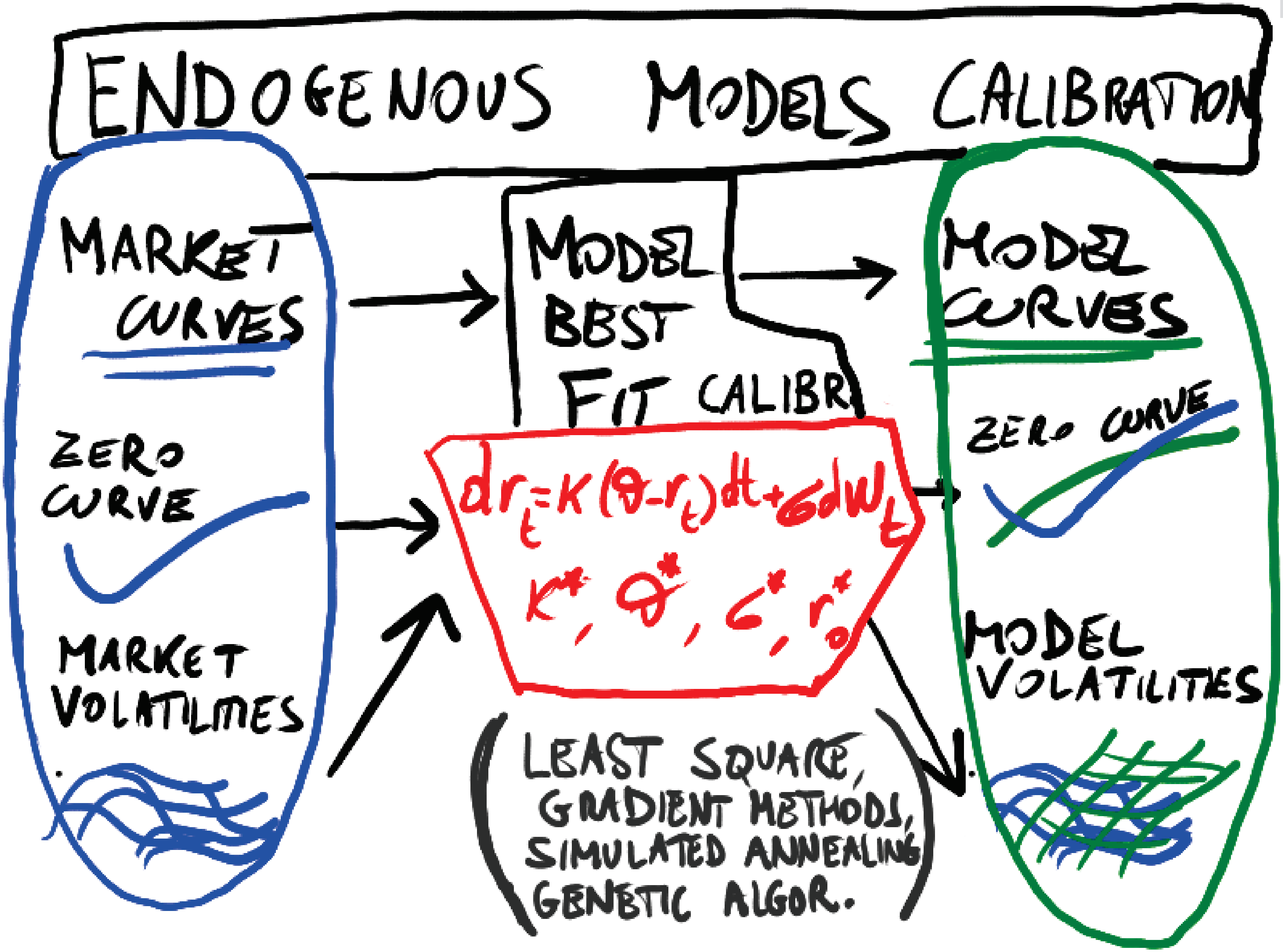
Too few parameters. Some shapes of $T \mapsto L^M(0,T)$ (like an inverted shape) can never be obtained, no matter the values of the parameters in the dynamics. To improve this situation and calibrate also **caplet** data, **exogenous** term structure models are usually considered.

THE MODEL CALIBRATION



Calibration

- A particularly important part of a model's operations is the calibration.
- Our aim is pricing, hedging and possibly risk managing a complex EXOTIC financial product whose quotations are not liquid or easily found
- To do so we plan to use a model
- The model needs to reflect as many available liquid market data as possible when these data are pertinent to the financial product to be analyzed
- In the interest rate market ususally one starts from the zero curve (FRA, Swaps) and a few vanilla options (Caps, Swaptions), imposing the model to fit them
- Once the model has been fit as well as possible to such data, the model is used to price the complex product


ENDOGENOUS MODELS CALIBRATION
MARKET CURVES
ZERO CURVE
MARKET VOLATILITIES
MODEL BEST FIT CALIBR
$dr_t = k(\theta - r_t)dt + \sigma dW_t$
$k^*, \theta^*, \sigma^*, r_0^*$
LEAST SQUARE, GRADIENT METHODS, SIMULATED ANNEALING GENETIC ALGOR.
MODEL CURVES
ZERO CURVE
MODEL VOLATILITIES

Endogenous Models Calibration Figure

- We are given a market zero coupon curve of interest rates at time 0, the blue curve ”zero curve” for $T \mapsto L^M(0,T)$.
- We are given also a number of options volatilities possibly, the blue surface ”market volatilities”
- We best fit the Red Vasicek model formula for the curve $L(0,T;k,\theta,\sigma,r_0)$ and perhaps a few options formulas to get the best parameters we can in matching the market data. These will be the red parameters $k^*,\theta^*,\sigma^*,r_0^*$.
- The best fit can occur through the grey optimization methods, either local (gradient method) or global (simulated annealing, genetic algorithms...)
- The resulting fit is usually poor. For example, Vasicek cannot reproduce an inverted curve, compare the green (model) and blue (market) zero curves on the right hand side of the figure...
- The volatility structure is also poorly fit, as you see comparing the blue and the green surfaces on the right hand side.

First Choice: Modeling r. Exogenous models. I

Exogenous short-rate models are built by suitably modifying the above endogenous models. The basic strategy that is used to transform an endogenous model into an exogenous model is the inclusion of "time-varying" parameters. Typically, in the Vasicek case, one does the following:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t) \longrightarrow dr(t) = k[\boxed{\vartheta(t)} - r(t)]dt + \sigma dW(t)\ .$$

Now the function of time $\vartheta(t)$ can be defined in terms of the market curve $T \mapsto L^M(0, T)$ in such a way that the model reproduces exactly the curve itself at time 0.

The remaining parameters may be used to calibrate CAPS/Swaptions data. We no longer price caps, since they are very liquid, but wish the model to "absorb" them to price more difficult things.

First Choice: Modeling r. Exogenous models. I

Dynamics of $r_t = x_t$ under the risk–neutral measure:

1. **Ho-Lee:**

$$dx_t = \theta(t)\, dt + \sigma\, dW_t.$$

2. **Hull-White (Extended Vasicek):**

$$dx_t = k(\theta(t) - x_t)dt + \sigma dW_t.$$

3. **Hull-White (Extended CIR):**

$$dx_t = k(\theta(t) - x_t)dt + \sigma \sqrt{x_t}\, dW_t\ .$$

4. **Black-Derman-Toy (Extended Dothan):**

$$x_t = x_0\, e^{u(t)+\sigma(t) W_t}$$

First Choice: Modeling r. Exogenous models. II

5. **Black-Karasinski (Extended exponential Vasicek):**

$$x_t = \exp(z_t), \quad dz_t = k\left[\theta(t) - z_t\right] dt + \sigma dW_t.$$

6. **CIR++ (Shifted CIR model, Brigo & Mercurio (2000)):**

$$r_t = x_t + \phi(t;\alpha), \quad dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t$$

Now parameters are used to fit volatility structures.
In general other parameters can be chosen to be time–varying so as to improve fitting of the volatility term–structure (but...)

Reference Model	Dist	ABP	AOP	Multif	M-R	$r > 0$?
Vasicek	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
CIR	n.c. χ^2	Yes	Yes	Yes	Yes	Yes
Dothan	$e^{\mathcal{N}}$	"Yes"	No	No	"Yes"	Yes
Exp. Vasicek	$e^{\mathcal{N}}$	No	No	No	Yes	Yes

Classical extended models:
Distribution (Distr)
Analytical bond prices (ABP)
Analytical bond–option prices (AOP)
Mean Reversion (MR)
Tractable Multi Factor Extension (Multif)

Extended Model	**Distr**	**ABP**	**AOP**	**Multif**	**M-R**	$r > 0$?
Ho-Lee	$\mathcal{N}$	Yes	Yes	Yes	No	No
Hull-White (Vas.)	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
Hull-White (CIR)	n.c. χ^2	No	No	No	Yes	Yes-but
BDT	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
Black Karasinski	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
CIR++ Brigo Mercurio	s.n.c. χ^2	Yes	Yes	Yes	Yes	Yes

Short rate models: Which model? I

Extended Model	**Distr**	**ABP**	**AOP**	**Multif**	**M-R**	$r > 0$?
Ho-Lee	$\mathcal{N}$	Yes	Yes	Yes	No	No
Hull-White (Vas.)	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
Hull-White (CIR)	n.c. χ^2	No	No	No	Yes	Yes-but
BDT	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
Black Karasinski	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
CIR++ Brigo Mercurio	s.n.c. χ^2	Yes	Yes	Yes	Yes	Yes

- Ho Lee: very tractable; stylized, simplistic, negative rates;
- Hull-White (Vasicek): Very tractable, formulas, easy to implement and calibrate, trees easy, Monte Carlo possible; possibly negative rates; can give pathological calibrations under certain market situations.
- Hull-White (CIR): Not tractable, numerical problems...

Short rate models: Which model? II

- BDT: No tractability, some mean reversion but linked to the volatility, excellent distribution and good calibration to the market rates implied distributions, explosion problem of bank account in continuos version: $EB(\epsilon) = E(\exp(\int_0^\epsilon r_u du)) = \infty$.
  Need trinomial trees (discretization in time and space) to have it work. No reasonable Monte Carlo simulation possible.

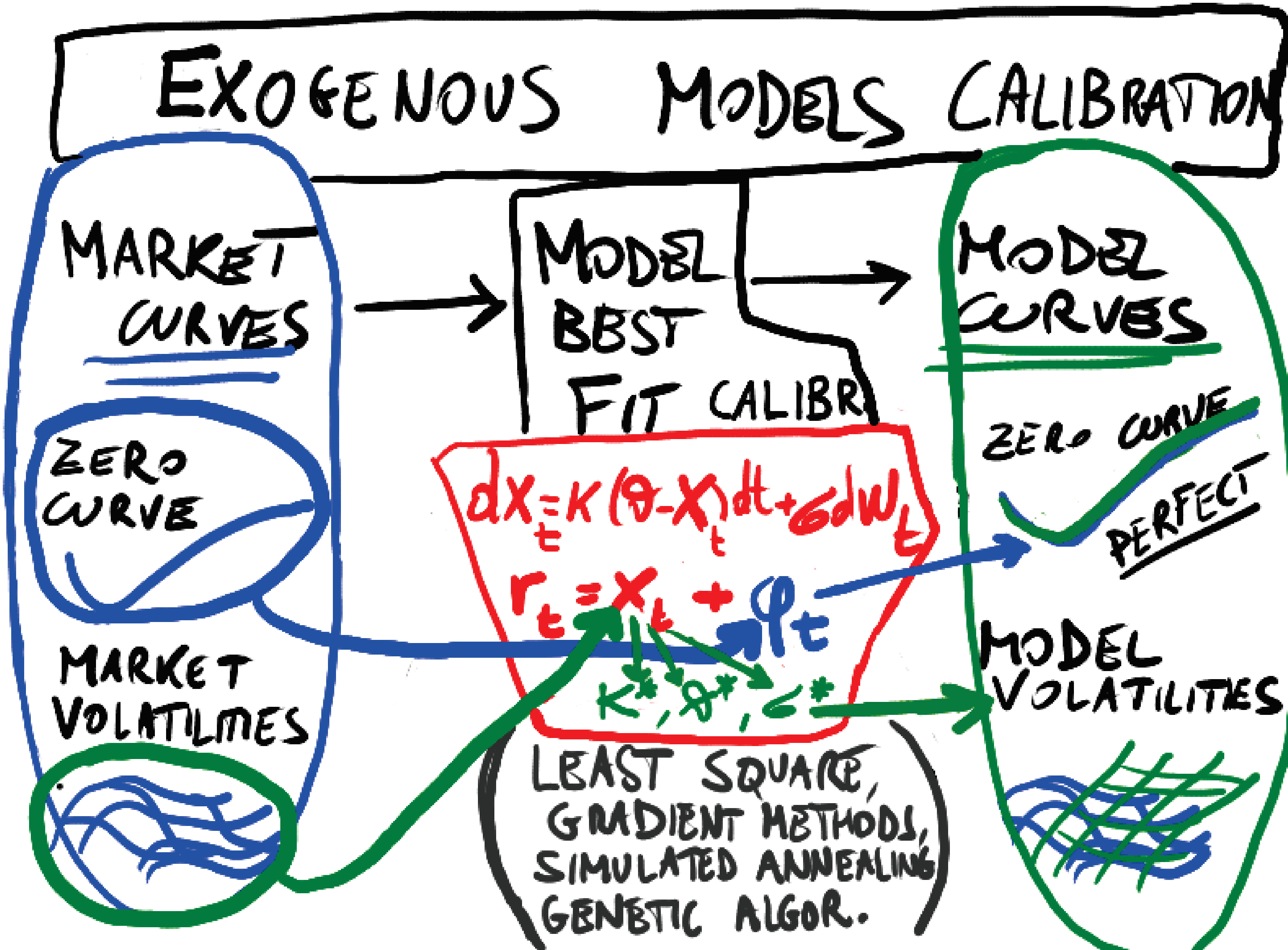
Short rate models: Which model? III

Extended Model	Distr	ABP	AOP	Multif	M-R	$r > 0$?
Ho-Lee	$\mathcal{N}$	Yes	Yes	Yes	No	No
Hull-White (Vas.)	$\mathcal{N}$	Yes	Yes	Yes	Yes	No
Hull-White (CIR)	n.c. χ^2	No	No	No	Yes	Yes-but
BDT	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
Black Karasinski	$e^{\mathcal{N}}$	No	No	No	Yes	Yes
CIR++ Brigo Mercurio	s.n.c. χ^2	Yes	Yes	Yes	Yes	Yes

- Black Karasinski: No tractability, mean reversion, excellent distribution and good calibration to the market rates distributions, explosion problem of bank account in continuos version (as in all lognormal short-rate models). Need trinomial trees (discretization in time and space) to have it work. No reasonable Monte Carlo simulation possible.

Short rate models: Which model? IV

- CIR++: Tractable, many formulas, easy to implement and calibrate, trees are not so easy but feasible, Monte Carlo possible, positive rates, can give pathological calibrations under certain market situations (as most one-dimensional short-rate models)


EXOGENOUS MODELS CALIBRATION
MARKET CURVES
ZERO CURVE
MARKET VOLATILITIES
MODEL BEST FIT CALIBR
dx_t = k(θ - x_t)dt + σdW_t
r_t = x_t + φ_t
k*, θ*, σ*
LEAST SQUARE, GRADIENT METHODS, SIMULATED ANNEALING GENETIC ALGOR.
MODEL CURVES
ZERO CURVE
PERFECT
MODEL VOLATILITIES

Exogenous Models Calibration Figure

- We are given a market zero coupon curve of interest rates at time 0, the blue curve ”zero curve” for $T \mapsto L^M(0,T)$.
- We are given also a number of vanilla options volatilities (typically caps and a few swaptions), possibly. This is the blue surface ”market volatilities”
- We now use a time dependent ”parameter” $\vartheta(t)$ or shift $\varphi(t)$ to fit the zero curve exactly, and this is represented by the blue arrow.
- Then we use the parameters k, θ, σ, r_0 in the x part of r to best fit the vanilla option data, and this is the green arrow.
- The best fit of the options data can occur through the grey optimization methods, either local (gradient method) or global (simulated annealing, genetic algorithms...)
- The resulting fit is usually not too good, as you see comparing the blue and the green surfaces on the right hand side. If we fit just a few options, the fit improves

Case study: Shifted Vasicek I

We have seen extensions of

$$dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t \ ,$$

obtained through time varying coefficients,

$$r_t = x_t, \quad dx_t = \mu(x_t;\alpha(t))dt + \sigma(x_t;\alpha(t))dW_t \ .$$

Instead, we propose the following alternative possibility:

$$r_t = x_t + \phi(t;\alpha), \quad dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t \ ,$$

with x_0 **a further parameter** we include augmenting α. We have the following bond and option prices

Case study: Shifted Vasicek. Bond and Option I

$$P^r(t,T,r_t;\alpha) = E_t\left\{\exp\left[-\int_t^T(\phi(s;\alpha)+x_s)ds\right]\right\}$$

$$= E_t\left\{\exp\left[-\int_t^T\phi(s;\alpha)ds\right]\exp\left[-\int_t^T x_s ds\right]\right\}$$

$$= \exp\left[-\int_t^T\phi(s;\alpha)ds\right]E_t\left\{\exp\left[-\int_t^T x_s ds\right]\right\}$$

$$= \exp\left[-\int_t^T\phi(s;\alpha)ds\right]P^x(t,T,x_t;\alpha)$$

Case study: Shifted Vasicek. Bond and Option II

$$\mathrm{ZBP}^r(0,T,s,K,r_0;\alpha) = E_0\left\{\exp\left[-\int_0^T r_u du\right](K - P^r(T,s,r_T;\alpha))^+\right\}$$

$$= \exp\left[-\int_0^s \phi(u;\alpha)du\right]\mathrm{ZBP}^x\left(0,T,s,K\exp\left[\int_T^s \phi(u;\alpha)\,du\right],x_0^\alpha;\alpha\right)$$

Calibration of the market zero curve ($T \mapsto P^M(0,T)$) and of Caplet data. How do we select α and $\phi(\cdot,\alpha)$ to calibrate the model?

Case study: Shifted Vasicek. Exact calibration of zero curve through ϕ I

$$r_t = x_t + \phi(t;\alpha), \quad dx_t = \mu(x_t;\alpha)dt + \sigma(x_t;\alpha)dW_t \ .$$

Fitting the initial term structure. Solve

$$P^r(0,T,r_t;\alpha) = P^M(0,T) \ \text{ for all } T, \quad \text{i.e.}$$

$$\exp\left[-\int_0^T \phi(s;\alpha)ds\right] P^x(0,T,x_0;\alpha) = P^M(0,T), \quad \text{and obtain}$$

$$\int_a^b \phi(u;\alpha)du = \ln\left(\frac{P^M(0,a)}{P^M(0,b)}\right) - \ln\left(\frac{P^x(0,a,r_0;\alpha)}{P^x(0,b,r_0;\alpha)}\right)$$

$$\phi(t;\alpha) = -\frac{\partial}{\partial t}\ln\left(\frac{P^M(0,t)}{P^x(0,t,r_0;\alpha)}\right) =: -\mathbf{f}^{\mathbf{x}}(\mathbf{0},\mathbf{t},\mathbf{r_0};\alpha) + \mathbf{f}^{\mathbf{M}}(\mathbf{0},\mathbf{t}).$$

Case study: Shifted Vasicek. Exact calibration of zero curve through ϕ II

If we select this ϕ, we fit the initial term structure, **no matter the value of** α. In the Vasicek case we obtain

$$\varphi^{VAS}(t;\alpha) = f^M(0,t) + (e^{-kt} - 1)\frac{k^2\theta - \sigma^2/2}{k^2}$$

$$-\frac{\sigma^2}{2k^2}e^{-kt}(1 - e^{-kt}) - x_0 e^{-kt} \ .$$

Case study: Shifted Vasicek. Exact calibration of zero curve through ϕ I

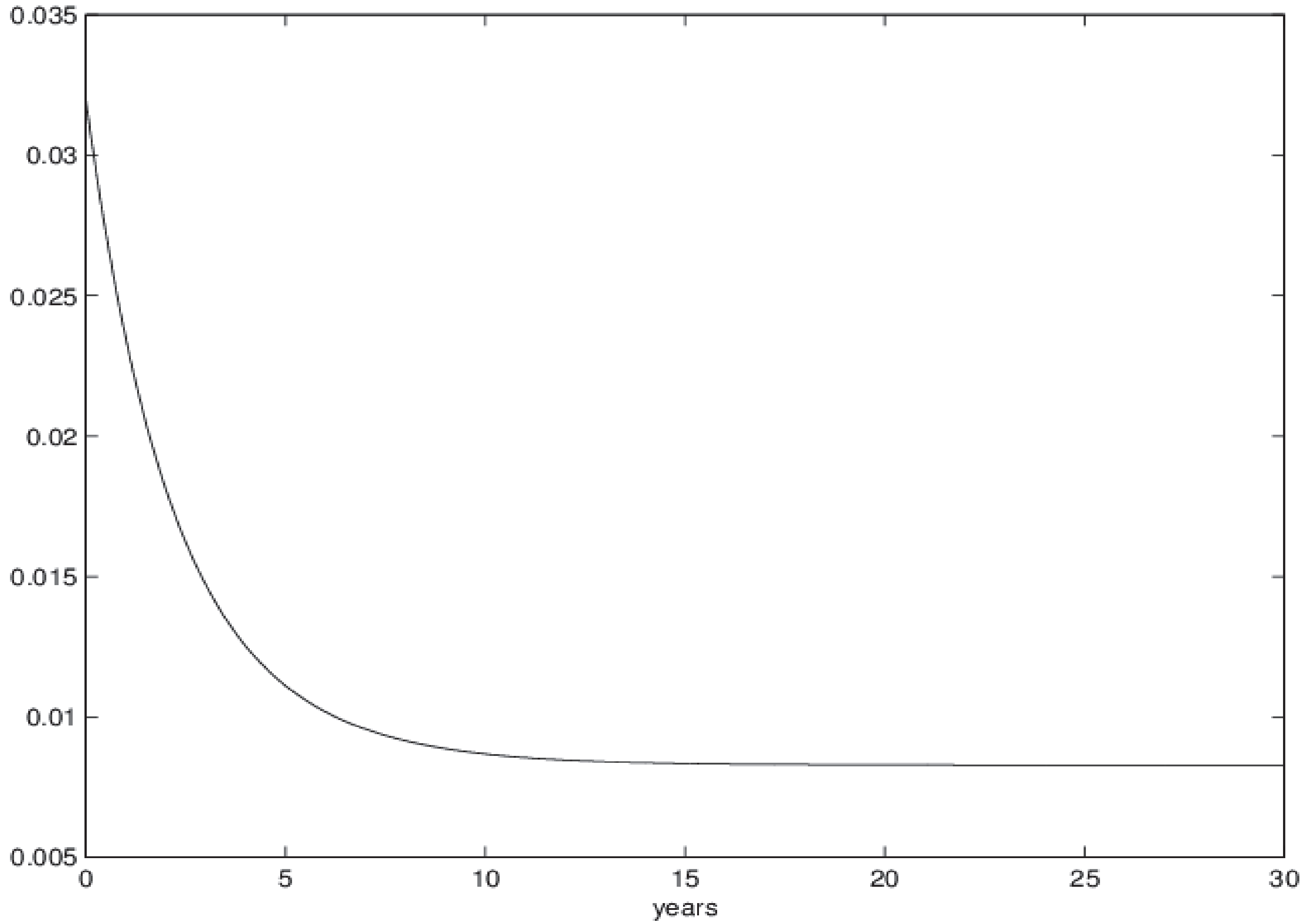
Notice that the parameters θ and x_0 are redundant. Indeed, we can easily see that such parameter can be reabsorbed in φ. We will therefore take, from now on, $\theta = 0$ in the above expressions, leading to

$$dx_t = -kx_t dt + \sigma dW_t, \; x_0 = 0, \;\; r_t = x_t + \varphi(t, \alpha), \;\; \alpha = [k, \sigma].$$

Indeed, when applied to the Vasicek model, our method is essentially equivalent to $\theta \mapsto \theta(t)$ and produces the Hull-White model, due to linearity of the equation for x. x_0 has no effect and we can assume it to be zero.

Case study: Shifted Vasicek. Exact calibration of zero curve through ϕ I

Case study: Shifted Vasicek. Exact calibration of zero curve through ϕ II



Case study: Shifted Vasicek. calibration of caplet market quotes through α I

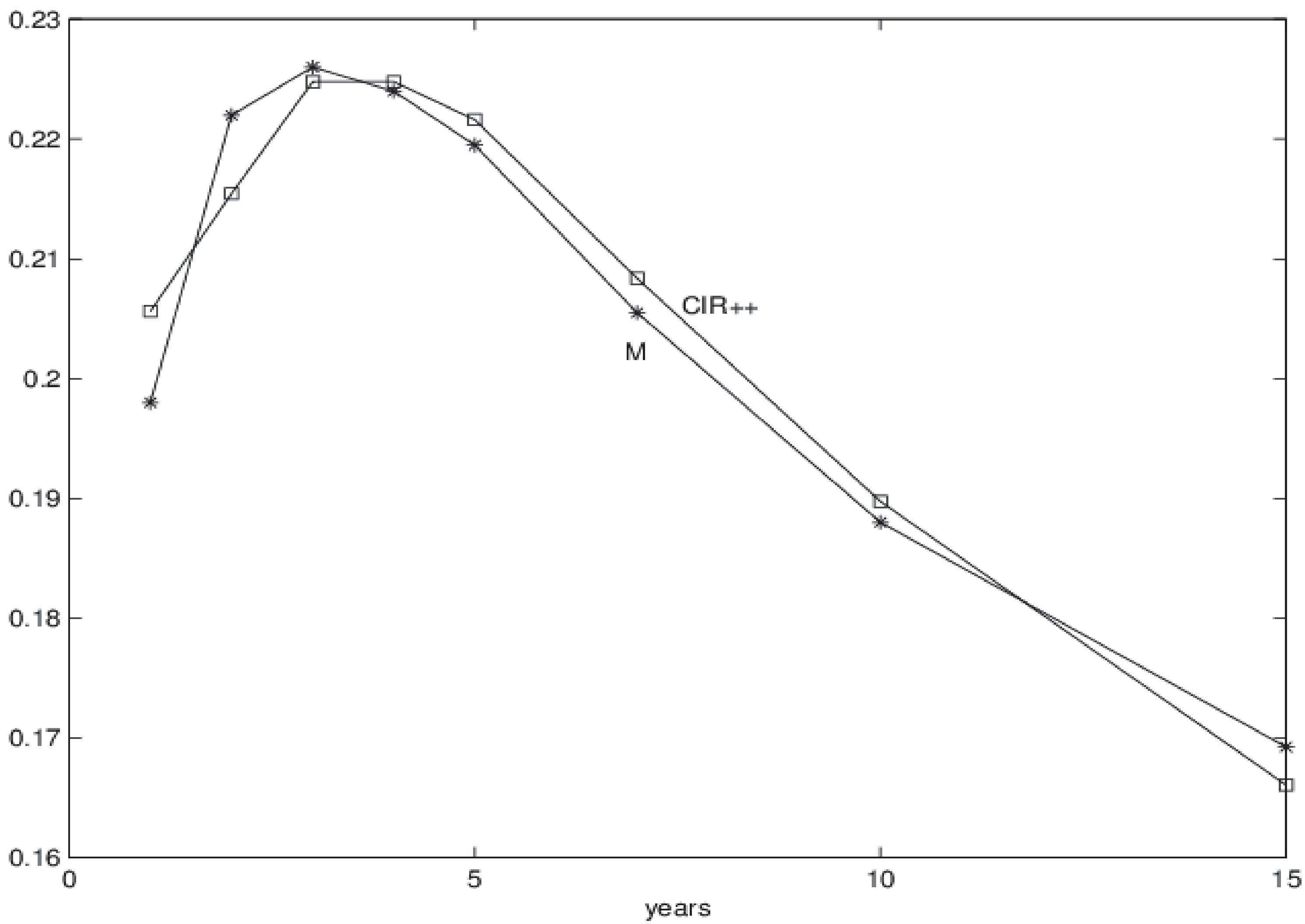
Choose α **to fit caplets (caps/floors) or a few swaptions** prices given analytically in terms of zero–bond option prices.
Find α (optimization) such that model prices

$$e^{-\int_0^s \phi(u;\alpha)du} \mathrm{ZBP}^x\left(0, T, s, K \exp\left[\int_T^s \phi(u;\alpha)\, du\right], x_0^\alpha; \alpha\right)$$

are as close as possible to market prices

$$\mathrm{ZBP}^{\mathrm{MKT}}(0, T, S, K)$$

Case study: Shifted Vasicek. calibration of caplet market quotes through α II



Monte Carlo and Trinomial Trees I

In the market there are products featuring path dependent payoffs and early exercise payoffs.
When we aim at pricing derivatives whose payout at final maturity T is a function not only of interest rates at a final time related to the final maturity T but also of interest rates related to earlier times $t_i < T$, then we say that we have a path dependent payout. More precisely, this happens if the payout cannot be decomposed into a sum of payouts each referencing a single maturity interest rate at the time.
For these path dependent payouts, except for a few exceptions, it may be necessary to price unsing Monte Carlo simulation.

There are also products that can be exercised at times t_i preceding the final maturity of the payout. The typical example is bermudan swaptions, which are swaptions that can be exercised every year rather than at a single maturity T_α. For such products Monte Carlo simulation

Monte Carlo and Trinomial Trees II

is not suitable. Indeed, simulating forward in time does not allow us to know or propagate the optimal exercise strategy for the option. On the contrary, this can be know at terminal time and be propagated backwards in time along a tree, similarly for how American options on equity are priced using binomial trees and backward induction.

Monte Carlo Simulation I

Since for Vasicek we know that
$r(t)$ conditional on r_s is normally distributed with mean and variance given respectively by

$$E\{r(t)|r_s\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$\mathrm{VAR}\{r(t)|r_s\} = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right],$$

this means that the short rate can be simulated exactly across large intervals t_{i-1}, t_i without further discretization. Monte Carlo simulation is easy because we know the exact normal distribution for the transition probability of the short rate between times t_{i-1} and t_i. A further advantage of the Vasicek model is that if we know the short rate at t_i we have a formula for the bond price $P(t_i, T)$ for every maturity T. Hence from the short rate simulation we can immediately get as a

Monte Carlo Simulation II

bonus Libor rates, forward and swap rates for any maturity. This makes the model handy in pricing path dependent payoffs via simulation. This reasoning of course applies as well to the shifted Vasicek model.

Trinomial Tree I

We now illustrate a procedure for the construction of a trinomial tree that approximates the evolution of the process x. It can be then extended to the shifted Vasicek model by suitably adjusting the tree (see for example Brigo and Mercurio 2006).
This is a two-stage procedure that is basically based on those suggested by Hull and White (1993d, 1994a).
Let us fix a time horizon T and the times $0 = t_0 < t_1 < \cdots < t_N = T$, and set $\Delta t_i = t_{i+1} - t_i$, for each i. The time instants t_i need not be equally spaced. This is an essential feature when employing the tree for practical purposes.
The first stage consists in constructing a trinomial tree for the process x

$$dx_t = -kx_t dt + \sigma dW_t$$

We explain how to build a tree for a generic diffusion process X first

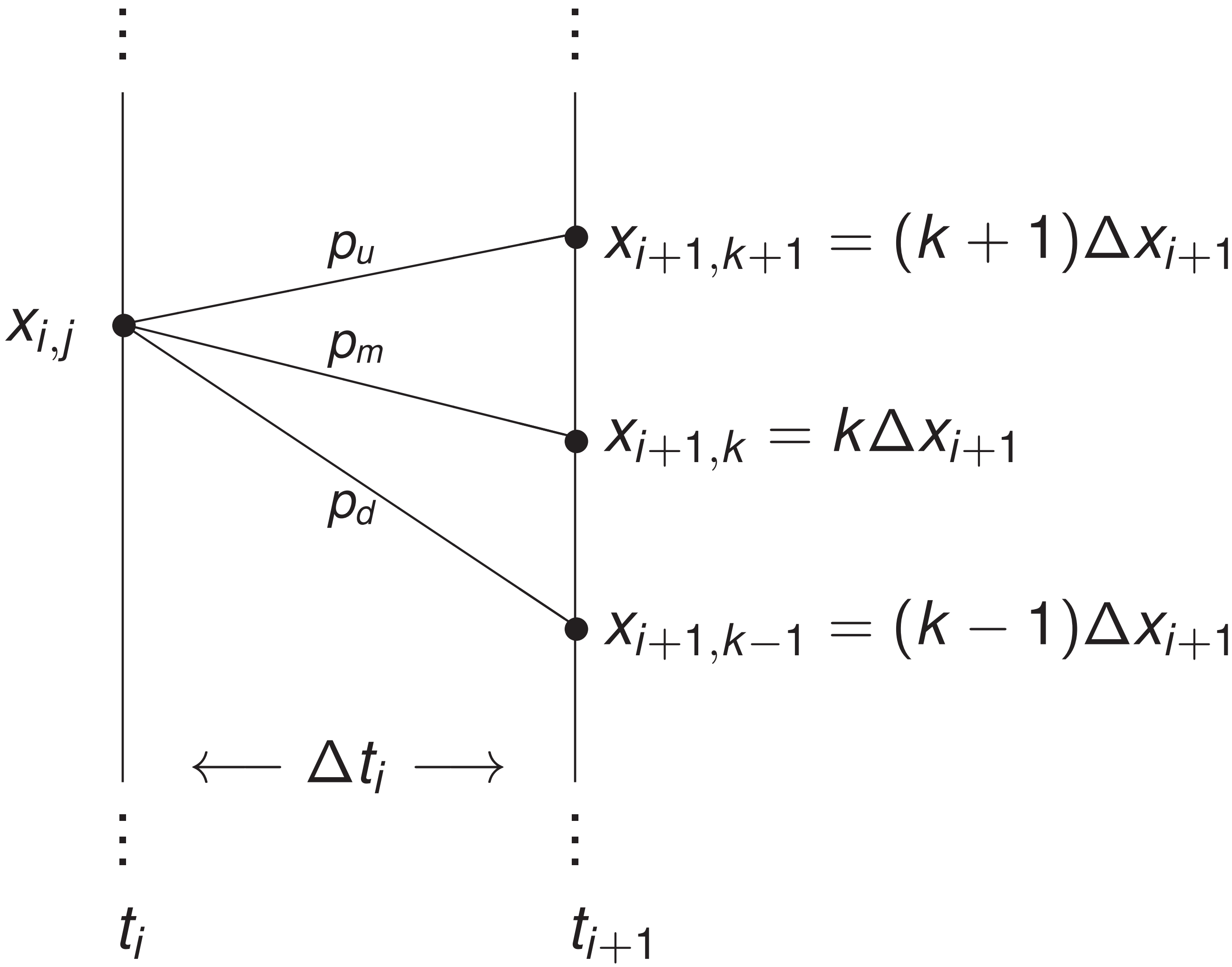
Trinomial Tree I

Let us consider the diffusion process X

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where μ and σ are smooth scalar real functions and W is a scalar standard Brownian motion.
We want to discretize this dynamics both in time and in space.
Precisely, we want to construct a trinomial tree that suitably approximates the evolution of the process X.
To this end, we fix a finite set of times $0 = t_0 < t_1 < \cdots < t_n = T$ and we set $\Delta t_i = t_{i+1} - t_i$. At each time t_i, we have a finite number of equispaced states, with constant vertical step Δx_i to be suitably determined. We set $x_{i,j} = j\Delta x_i$.

Trinomial Tree I



Trinomial Tree II

Assuming that at time t_i we are on the j-th node with associated value $x_{i,j}$, the process can move to $x_{i+1,k+1}$, $x_{i+1,k}$ or $x_{i+1,k-1}$ at time t_{i+1} with probabilities p_u, p_m and p_d, respectively. The central node is therefore the k-th node at time t_{i+1}, where also the level k is to be suitably determined.
Denoting by $M_{i,j}$ and $V_{i,j}^2$ the mean and the variance of X at time t_{i+1} conditional on $X(t_i) = x_{i,j}$, i.e.,

$$E\left\{X(t_{i+1})|X(t_i) = x_{i,j}\right\} = M_{i,j}$$
$$\text{Var}\left\{X(t_{i+1})|X(t_i) = x_{i,j}\right\} = V_{i,j}^2,$$

we want to find p_u, p_m and p_d such that these conditional mean and variance match those in the tree.

Trinomial Tree III

Precisely, noting that $x_{i+1,k+1} = x_{i+1,k} + \Delta x_{i+1}$ and $x_{i+1,k-1} = x_{i+1,k} - \Delta x_{i+1}$, we look for positive constants p_u, p_m and p_d summing up to one and satisfying

$$\begin{cases} p_u(x_{i+1,k} + \Delta x_{i+1}) + p_m x_{i+1,k} + p_d(x_{i+1,k} - \Delta x_{i+1}) = M_{i,j} \\ p_u(x_{i+1,k} + \Delta x_{i+1})^2 + p_m x_{i+1,k}^2 + p_d(x_{i+1,k} - \Delta x_{i+1})^2 = \\ = V_{i,j}^2 + M_{i,j}^2. \end{cases}$$

Simple algebra leads to

$$\begin{cases} x_{i+1,k} + (p_u - p_d)\Delta x_{i+1} = M_{i,j} \\ x_{i+1,k}^2 + 2x_{i+1,k}\Delta x_{i+1}(p_u - p_d) + \Delta x_{i+1}^2(p_u + p_d) \\ = V_{i,j}^2 + M_{i,j}^2. \end{cases}$$

Trinomial Tree IV

Setting $\eta_{j,k} = M_{i,j} - x_{i+1,k}$ (we omit to express the dependence on the index i to lighten the notation) we finally obtain

$$\begin{cases} (p_u - p_d)\Delta x_{i+1} = \eta_{j,k} \\ (p_u + p_d)\Delta x_{i+1}^2 = V_{i,j}^2 + \eta_{j,k}^2, \end{cases}$$

so that, remembering that $p_m = 1 - p_u - p_d$, the candidate probabilities are

$$\begin{cases} p_u = \dfrac{V_{i,j}^2}{2\Delta x_{i+1}^2} + \dfrac{\eta_{j,k}^2}{2\Delta x_{i+1}^2} + \dfrac{\eta_{j,k}}{2\Delta x_{i+1}}, \\ p_m = 1 - \dfrac{V_{i,j}^2}{\Delta x_{i+1}^2} - \dfrac{\eta_{j,k}^2}{\Delta x_{i+1}^2}, \\ p_d = \dfrac{V_{i,j}^2}{2\Delta x_{i+1}^2} + \dfrac{\eta_{j,k}^2}{2\Delta x_{i+1}^2} - \dfrac{\eta_{j,k}}{2\Delta x_{i+1}}. \end{cases}$$

Trinomial Tree V

In general, there is no guarantee that p_u, p_m and p_d are actual probabilities, because the expressions defining them could be negative. We then have to exploit the available degrees of freedom in order to obtain quantities that are always positive. To this end, we make the assumption that $V_{i,j}$ is independent of j, so that from now on we simply write V_i instead of $V_{i,j}$. We then set $\Delta x_{i+1} = V_i\sqrt{3}$ (this choice, motivated by convergence purposes, is a standard one. See for instance Hull and White (1993, 1994)) and we choose the level k, and hence $\eta_{j,k}$, in such a way that $x_{i+1,k}$ is as close as possible to $M_{i,j}$. As a consequence,

$$k = \text{round}\left(\frac{M_{i,j}}{\Delta x_{i+1}}\right), \tag{20}$$

Trinomial Tree VI

where round(x) is the closest integer to the real number x. Moreover,

$$\begin{cases} p_u = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i}, \\ p_m = \frac{2}{3} - \frac{\eta_{j,k}^2}{3V_i^2}, \\ p_d = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i}. \end{cases} \qquad (21)$$

It is easily seen that both p_u and p_d are positive for every value of $\eta_{j,k}$, whereas p_m is positive if and only if $|\eta_{j,k}| \leq V_i\sqrt{2}$. However, defining k as above implies that $|\eta_{j,k}| \leq V_i\sqrt{3}/2$, hence the condition for the positivity of p_m is satisfied, too.

As a conclusion, the above are actual probabilities such that the corresponding trinomial tree has conditional (local) mean and variance that match those of the continuous-time process X.

Trinomial Tree VII

Going back to our x_t as in Vasicek, we have

$$
\begin{aligned}
E\{x(t_{i+1})|x(t_i) = x_{i,j}\} &= x_{i,j}e^{-a\Delta t_i} =: M_{i,j} \\
\mathrm{Var}\{x(t_{i+1})|x(t_i) = x_{i,j}\} &= \frac{\sigma^2}{2a}\left[1 - e^{-2a\Delta t_i}\right] =: V_i^2.
\end{aligned}
\tag{22}
$$

We then set $x_{i,j} = j\Delta x_i$, where

$$
\Delta x_i = V_{i-1}\sqrt{3} = \sigma\sqrt{\frac{3}{2a}\left[1 - e^{-2a\Delta t_{i-1}}\right]}. \tag{23}
$$

and we apply the above procedure.

Trinomial Tree VIII

Once we have the tree, pricing of (early exercise) Bermudan swaptions occurs by backward induction.

One first computes the final payout at each final node in the tree at T, and then starts rolling back the payout along the tree in time.

At each time where exercise is available one then compares the rolled back price down to that point/node (continuation value) to the price of exercise in that specific node, and takes the maximum.

This maximum is then rolled further backwards in the tree, discounting at the local tree interest rate, and then compared to immediate exercise; maximum is then taken and the backwards propagation continues down to time 0, when the price is obtained at the single initial node of the tree.

Trinomial Tree IX

This way we make the optimal choice every time early exercise is available. This is easily implemented once the tree is built.

A (forward looking) monte carlo simulation would not work here, since we would not know, in a specific path at a point in time, the continuaton value, which can be computed going backwards but not forward

Special versions of the Monte Carlo method that approximate the continuation value as a function of the present state variables can be used. This is called Least Squared Monte Carlo.

The student has certainly seen continuation value calculations in trees for simple option pricing theory. This is completely analogous to the binomial-tree model of Cox Ross Rubinstein for american options on a stock.

Trinomial Tree X

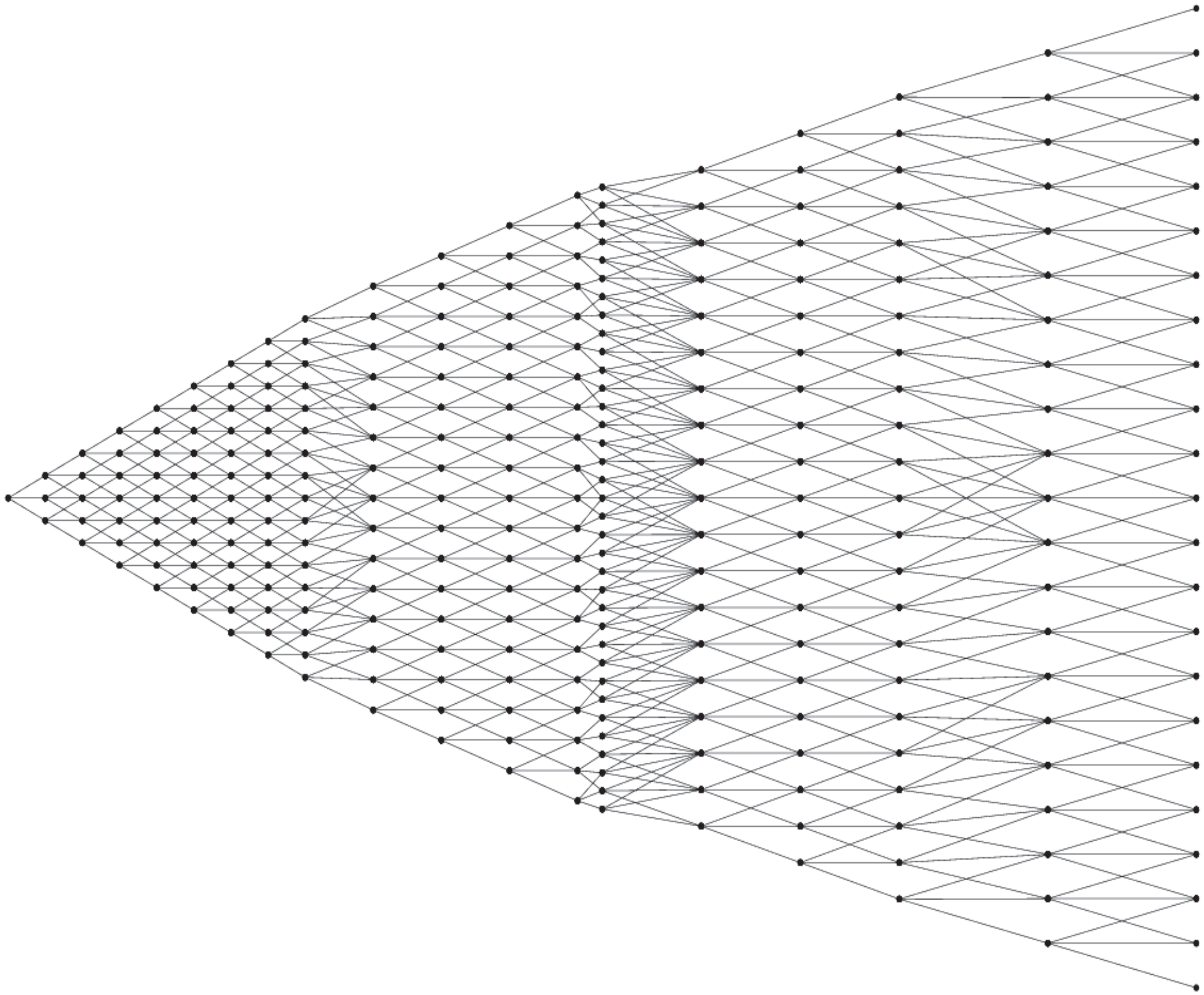


Figure: A possible geometry for the tree approximating x.

First choice: Modeling r. Multidimensional models I

In these models, typically (e.g. shifted two-factor Vasicek)

$$\begin{aligned} dx_t &= k_x(\theta_x - x_t)dt + \sigma_x dW_1(t), \\ dy_t &= k_y(\theta_y - y_t)dt + \sigma_y dW_2(t), \quad dW_1\, dW_2 = \rho\, dt, \\ r_t &= x_t + y_t + \phi(t,\alpha), \quad \alpha = (k_x, \theta_x, \sigma_x, x_0, k_y, \theta_y, \sigma_y, y_0) \end{aligned}$$

More parameters, can capture more flexible caps or swaptions structures in the market and especially gives **less correlated** rates at future times.
Indeed, suppose we define Continuously Compounded Spot Rates at time t for the maturity T as

$$R(t,T) := -\frac{1}{T-t} \ln P(t,T) \;\Rightarrow\; P(t,T) = e^{-R(t,T)(T-t)}.$$

First choice: Modeling r. Multidimensional models II

This is an alternative definition to the Simply Compounded (Libor) Spot rates we have seen earlier:

$$L(t,T) := \frac{1}{T-t}\left[\frac{1}{P(t,T)} - 1\right]$$

One dimensional models have
$\text{corr}_0(R(1y,2y), R(1y,30y)) = 1$,
due to the unique source of randomness dW.
Multidimensional models can lower this perfect correlation by playing with the instantaneous correlation ρ in the **two** sources of randomness W_1 and W_2.
We may retain analytical tractability.

What do we model? Second Choice: instantaneous forward rates $f(t,T)$ I

Recall the forward LIBOR rate at time t between T and S, $F(t;T,S) = (P(t,T)/P(t,S) - 1)/(S-T)$, which makes the FRA contract to lock in at time t interest rates between T and S fair. When S collapses to T we obtain *instantaneous* forward rates:

$$f(t,T) = \lim_{S \to T^+} F(t;T,S) \approx -\frac{\partial \ln P(t,T)}{\partial T}, \quad \lim_{T \to t} f(t,T) = r_t.$$

Why should one be willing to model the f's at all? The f's are not observed in the market, so that there is no improvement with respect to modeling r in this respect. Moreover notice that f's are more structured quantities:

$$f(t,T) = -\frac{\partial \ln E_t\left[\exp\left(-\int_t^T r(s)ds\right)\right]}{\partial T},$$

What do we model? Second Choice: instantaneous forward rates $f(t, T)$ II

$$P(t, T) = e^{-\int_t^T f(t,u)du}$$

Given the structure in r, we may expect some restrictions on the risk-neutral dynamics that are allowed for f.

What do we model? Second Choice: instantaneous forward rates $f(t, T)$ I

Indeed, there is a fundamental theoretical result: Set $f(0, T) = f^M(0, T)$. We have

$$df(t, T) = \boxed{\sigma(t, T)\left(\int_t^T \sigma(t, s)ds\right)} dt + \sigma(t, T)dW(t),$$

under the risk neutral world measure, if no arbitrage has to hold. Thus we find that the no-arbitrage property of interest rates dynamics is here clearly expressed as a **link between the local standard deviation (volatility or diffusion coefficient) and the local mean (drift)** in the dynamics. We will prove easily this result later on, after we introduce more detailed tools for the change of numeraire.

What do we model? Second Choice: instantaneous forward rates $f(t, T)$ II

Going back to the result itself, this is saying that given the volatility, there is no freedom in selecting the drift, contrary to the more fundamental models based on dr_t, where the whole risk neutral dynamics was free:

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t$$

b and σ had no link due to no-arbitrage.

Second Choice, modeling f (HJM): is it worth it? I

$$df(t,T) = \boxed{\sigma(t,T)\left(\int_t^T \sigma(t,s)ds\right)} dt + \sigma(t,T)dW(t),$$

This can be useful to study arbitrage free properties of models, but when in need of writing a concrete model to price and hedge financial products, most useful models coming out of HJM are the already known short rate models seen earlier and their multifactor extensions we see next (these are particular HJM models, especially Gaussian models) or the market models we are going to see later.
Even though market models do not necessarily need the HJM framework to be derived, HJM may serve as a unifying framework in which all categories of no-arbitrage interest-rate models can be expressed.

Multidimensional models and correlations I

Before turning to the third choice on what to model, we go back to the second one and consider multidimensional models more in detail. Recall that the Vasicek model assumes the evolution of the short-rate process r to be given by the linear-Gaussian SDE

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \ .$$

Recall also the bond price formula $P(t,T) = A(t,T)\exp(-B(t,T)r_t)$, from which all rates can be computed in terms of r. In particular, continuously-compounded spot rates are given by the following affine transformation of the fundamental quantity r

$$R(t,T) = -\ln(P(t,T))/(T-t) = -\frac{\ln A(t,T)}{T-t} + \frac{B(t,T)}{T-t} r_t =$$

$$=: a(t,T) + b(t,T) r_t \ .$$

Multidimensional models and correlations II

Consider now a payoff depending on the joint distribution of two such rates at time t. For example, we may set $T_1 = t + 1$ years and $T_2 = t + 10$ years. The payoff would then depend on the joint distribution of the one-year and ten-year continuously-compounded spot interest rates at "terminal" time t. In particular, since the joint distribution is involved, the correlation between the two rates plays a crucial role. With the Vasicek model such terminal correlation is easily computed as

$$\mathrm{Corr}(R(t,T_1), R(t,T_2)) =$$

$$= \mathrm{Corr}(a(t,T_1) + b(t,T_1)r_t, a(t,T_2) + b(t,T_2)r_t) = 1$$

so that at every time instant rates for all maturities in the curve are perfectly correlated. For example, the thirty-year interest rate at a given instant is perfectly correlated with the three-month rate at the same instant. This means that a shock to the interest rate curve at

Multidimensional models and correlations III

time t is transmitted equally through all maturities, and the curve, when its initial point (the short rate r_t) is shocked, moves almost rigidly in the same direction. Clearly, it is hard to accept this perfect-correlation feature of the model. Truly, interest rates are known to exhibit some decorrelation (i.e. non-perfect correlation), so that a more satisfactory model of curve evolution has to be found.

One-factor models such as HW, BK, CIR++, EEV may still prove useful when the product to be priced does not depend on the correlations of different rates but depends at every instant on a single rate of the whole interest-rate curve (say for example the six-month rate).

Otherwise, the approximation can still be acceptable, especially for “risk-management-like” purposes, when the rates that jointly influence the payoff at every instant are close (say for example the six-month and one-year rates). Indeed, the real correlation between such near

Multidimensional models and correlations IV

rates is likely to be rather high anyway, so that the perfect correlation induced by the one-factor model will not be unacceptable in principle. But in general, whenever the correlation plays a more relevant role, or when a higher precision is needed anyway, we need to move to a model allowing for more realistic correlation patterns. This can be achieved with multifactor models, and in particular with two-factor models. Indeed, suppose for a moment that we replace the Gaussian Vasicek model with its hypothetical two-factor version (G2):

$$r_t = x_t + y_t,$$

$$dx_t = k_x(\theta_x - x_t)dt + \sigma_x dW_1(t),$$

$$dy_t = k_y(\theta_y - y_t)dt + \sigma_y dW_2(t),$$

with instantaneously-correlated sources of randomness, $dW_1 \, dW_2 = \rho \, dt$. Again, we will see later on that also for this kind of

Multidimensional models and correlations V

models the bond price is an affine function, this time of the two factors x and y,

$$P(t,T) = A(t,T)\exp(-B^x(t,T)x_t - B^y(t,T)y_t),$$

where quantities with the superscripts "x" or "y" denote the analogous quantities for the one-factor model where the short rate is given by x or y, respectively. Taking this for granted at the moment, we can see easily that now

Multidimensional models and correlations VI

$$\mathrm{Corr}(R(t,T_1),R(t,T_2)) =$$

$$= \mathrm{Corr}(b^x(t,T_1)x_t + b^y(t,T_1)y_t, b^x(t,T_2)x_t + b^y(t,T_2)y_t),$$

and this quantity is not identically equal to one, but depends crucially on the correlation between the two factors x and y, which in turn depends, among other quantities, on the instantaneous correlation ρ in their joint dynamics.
How much flexibility is gained in the correlation structure and whether this is sufficient for practical purposes will be debated. It is however clear that the choice of a multi-factor model is a step forth in that correlation between different rates of the curve at a given instant is not necessarily equal to one.
Another question that arises naturally is: How many factors should one use for practical purposes? Indeed, what we have suggested with two

Multidimensional models and correlations VII

factors can be extended to three or more factors. The choice of the number of factors then involves a compromise between numerically-efficient implementation and capability of the model to represent realistic correlation patterns (and covariance structures in general) and to fit satisfactorily enough market data in most concrete situations.

Multidimensional models: how many factors? I

Usually, historical analysis of the whole yield curve, based on principal component analysis or factor analysis, suggests that under the objective measure two components can explain 85% to 90% of variations in the yield curve, as illustrated for example by Jamshidian and Zhu (1997, Finance and Stochastics 1, in their Table 1), who consider JPY, USD and DEM data. They show that one principal component explains from 68% to 76% of the total variation, whereas three principal components can explain from 93% to 94%. A related analysis is carried out in Chapter 3 of Rebonato (book on interest rate models, 1998, in his Table 3.2) for the UK market, where results seem to be more optimistic: One component explains 92% of the total variance, whereas two components already explain 99.1% of the total variance. In some works an interpretation is given to the components in terms of average level, slope and curvature of the zero-coupon curve, see for example again Jamshidian and Zhu (1997).

Multidimensional models: how many factors? II

What we learn from these analyses is that, in the objective world, a while back a two- or three-dimensional process was needed to provide a realistic evolution of the whole zero-coupon curve. Since the instantaneous-covariance structure of the same process when moving from the objective probability measure to the risk-neutral probability measure does not change, we may guess that also in the risk-neutral world a two- or three-dimensional process may be needed in order to obtain satisfactory results. This is a further motivation for introducing a two- or three-factor model for the short rate. Here, we have decided to focus on two-factor models for their better tractability and implementability. In particular, we will consider additive models of the form

$$r_t = x_t + y_t + \varphi(t), \tag{24}$$

where φ is a deterministic shift which is added in order to fit exactly the initial zero-coupon curve, as in the one-factor case. This formulation

Multidimensional models: how many factors? III

encompasses the classical Hull and White two-factor model as a deterministically-shifted two-factor Vasicek (G2++), and an extension of the Longstaff and Schwartz (LS) model that is capable of fitting the initial term structure of rates (CIR2++), where the basic LS model is obtained as a two-factor additive CIR model.

Multidimensional models: volatility shape I

These are the two-factor models we will consider, and we will focus especially on the two-factor additive Gaussian model G2++. The main advantage of the G2++ model over the shifted Longstaff and Schwartz CIR2++ with x and y as in

$$dx_t = k_x(\theta_x - x_t)dt + \sigma_x\sqrt{x_t}dW_1(t),$$

$$dy_t = k_y(\theta_y - y_t)dt + \sigma_y\sqrt{y_t}dW_2(t),$$

is that in the latter we are forced to take $dW_1\,dW_2 = 0\,dt$ in order to maintain analytical tractability, whereas in the former we do not need to do so. The reason why we are forced to take $\rho = 0$ in the CIR2++ case lies in the fact that square-root non-central chi-square processes do not work as well as linear-Gaussian processes when adding nonzero instantaneous correlations. Requiring $dW_1\,dW_2 = \rho\,dt$ with $\rho \neq 0$ in the above CIR2++ model would indeed destroy analytical tractability: It

Multidimensional models: volatility shape II

would no longer be possible to compute analytically bond prices and rates starting from the short-rate factors. Moreover, the distribution of r would become more involved than that implied by a simple sum of independent non-central chi-square random variables. Why is the possibility that the parameter ρ be different than zero so important as to render G2++ preferable to CIR2++? As we said before, the presence of the parameter ρ renders the correlation structure of the two-factor model more flexible. Moreover, $\rho < 0$ allows for a humped volatility curve of the instantaneous forward rates. Indeed, if we consider at a given time instant t the graph of the T function

$$T \mapsto \sqrt{\mathrm{Var}[d\ f(t,T)]/dt}$$

where the instantaneous forward rate $f(t,T)$ comes from the G2++ model, it can be seen that for $\rho = 0$ this function is decreasing and upwardly concave. This function can assume a humped shape for

Multidimensional models: volatility shape III

suitable values of k_x and k_y only when $\rho < 0$. Since such a humped shape is a desirable feature of the model which is in agreement with market behaviour, it is important to allow for nonzero instantaneous correlation in the G2++ model. The situation is somewhat analogous in the CIR2++ case: Choosing $\rho = 0$ does not allow for humped shapes in the curve

$$T \mapsto \sqrt{\text{Var}[d\, f(t, T)]/dt},$$

which consequently results monotonically decreasing and upwardly concave, exactly as in the G2++ case with $\rho = 0$, as we will see later on in the chapter.

Multidimensional models: G2++ vs CIR2++ I

In turn, the advantage of CIR2++ over G2++ is that, as in the one-factor case where HW is compared to CIR++, it can maintain positive rates through reasonable restrictions on the parameters. Moreover, the distribution of the short rate is the distribution of the sum of two independent noncentral chi-square variables, and as such it has fatter tails than the Gaussian distribution in G2++. This is considered a desirable property, especially because in such a way (continuously-compounded) spot rates for any maturity are affine transformations of such non-central chi-squared variables and are closer to the lognormal distribution than the Gaussian distribution for the same rates implied by the G2++ model. Therefore, both from a point of view of positivity and distribution of rates, the CIR2++ model would be preferable to the G2++ model. However, the humped shape for the instantaneous forward rates volatility curve is very important for the model to be able to fit market data in a satisfactory way.

Multidimensional models: G2++ vs CIR2++ II

Furthermore, the G2++ model is more analytically tractable and easier to implement. These overall considerations then imply that the G2++ model is more suitable for practical applications, even though we should not neglect the advantages that a model like CIR2++ may have. In general, when analyzing an interest rate model from a practical point of view, one should try to answer questions like the following. Is a two-factor model like G2++ flexible enough to be calibrated to a large set of swaptions, or even to caps and swaptions at the same time? How many swaptions can be calibrated in a sufficiently satisfactory way? What is the evolution of the term structure of volatilities as implied by the calibrated model? Is this realistic? How can one implement trees for models such as G2++? Is Monte Carlo simulation feasible? Can the model be profitably used for quanto-like products and for products depending on more than an interest rate curve when

Multidimensional models: G2++ vs CIR2++ III

taking into account correlations between different interest-rate curves and also with exchange rates?

Here we will focus mainly on the G2++ model and we will try to deal with some of the above questions.

The G2++ model I

We assume that the dynamics of the instantaneous-short-rate process under the risk-adjusted measure Q is given by

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0, \tag{25}$$

where the processes $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ satisfy

$$dx(t) = -ax(t)dt + \sigma dW_1(t), \quad x(0) = 0,$$

$$dy(t) = -by(t)dt + \eta dW_2(t), \quad y(0) = 0,$$

where (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ as from

$$dW_1(t)dW_2(t) = \rho dt,$$

The G2++ model II

where r_0, a, b, σ, η are positive constants, and where $-1 \leq \rho \leq 1$. The function φ is deterministic and well defined in the time interval $[0, T^*]$, with T^* a given time horizon, typically 10, 30 or 50 (years). In particular, $\varphi(0) = r_0$. We denote by $\mathcal{F}_t$ the sigma-field generated by the pair (x, y) up to time t.

The G2++ model I

Simple integration of these equations implies that for each $s < t$

$$r(t) = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)}$$

$$+\sigma \int_s^t e^{-a(t-u)} dW_1(u)$$

$$+\eta \int_s^t e^{-b(t-u)} dW_2(u) + \varphi(t),$$

meaning that $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given respectively by

$$E\{r(t)|\mathcal{F}_s\} = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \varphi(t),$$

$$\mathrm{Var}\{r(t)|\mathcal{F}_s\} = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right] + \frac{\eta^2}{2b}\left[1 - e^{-2b(t-s)}\right]$$

The G2++ model II

$$+2\rho\frac{\sigma\eta}{a+b}\left[1-e^{-(a+b)(t-s)}\right].$$

In particular

$$r(t)=\sigma\int_0^t e^{-a(t-u)}dW_1(u)+\eta\int_0^t e^{-b(t-u)}dW_2(u)+\varphi(t). \quad (26)$$

Bond pricing I

We denote by $P(t, T)$ the price at time t of a zero-coupon bond maturing at T and with unit face value, so that

$$P(t, T) = E\left\{e^{-\int_t^T r_s ds} | \mathcal{F}_t\right\},$$

where as usual E denotes the expectation under the risk-adjusted measure Q. In order to explicitly compute this expectation, we need the following

Lemma. For each t, T the random variable

$$I(t, T) := \int_t^T [x(u) + y(u)]du$$

Bond pricing II

conditional to the sigma-field $\mathcal{F}_t$ is normally distributed with mean $M(t,T)$ and variance $V(t,T)$, respectively given by

$$M(t,T) = \frac{1-e^{-a(T-t)}}{a}x(t) + \frac{1-e^{-b(T-t)}}{b}y(t) \tag{27}$$

and

Bond pricing III

$$V(t,T) = \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]$$
$$+\frac{\eta^2}{b^2}\left[T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right]$$
$$+2\rho\frac{\sigma\eta}{ab}\left[T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b}\right.$$
$$\left. - \frac{e^{-(a+b)(T-t)} - 1}{a+b}\right].$$

Proof is not too difficult but is omitted.

Bond pricing IV

The price at time t of a zero-coupon bond maturing at time T and with unit face value is

$$P(t,T) = \exp\left\{-\int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t) + \frac{1}{2}V(t,T)\right\}. \qquad (28)$$

Proof: Being φ a deterministic function, the theorem follows from straightforward application of the Lemma and the fact that if Z is a normal random variable with mean m_Z and variance σ_Z^2, then $E\{\exp(Z)\} = \exp(m_Z + \frac{1}{2}\sigma_Z^2)$.

Bond pricing V

Let us now assume that the term structure of discount factors that is currently observed in the market is given by the sufficiently smooth function $T \mapsto P^M(0,T)$.
If we denote by $f^M(0,T)$ the instantaneous forward rate at time 0 for a maturity T implied by the term structure $T \mapsto P^M(0,T)$, i.e.,

$$f^M(0,T) = -\frac{\partial \ln P^M(0,T)}{\partial T},$$

we then have the following:

Bond pricing VI

The G++ model fits the currently-observed term structure of discount factors if and only if, for each T,

$$\begin{aligned}\varphi(T) \;&=\; f^M(0,T) + \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 \\ &+\frac{\eta^2}{2b^2}\left(1-e^{-bT}\right)^2 + \qquad (29)\\ &+\rho\frac{\sigma\eta}{ab}\left(1-e^{-aT}\right)\left(1-e^{-bT}\right), \qquad (30)\end{aligned}$$

i.e., if and only if

$$\exp\left\{-\int_t^T \varphi(u)du\right\} =$$

$$= \frac{P^M(0,T)}{P^M(0,t)}\exp\left\{-\frac{1}{2}[V(0,T)-V(0,t)]\right\},$$

Bond pricing VII

so that the corresponding zero-coupon-bond prices at time t are given by

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\{\mathcal{A}(t,T)\}$$

$$\mathcal{A}(t,T) := \frac{1}{2}[V(t,T) - V(0,T) + V(0,t)]$$

$$-\frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t).$$

Proof is omitted.

(Is it really necessary to derive the market instantaneous forward curve?) Notice that, at a first sight, one may have the impression that in order to implement the G2++ model we need to derive the whole φ curve, and therefore the market instantaneous forward curve

Bond pricing VIII

$T \mapsto f^M(0, T)$. Now, this curve involves differentiating the market discount curve $T \mapsto P^M(0, T)$, which is usually obtained from a finite set of maturities via interpolation. Interpolation and differentiation may induce a certain degree of approximation, since the particular interpolation technique being used has a certain impact on (first) derivatives.

However, it turns out that one does not really need the whole φ curve. Indeed, what matters is the integral of φ between two given instants. This integral has been computed above. From this expression, we see that the only curve needed is the market discount curve, which need not be differentiated, and only at times corresponding to the maturities of the bond prices and rates desired, thus limiting also the need for interpolation.

Bond pricing IX

(Short-rate distribution and probability of negative rates). By fitting the currently-observed term structure of discount factors, we obtain that the expected instantaneous short rate at time t, $\mu_r(t)$, is

$$\mu_r(t) := E\{r(t)\} =$$

$$= f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 + \frac{\eta^2}{2b^2}\left(1-e^{-bt}\right)^2$$

$$+\rho\frac{\sigma\eta}{ab}(1-e^{-at})\left(1-e^{-bt}\right),$$

while the variance $\sigma_r^2(t)$ of the instantaneous short rate at time t is

$$\sigma_r^2(t) = \text{Var}\{r(t)\} = \frac{\sigma^2}{2a}\left(1-e^{-2at}\right) + \frac{\eta^2}{2b}\left(1-e^{-2bt}\right)$$

$$+2\frac{\rho\sigma\eta}{a+b}\left(1-e^{-(a+b)t}\right).$$

Bond pricing X

This implies that the risk-neutral probability of negative rates at time t is

$$Q\{r(t) < 0\} = \Phi\left(-\frac{\mu_r(t)}{\sigma_r(t)}\right),$$

which is often negligible in many concrete situations, with Φ denoting the standard normal cumulative distribution function.

Warning. When trying to use G2++ or even the one factor model after the beginning of the crisis in 2007, one often finds that the probability of negative rates has increased dramatically. This is due to the large market volatilities and the low levels of rates.

We have that the limit distribution of the process r is Gaussian with mean $\mu_r(\infty)$ and variance $\sigma_r^2(\infty)$ given by

$$\mu_r(\infty) := \lim_{t\to\infty} E\{r(t)\} =$$

Bond pricing XI

$$= f^M(0,\infty) + \frac{\sigma^2}{2a^2} + \frac{\eta^2}{2b^2} + \rho\frac{\sigma\eta}{ab},$$

$$\sigma_r^2(\infty) := \lim_{t\to\infty} \text{Var}\{r(t)\} =$$

$$= \frac{\sigma^2}{2a} + \frac{\eta^2}{2b} + 2\rho\frac{\sigma\eta}{a+b},$$

where

$$f^M(0,\infty) = \lim_{t\to\infty} f^M(0,t).$$

Vol and Correl Structures in 2-Factor Models I

We now derive the dynamics of forward rates under the risk-neutral measure to obtain an equivalent formulation of the two-additive-factor Gaussian model in the Heath-Jarrow-Morton (1992) framework. In particular, we explicitly derive the volatility structure of forward rates. This also allows us to understand which market-volatility structures can be fitted by the model.

Let us define $A(t,T)$ and $B(z,t,T)$ by

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{ \frac{1}{2}[V(t,T) - V(0,T) + V(0,t)] \right\},$$

$$B(z,t,T) = \frac{1 - e^{-z(T-t)}}{z},$$

so that we can write

$$P(t,T) = A(t,T) \exp\{-B(a,t,T)x(t) - B(b,t,T)y(t)\}. \tag{31}$$

Vol and Correl Structures in 2-Factor Models II

The (continuously-compounded) instantaneous forward rate at time t for the maturity T is then given by

$$
\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}\ln P(t,T) \\
&= -\frac{\partial}{\partial T}\ln A(t,T) + \frac{\partial B}{\partial T}(a,t,T)x(t) \\
&\quad + \frac{\partial B}{\partial T}(b,t,T)y(t),
\end{aligned}
$$

whose differential form can be written as

$$
df(t,T) = \ldots dt + \frac{\partial B}{\partial T}(a,t,T)\sigma dW_1(t) + \frac{\partial B}{\partial T}(b,t,T)\eta dW_2(t).
$$

Vol and Correl Structures in 2-Factor Models III

Therefore

$$\frac{\mathrm{Var}(df(t,T))}{dt} = \left(\frac{\partial B}{\partial T}(a,t,T)\sigma\right)^2 + \left(\frac{\partial B}{\partial T}(b,t,T)\eta\right)^2 + 2\rho\sigma\eta\frac{\partial B}{\partial T}(a,t,T)\frac{\partial B}{\partial T}(b,t,T)$$

$$= \sigma^2 e^{-2a(T-t)} + \eta^2 e^{-2b(T-t)} + 2\rho\sigma\eta e^{-(a+b)(T-t)},$$

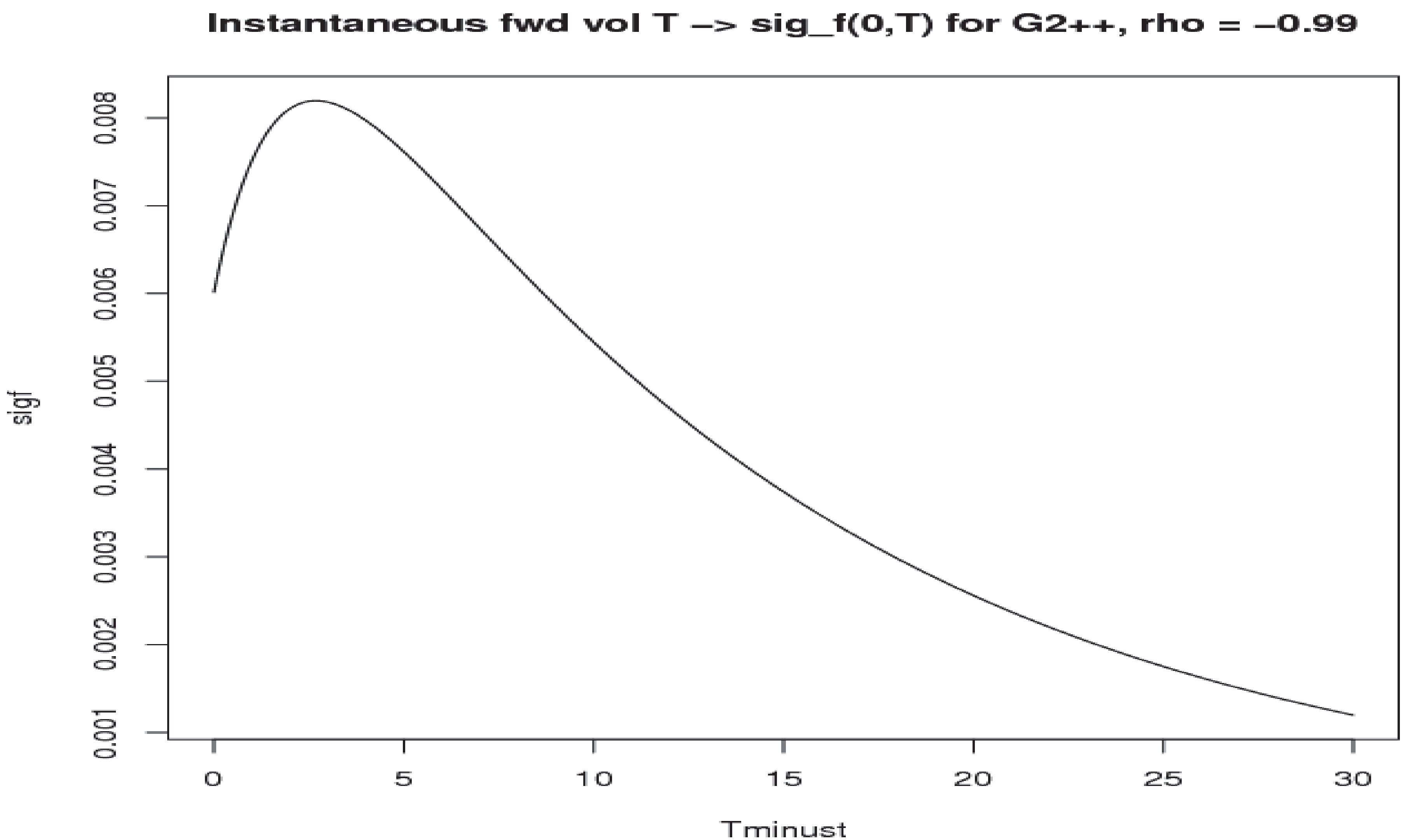
which implies that the absolute volatility of the instantaneous forward rate $f(t,T)$ is

$$\sigma_f(t,T) = \sqrt{\sigma^2 e^{-2a(T-t)} + \eta^2 e^{-2b(T-t)} + 2\rho\sigma\eta e^{-(a+b)(T-t)}}.$$

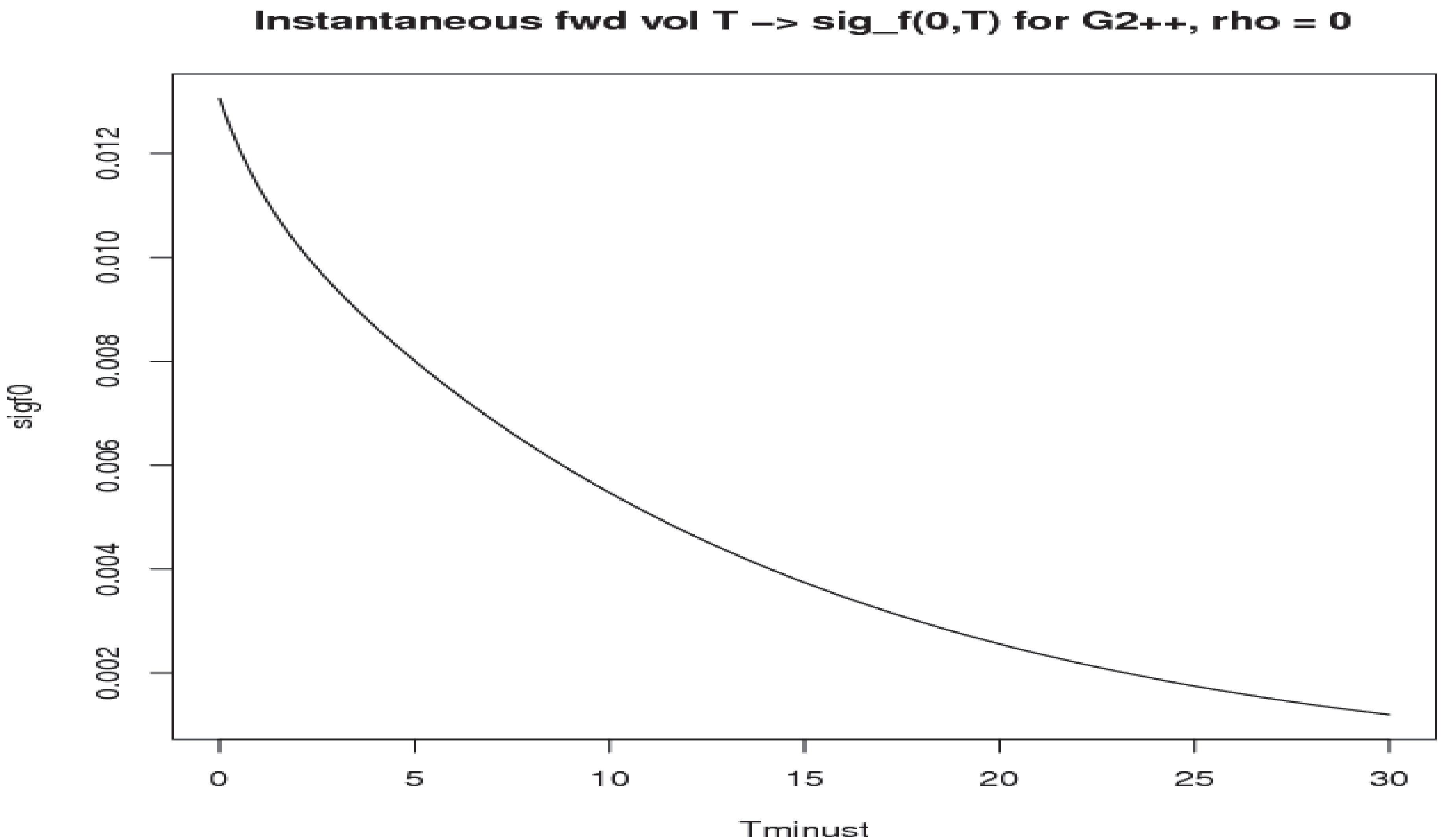
Vol and Correl Structures in 2-Factor Models IV

We immediately see that the desirable feature, as far as calibration to the market is concerned, of a humped volatility structure similar to what is commonly observed in the market for the caplets volatility, may be only reproduced for negative values of ρ. Notice indeed that if ρ is positive, the terms $\sigma^2 e^{-2a(T-t)}$, $\eta^2 e^{-2b(T-t)}$ and $2\rho\sigma\eta e^{-(a+b)(T-t)}$ are all decreasing functions of the time to maturity $T - t$ and no hump is possible. This does not mean, in turn, that every combination of the parameter values with a negative ρ leads to a volatility hump. A simple study of $\sigma_f(t, T)$ as a function of $T - t$, however, shows that there exist suitable choices of the parameter values that produce the desired shape.

$T \mapsto \sigma_f(0, T)$ for G2 calibrated on 13 02 2001. $a = .54, b = .076, \sigma = .0058, \eta = .0117, \rho = -0.99$



$T \mapsto \sigma_f(0, T)$ for G2 calibrated on 13 02 2001.
$a = 0.54, b = 0.076, \sigma = 0.0058, \eta = 0.0117, \rho = 0$



Options in G2++ I

Given the current time t and the future times T_1 and T_2, a caplet pays off at time T_2

$$[L(T_1, T_2) - X]^+ \alpha(T_1, T_2) N,$$

where N is the nominal value, X is the caplet rate (strike), $\alpha(T_1, T_2)$ is the year fraction between times T_1 and T_2 and $L(T_1, T_2)$ is the LIBOR rate at time T_1 for the maturity T_2, i.e.,

$$L(T_1, T_2) = \frac{1}{\alpha(T_1, T_2)} \left[\frac{1}{P(T_1, T_2)} - 1 \right].$$

By setting

$$X' = \frac{1}{1 + X\alpha(T_1, T_2)}, N' = N(1 + X\alpha(T_1, T_2)),$$

Options in G2++ II

we have

$$\mathrm{Cpl}(t, T_1, T_2, N, X) = E[D(t, T_2)\,(L(T_1, T_2) - X)^+\,\alpha(T_1, T_2)N] \tag{32}$$

$$= -N'P(t, T_2)\Phi\left(\frac{\ln \frac{NP(t,T_1)}{N'P(t,T_2)}}{\Sigma(t, T_1, T_2)} - \frac{1}{2}\Sigma(t, T_1, T_2)\right) \tag{33}$$

$$+P(t, T_1)N\Phi\left(\frac{\ln \frac{NP(t,T_1)}{N'P(t,T_2)}}{\Sigma(t, T_1, T_2)} + \frac{1}{2}\Sigma(t, T_1, T_2)\right). \tag{34}$$

Options in G2++ III

where

$$\Sigma(t,T,S)^2 = \frac{\sigma^2}{2a^3}\left[1-e^{-a(S-T)}\right]^2\left[1-e^{-2a(T-t)}\right] + \frac{\eta^2}{2b^3}\left[1-e^{-b(S-T)}\right]^2\left[1-e^{-2b(T-t)}\right] + 2\rho\frac{\sigma\eta}{ab(a+b)}\left[1-e^{-a(S-T)}\right]\left[1-e^{-b(S-T)}\right]\left[1-e^{-(a+b)(T-t)}\right].$$

From caplets one gets caps by adding up. Floorlets and floor are completely analogous. For the details see Brigo and Mercurio (2006).

First Choice: short rate r I

This approach is based on the fact that the zero coupon curve at any instant, or the (informationally equivalent) zero bond curve

$$T \mapsto P(t,T) = E_t^Q \exp\left(-\int_t^T \boxed{r_s}\, ds\right)$$

is completely characterized by the probabilistic/dynamical properties of r. So we write a model for r, the initial point of the curve $T \mapsto L(t,T)$ for $T = t$ at every instant t.

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t$$

- Unrealistic correlation patterns between points of the curve with different maturities. for example, in one-factor short-rate models

$$\mathrm{Corr}(dF_i(t), dF_j(t)) = 1;$$

First Choice: short rate r II

- Poor calibration capabilities: can only fit a low number of caps and swaptions unless dangerous and uncontrollable extensions are taken into account;
- Difficulties in expressing market views and quotes in terms of model parameters;
- Related lack of agreement with market valuation formulas for basic derivatives.
- Models that are good as distribution (lognormal models) are not analytically tractable and have problems of explosion for the bank account.

What? 3d choice: MARKET MODELS. Intro I

Before market models were introduced, short-rate models used to be the main choice for pricing and hedging interest-rate derivatives. Short-rate models are still chosen for many applications and are based on modeling the instantaneous spot interest rate ("short rate" r_t) via a (possibly multi-dimensional) diffusion process. This diffusion process characterizes the evolution of the complete yield curve in time. To introduce market models, recall the forward LIBOR rate at time t between T and S,

$$F(t; T, S) = \frac{1}{(S-T)}(P(t,T)/P(t,S) - 1),$$

which makes the FRA contract to lock in at time t interest rates between T and S fair (=0). **A family of such rates for $(T, S) = (T_{i-1}, T_i)$ spanning $T_0, T_1, T_2, \ldots, T_M$ is modeled in the LIBOR market model.**

What? 3d choice: MARKET MODELS. Intro II

These are rates associated to market payoffs (FRA's) and not abstract rates such as r_t or $f(t, T)$ (rates on infinitesimal maturities/tenors).

To further motivate market models, let us consider the time-0 price of a T_2-maturity caplet resetting at time T_1 $(0 < T_1 < T_2)$ with strike X and a notional amount of 1. Let τ denote the year fraction between T_1 and T_2. Such a contract pays out at time T_2 the amount

$$\tau(L(T_1, T_2) - X)^+ = \tau(F_2(T_1) - X)^+.$$

On the other hand, the market has been pricing caplets (actually caps) with Black's formula for years. Let us see how this formula is rigorously derived under the LIBOR model dynamics, the only dynamical model that is consistent with it.

What? 3d choice: MARKET MODELS. Intro III

FACT ONE. *The price of any asset divided by a reference asset (called numeraire) is a martingale (no drift) under the measure associated with that numeraire.*
In particular,

$$F_2(t) = \frac{(P(t,T_1) - P(t,T_2))/(T_2 - T_1)}{P(t,T_2)},$$

is a portfolio of two zero coupon bonds divided by the zero coupon bond $P(\cdot, T_2)$. If we take the measure Q^2 associated with the numeraire $P(\cdot, T_2)$, by FACT ONE F_2 will be a martingale (no drift) under that measure.
F_2 is a martingale (no drift) under that Q^2 measure associated with numeraire $P(\cdot, T_2)$.

What? 3d choice: MARKET MODELS. Intro IV

FACT TWO: THE TIME-t RISK NEUTRAL PRICE

$$\text{Price}_t = E_t^{\boxed{B}}\left[\boxed{B(t)}\frac{\text{Payoff}(T)}{\boxed{B(T)}}\right]$$

IS INVARIANT BY CHANGE OF NUMERAIRE: IF S IS ANY OTHER NUMERAIRE, WE HAVE

$$\text{Price}_t = E_t^{\boxed{S}}\left[\boxed{S_t}\frac{\text{Payoff}(T)}{\boxed{S_T}}\right].$$

IN OTHER TERMS, IF WE SUBSTITUTE THE THREE OCCURRENCES OF THE NUMERAIRE WITH A NEW NUMERAIRE THE PRICE DOES NOT CHANGE.

What? 3d choice: MARKET MODELS. Intro V

Consider now the caplet price and apply FACT TWO: Replace B with $P(\cdot, 2)$

$$E^B\left[\frac{B(0)}{B(T_2)}\tau(F_2(T_1) - X)^+\right] =$$

$$= E^{Q^2}\left[\frac{P(0,T_2)}{P(T_2,T_2)}\tau(F_2(T_1) - X)^+\right]$$

Take out $P(0, T_2)$ and recall that $P(T_2, T_2) = 1$. We have

$$= P(0, T_2)E^{Q^2}\tau\left[(F_2(T_1) - X)^+,\right]$$

By fact ONE F_2 is a martingale (no drift) under Q_2. Take a geometric Brownian motion

$$dF(t; T_1, T_2) = \boxed{\sigma_2(t)}\, F(t; T_1, T_2)dW_2(t), \quad \text{mkt } F(0; T_1, T_2)$$

What? 3d choice: MARKET MODELS. Intro VI

where σ_2 is the instantaneous volatility, assumed here to be constant for simplicity, and W_2 is a standard Brownian motion under the measure Q^2. **The forward LIBOR rates F's are the quantities that are modeled instead of r and f in the LIBOR market model**.

$$dF_2(t) = \boxed{\sigma_2(t)}\, F_2(t)dW_2(t), \quad \text{mkt } F_2(0)$$

Let us solve this equation and compute $E^{Q^2}[(F_2(T_1) - X)^+,]$. By Ito's formula:

What? 3d choice: MARKET MODELS. Intro VII

$$
\begin{aligned}
d\ln(F_2(t)) &= \ln'(F_2)dF_2 + \frac{1}{2}\ln''(F_2)\,dF_2\,dF_2 \\
&= \frac{1}{F_2}dF_2 + \frac{1}{2}\left(-\frac{1}{(F_2)^2}\right)dF_2\,dF_2 = \\
= \frac{1}{F_2}\sigma_2 F_2 dW_2 - \frac{1}{2}\frac{1}{(F_2)^2}&(\sigma_2 F_2 dW_2)(\sigma_2 F_2 dW_2) = \\
&= \sigma_2 dW_2 - \frac{1}{2}\frac{1}{(F_2)^2}\sigma_2^2 F_2^2 dW_2 dW_2 = \\
&= \sigma_2(t)dW_2(t) - \frac{1}{2}\sigma_2^2(t)dt
\end{aligned}
$$

(we used $dW_2\,dW_2 = dt$). So we have

$$d\ln(F_2(t)) = \sigma_2(t)dW_2(t) - \frac{1}{2}\sigma_2^2(t)dt$$

What? 3d choice: MARKET MODELS. Intro VIII

Integrate both sides:

$$\int_0^T d\ln(F_2(t)) = \int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt$$

$$\ln(F_2(T)) - \ln(F_2(0)) = \int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt$$

$$\ln\frac{F_2(T)}{F_2(0)} = \int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt$$

$$\frac{F_2(T)}{F_2(0)} = \exp\left(\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt\right)$$

$$F_2(T) = F_2(0)\exp\left(\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt\right)$$

What? 3d choice: MARKET MODELS. Intro IX

$$F_2(T) = F_2(0)\exp\left(\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt\right)$$

Compute the distribution of the random variable in the exponent.
It is Gaussian, since it is a stochastic integral of a deterministic function times a Brownian motion (sum of independent Gaussians is Gaussian).
Compute the expectation:

$$E[\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt] = 0 - \frac{1}{2}\int_0^T \sigma_2^2(t)dt$$

What? 3d choice: MARKET MODELS. Intro X

and the variance

$$\mathrm{Var}\left[\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt\right] =$$

$$= \mathrm{Var}\left[\int_0^T \sigma_2(t)dW_2(t)\right]$$

$$= E\left[\left(\int_0^T \sigma_2(t)dW_2(t)\right)^2\right] - 0^2 = \int_0^T \sigma_2(t)^2 dt$$

where we have used Ito's isometry in the last step.

What? 3d choice: MARKET MODELS. Intro XI

We thus have

$$I(T) := \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2}\int_0^T \sigma_2^2(t) dt \sim$$

$$\sim m + V\mathcal{N}(0,1), \quad m = -\frac{1}{2}\int_0^T \sigma_2(t)^2 dt, \; V^2 = \int_0^T \sigma_2(t)^2 dt$$

Recall that we have

$$F_2(T) = F_2(0)\exp(I(T)) = F_2(0)e^{m+V\mathcal{N}(0,1)}$$

Compute now the option price

$$E^{Q^2}[(F_2(T_1) - X)^+] = E^{Q^2}[(F_2(0)e^{m+V\mathcal{N}(0,1)} - X)^+]$$
$$= \int_{-\infty}^{+\infty} (F_2(0)e^{m+Vy} - X)^+ p_{\mathcal{N}(0,1)}(y) dy = \ldots$$

What? 3d choice: MARKET MODELS. Intro XII

Note that $F_2(0)\exp(m+Vy) - X > 0$ if and only if

$$y > \frac{-\ln\left(\frac{F_2(0)}{X}\right) - m}{V} =: \bar{y}$$

so that

$$\ldots = \int_{\bar{y}}^{+\infty} (F_2(0)\exp(m+Vy) - X) p_{\mathcal{N}(0,1)}(y) dy =$$

$$= F_2(0) \int_{\bar{y}}^{+\infty} e^{m+Vy} p_{\mathcal{N}(0,1)}(y) dy - X \int_{\bar{y}}^{+\infty} p_{\mathcal{N}(0,1)}(y) dy =$$

What? 3d choice: MARKET MODELS. Intro XIII

$$= F_2(0)\frac{1}{\sqrt{2\pi}}\int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}y^2+Vy+m}dy - X(1-\Phi(\bar{y}))$$

$$= F_2(0)\frac{1}{\sqrt{2\pi}}\int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}(y-V)^2+m+\frac{1}{2}V^2}dy - X(1-\Phi(\bar{y})) =$$

$$= F_2(0)e^{m+\frac{1}{2}V^2}\frac{1}{\sqrt{2\pi}}\int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}(y-V)^2}dy - X(1-\Phi(\bar{y})) =$$

$$= F_2(0)e^{m+\frac{1}{2}V^2}\frac{1}{\sqrt{2\pi}}\int_{\bar{y}-V}^{+\infty} e^{-\frac{1}{2}z^2}dz - X(1-\Phi(\bar{y})) =$$

$$= F_2(0)e^{m+\frac{1}{2}V^2}(1-\Phi(\bar{y}-V)) - X(1-\Phi(\bar{y})) =$$

$$= F_2(0)e^{m+\frac{1}{2}V^2}\Phi(-\bar{y}+V) - X\Phi(-\bar{y}) =$$

$$= F_2(0)\Phi(d_1) - X\Phi(d_2), \quad d_{1,2} = \frac{\ln\frac{F_2(0)}{X} \pm \frac{1}{2}\int_0^{T_1}\sigma_2^2(t)dt}{\sqrt{\int_0^{T_1}\sigma_2^2(t)dt}}$$

What? 3d choice: MARKET MODELS. Intro XIV

$$\mathrm{Cpl}(0, T_1, T_2, X) = P(0, T_2)\tau[F_2(0)\Phi(d_1) - X\Phi(d_2)],$$

$$d_{1,2} = \frac{\ln \frac{F_2(0)}{X} \pm \frac{1}{2}\int_0^{T_1} \sigma_2^2(t)dt}{\sqrt{\int_0^{T_1} \sigma_2^2(t)dt}}$$

This is exactly the classic market Black's formula for the $T_1 - T_2$ caplet. The term in squared brackets can be also written as

$$= F_2(0)\Phi(d_1) - X\Phi(d_2), \quad d_{1,2} = \frac{\ln \frac{F_2(0)}{X} \pm \frac{1}{2} T_1 v_1(T_1)^2}{\sqrt{T_1} v_1(T_1)}$$

where $v_1(T_1)$ is the time-averaged quadratic volatility

$$v_1(T_1)^2 = \frac{1}{T_1}\int_0^{T_1} \sigma_2(t)^2 dt.$$

What? 3d choice: MARKET MODELS. Intro XV

Notice that in case $\sigma_2(t) = \sigma_2$ is constant we have $v_1(T_1) = \sigma_2$. Summing up: take

$$dF(t; T_1, T_2) = \sigma_2\, F(t; T_1, T_2) dW_2(t), \quad \text{mkt } F(0; T_1, T_2)$$

The current zero-curve $T \mapsto L(0, T)$ is calibrated through **the initial market** $F(0; T, S)$**'s.** This dynamics in **under the numeraire** $P(\cdot, T_2)$ (measure Q^2), where W_2 is a Brownian motion. We wish to compute

$$E\left[\frac{B(0)}{B(T_2)}\tau(F(T_1; T_1, T_2) - X)^+\right]$$

What? 3d choice: MARKET MODELS. Intro XVI

We obtain from the change of numeraire and under Q^2, assuming **lognormality of F**:

$$
\begin{aligned}
\mathrm{Cpl}(0, T_1, T_2, X) &:= P(0, T_2)\tau E(F(T_1; T_1, T_2) - X)^+ \\
= P(0, T_2)\tau[F(0; T_1, T_2)\Phi(d_1(X, F(0; T_1, T_2), \sigma_2\sqrt{T_1})) \\
&\quad -X\Phi(d_2(X, F(0; T_1, T_2), \sigma_2\sqrt{T_1}))], \\
d_{1,2}(X, F, u) &= \frac{\ln(F/X) \pm u^2/2}{u},
\end{aligned}
$$

This is the Black formula used in the market to convert Cpl prices in volatilities σ and vice-versa. This dynamical model is thus compatible with Black's market formula. The key property is **lognormality** of F when taking the expectation.

What? 3d choice: MARKET MODELS. Intro XVII

The example just introduced is a simple case of what is known as "lognormal forward-LIBOR model". It is known also as Brace-Gatarek-Musiela (1997) model, from the name of the authors of one of the first papers where it was introduced rigorously. This model was also introduced earlier by Miltersen, Sandmann and Sondermann (1997). Jamshidian (1997) also contributed significantly to its development. However, a common terminology is now emerging and the model is generally known as "LIBOR Market Model" (LMM).

What? 3d choice: MARKET MODELS. Intro XVIII

Question: Can this model be obtained as a special **short rate model?** Is there a choice for the equation of r that is consistent with the above market formula, or with the lognormal distribution of F's?
Again to fix ideas, let us choose a specific short-rate model and assume we are using the Vasicek model. The parameters k, θ, σ, r_0 are denoted by α.

$$r_t = x_t, \quad dx_t = k(\theta - x_t)dt + \sigma dW_t.$$

Such model allows for an analytical formula for forward LIBOR rates F,

$$F(t; T_1, T_2) \quad = \quad F^{VAS}(t; T_1, T_2; x_t, \alpha).$$

At this point one can try and price a caplet. To this end, one can compute the risk-neutral expectation

$$E\left[\frac{B(0)}{B(T_2)}\tau(F^{VAS}(T_1; T_1, T_2, x_{T_1}, \alpha) - X)^+\right].$$

What? 3d choice: MARKET MODELS. Intro XIX

This too turns out to be feasible, and leads to a function

$$U_C^{VAS}(0, T_1, T_2, X, \alpha).$$

Question: Is there a short-rate model compatible with the Market model? For VASICEK $dx_t = k(\theta - x_t)dt + \sigma dW_t$, rewritten under Q^2, we have

$$dF^{VAS}(t; T_1, T_2; x_t, \alpha) = \frac{\partial F^{VAS}}{\partial [t, x]} d[t\ x_t]' + \frac{1}{2} \frac{\partial^2 F^{VAS}}{\partial x^2} (dx_t)^2,$$

VS Lognormal $dF(t; T_1, T_2) = vF(t; T_1, T_2)dW_2(t)$.

F^{VAS} is not lognormal, nor are F's associated to other known short rate models. So no known short rate model is consistent with the market formula. Short rate models are calibrated through their

What? 3d choice: MARKET MODELS. Intro XX

particular formulas for caplets, but these formulas are not Black's market formula (although some are close).
When Hull and White (extended VASICEK) is calibrated to caplets one has the values of k, θ, σ, x_0 consistent with caplet prices, but these parameters don't have an **immediate intuitive meaning** for traders, who don't know **how to relate them to Black's market formula**. On the contrary, the parameter σ_2 in the mkt model **has an immediate meaning as the Black caplet volatility of the market. There is an immediate link between model parameters and market quotes. Language is important**.

What? 3d choice: MARKET MODELS. Intro XXI

When dealing with several caplets involving different forward rates,

$$F_2(t) = F(t; T_1, T_2), \quad F_3(t) = F(t; T_2, T_3), \ldots, F_\beta(t) := F(t; T_{\beta-1}, T_\beta),$$

or with swaptions, different structures of instantaneous volatilities can be employed. One can select a different σ for each forward rate by assuming each forward rate to have a constant instantaneous volatility. Alternatively, one can select piecewise-constant instantaneous volatilities for each forward rate. Moreover, different forward rates can be modeled as each having different random sources Z that are **instantaneously correlated**. This implies that we have great freedom in modeling

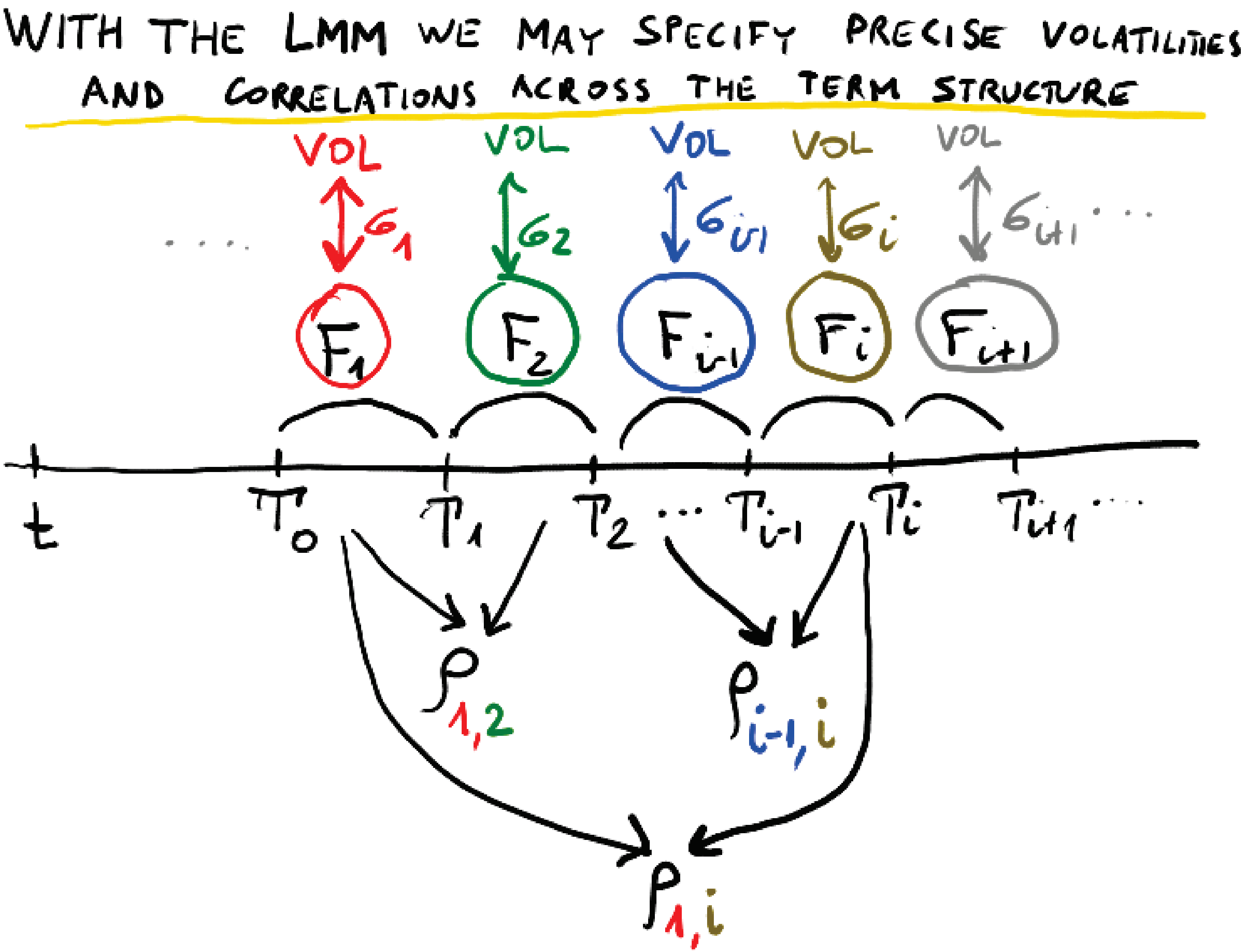
$$corr(dF_i(t), dF_j(t)) = \rho_{i,j}$$

whereas in one-factor short rate models dr these correlations were fixed practically to 1.

What? 3d choice: MARKET MODELS. Intro XXII

Modeling correlation is necessary for pricing payoffs depending on more than a single rate at a given time, such as swaptions.

What? 3d choice: MARKET MODELS. Intro XXIII

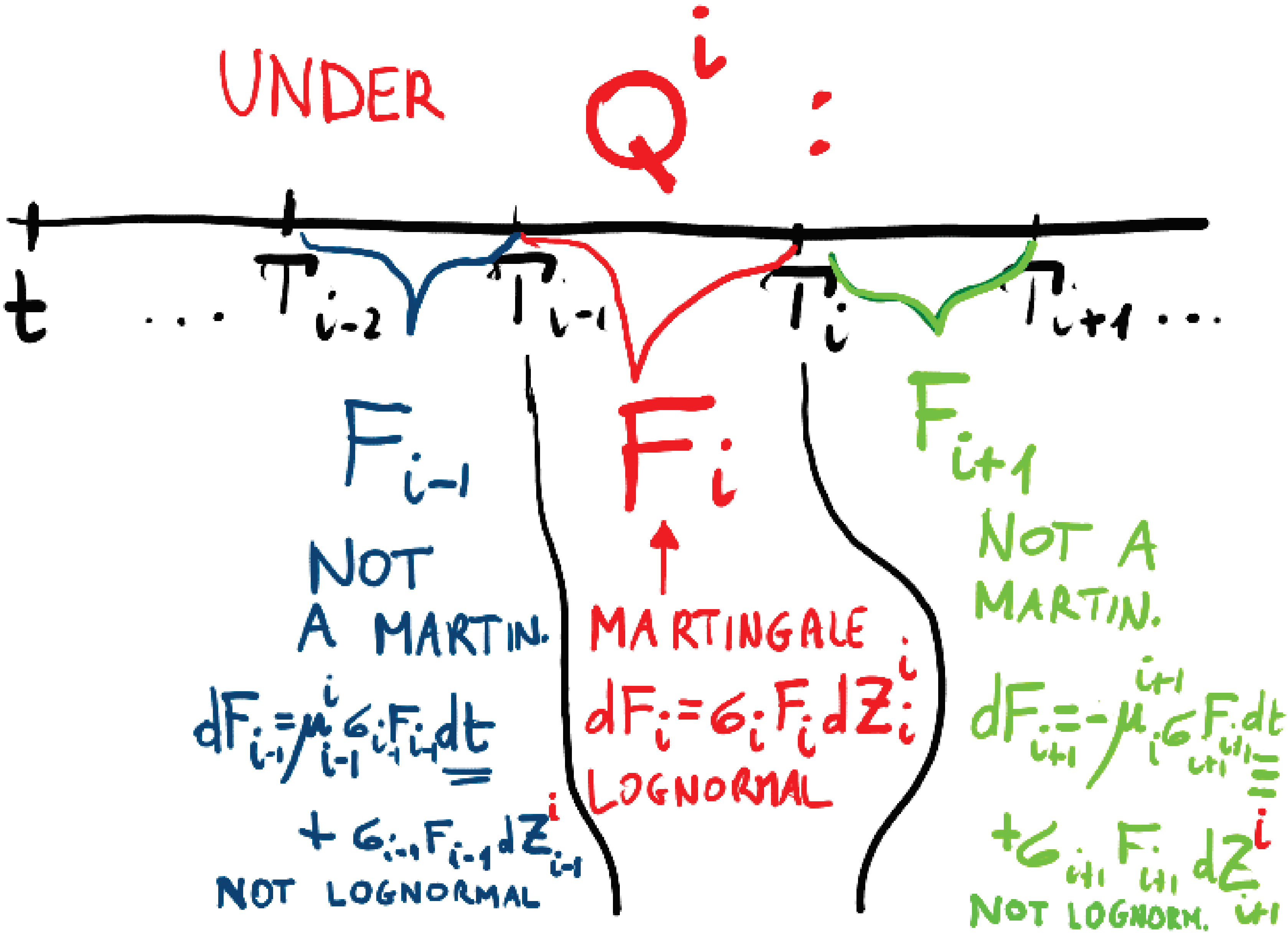


What? 3d choice: MARKET MODELS. Intro XXIV

Dynamics of $F_k(t) = F(t, T_{k-1}, T_k)$ under Q^k (numeraire $P(\cdot, T_k)$) is $dF_k(t) = \sigma_k(t) F_k \, dZ_k(t)$, lognormal distrib. (we have seen the example $k = 2$ above).

Dynamics of F_k under $Q^i \neq Q^k$ for $i < k$ and $i > k$ is more involved, has a complicated drift (local mean) and does not lead to a known distribution of F_k under such measures. Hence the model needs to be used with simulations (no PDE's) or approximations (drift freezing).

What? 3d choice: MARKET MODELS. Intro XXV



What? 3d choice: MARKET MODELS. Intro I

Precisely because the dynamics of $F_k(t) = F(t, T_{k-1}, T_k)$ under Q^k (numeraire $P(\cdot, T_k)$) is $dF_k(t) = \sigma_k(t) F_k \, dZ_k(t)$, lognormally distributed, the LIBOR market model is calibrated to caplets **automatically** through integrals of the squared deterministic functions $\sigma_k(t)$.
For example, if one takes constant $\sigma_k(t) = \sigma_k$ (constant), then σ_k is the market caplet volatility for the caplet resetting at T_{k-1} and paying at T_k.

What? 3d choice: MARKET MODELS. Intro II

No effort or complicated nonlinear inversion / minimization is involved to solve the "reverse engineering" problem
MarketCplPrice$(0, T_1, T_2, X_2) =$LIBORModelCplPrice$(\sigma_2?)$;
MarketCplPrice$(0, T_2, T_3, X_3) =$LIBORModelCplPrice$(\sigma_3?)$;
MarketCplPrice$(0, T_3, T_4, X_4) =$LIBORModelCplPrice$(\sigma_4?)$;
....
Whereas it is complicated to solve
MarketCplPrice$(0, T_1, T_2, X_2) =$VasicekModelCplPrice$(k?, \theta?, \sigma?)$;
MarketCplPrice$(0, T_2, T_3, X_3) =$VasicekModelCplPrice$(k?, \theta?, \sigma?)$;
MarketCplPrice$(0, T_3, T_4, X_4) =$VasicekModelCplPrice$(k?, \theta?, \sigma?)$;
....
Swaptions can be calibrated through some algebraic formulas under some good approximations, and the swaptions market formula is almost compatible with the model.

What? 3d choice: MARKET MODELS. Intro III

The LIBOR market model for F's allows for:

- immediate and intuitive calibration of caplets (better than any short rate model)
- easy calibration to swaptions through algebraic approximation (again better than most short rate models)
- can virtually calibrate a high number of market products exactly or with a precision impossible to short rate models;
- clear correlation parameters, since these are intantaneous correlations of market forward rates;
- Powerful diagnostics: can check **future** volatility and terminal correlation structures (Diagnostics impossibile with most short rate models);
- Can be used for monte carlo simulation;
- High dimensionality (many F are evolving jointly).

What? 3d choice: MARKET MODELS. Intro IV

- Unknown joint distribution of the F's (although each is lognormal under its canonical measure)
- Difficult to use with partial differential equations or lattices/trees, but recent Monte Carlo approaches such as Least Square Monte Carlo make trees and PDE's less necessary.

What? 3d choice: MARKET MODELS. Intro V

However the LIBOR market model is not the only market model. The simple market options on interest rates are divided in two markets CAPS/FLOORS and SWAPTION.

The LIBOR market model is the model of choice for caplets, as we have seen, since it produces the Black-Scholes type (Black's) caplet formula the market uses to quote implied volatilities.

But what about SWAPTIONS?

SWAPTIONS can be managed well in the LIBOR model only through approximations like drift freezing. To properly deal with swaptions, one would have to use a different market model, the SWAP market model (SMM).

We now present it briefly.

What? 3d choice: MARKET MODELS. Intro VI

Consider the payer swaption giving the right (and no obligation) to enter into the swap first resetting in T_α and paying at $T_{\alpha+1}, T_{\alpha+2} \ldots$ up to T_β, for a fixed rate K.

Recall that one way to write the payout of such option at maturity T_α is

$$(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i).$$

Let's define the annuity numeraire, also known as Present Value per Basis Point (PVPBP), PV01 or DV01, and the related measure:

$$U = C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i), \quad Q^U = Q^{\alpha,\beta}$$

What? 3d choice: MARKET MODELS. Intro VII

By FACT ONE the forward swap rate $S_{\alpha,\beta}$ is then a martingale under $Q^{\alpha,\beta}$:

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)} = \frac{P(t,T_\alpha) - P(t,T_\beta)}{C_{\alpha,\beta}(t)}$$

Take the usual martingale (zero drift) lognormal geometric brownian motion

$$d\,S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t)\, dW_t^{\alpha,\beta}, \quad Q^{\alpha,\beta} \quad \textbf{(SMM)},$$

What? 3d choice: MARKET MODELS. Intro VIII

BY FACT TWO on the change of numeraire

$$E^{\boxed{B}}\left((S_{\alpha,\beta}(T_\alpha)-K)^+ \; C_{\alpha,\beta}(T_\alpha)\frac{\boxed{B(0)}}{\boxed{B(T_\alpha)}}\right) =$$

$$= \; E^{\boxed{\alpha,\beta}}\left[(S_{\alpha,\beta}(T_\alpha)-K)^+ C_{\alpha,\beta}(T_\alpha)\frac{\boxed{C_{\alpha,\beta}(0)}}{\boxed{C_{\alpha,\beta}(T_\alpha)}}\right]$$

$$= C_{\alpha,\beta}(0) \; E^{\alpha,\beta}\left[(S_{\alpha,\beta}(T_\alpha)-K)^+\right]$$

$$= C_{\alpha,\beta}(0)\;[S_{\alpha,\beta}(0)\Phi\,(d_1)-K\Phi(d_2)], \;\; d_{1,2}=\frac{\ln\frac{S_{\alpha,\beta}(0)}{K}\pm\frac{1}{2}T_\alpha v^2_{\alpha,\beta}(T_\alpha)}{\sqrt{T_\alpha}v_{\alpha,\beta}(T_\alpha)}$$

$$v^2_{\alpha,\beta}(T)=\frac{1}{T}\int_0^T(\sigma^{(\alpha,\beta)}(t))^2\,dt\,.$$

What? 3d choice: MARKET MODELS. Intro IX

This is the well known Black's formula for swaptions.

It is a Black Scholes type formula for swaptions.

It is the formula the market uses to convert swaptions prices into swaptions implied volatilities v.

SMM is the only model that is consistent with this market formula.

LMM is not compatible with the Black formula for Swaptions.

The SMM is not used as much as the LMM. The reason is that swap rates do not recombine as well as forward rates in describing other rates. Also, swaptions can be priced easily in the LMM through drift freezing with formulas that are very similar to the market swaptions formula. It follows that, even if in principle the two models are not compatible and consistent, in practice the LMM is quite close to the SMM even in terms of swap rate dynamics.

What? 3d choice: MARKET MODELS. Intro X

Hence we will focus on the LMM only in the following.

End of the guided tour to the LIBOR model
Now we begin the detailed presentation.

Giving rigor to Black's formulas: The LMM market model in general I

End of the guided tour to the LIBOR model
Now we begin the detailed presentation.
Recall measure Q^U associated with numeraire U
(Risk–neutral measure $Q = Q^B$).
FACT 1: A/U, with A a tradable asset, is a Q^U-martingale
Caps: Rigorous derivation of Black's formula.
Take $U = P(\cdot, T_i)$, $Q^U = Q^i$. Since

$$F(t; T_{i-1}, T_i) = (1/\tau_i)(P(t, T_{i-1}) - P(t, T_i))/P(t, T_i),$$

$F(t; T_{i-1}, T_i) =: F_i(t)$ is a Q^i-martingale. Take

$$dF_i(t) = \sigma_i(t) F_i(t) dZ_i(t), \quad Q^i, \quad t \le T_{i-1}.$$

Giving rigor to Black's formulas: The LMM market model in general II

This is the **L**ognormal **F**orward–Libor **M**odel (LMM). Consider the discounted T_{k-1}–*caplet*

$$(F_k(T_{k-1}) - K)^+ B(0)/B(T_k)$$

The LMM model dynamics in general I

$$\textbf{LMM:}\ \ dF_k(t) = \sigma_k(t) F_k(t) dZ_k(t), \quad Q^k, \quad t \le T_{k-1}.$$

The price at the time 0 of the single caplet is (use FACT 2)

$$\boxed{B(0)}\, E^{\boxed{Q^B}} \left[(F_k(T_{k-1}) - K)^+ / \boxed{B(T_k)} \right] =$$

$$= \boxed{P(0, T_k)}\, E^{\boxed{k}} [(F_k(T_{k-1}) - K)^+ / \boxed{P(T_k, T_k)}\,] = \dots$$

$$= P(0, T_k)\ \ \text{B\&S}(F_k(0), K, v_{T_{k-1}-\text{caplet}} \sqrt{T_{k-1}})$$

$$v^2_{T_{k-1}-\text{caplet}} = \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_k(t)^2 dt$$

The dynamics of F_k is easy under Q^k. But if we price a product depending on several forward rates at the same time, we need to fix a

The LMM model dynamics in general II

pricing measure, say Q^i, and model all rates F_k under this same measure Q^i.

In this case we are lucky when $k = i$, since things are easy, but we are in troubles when $i < k$ or $i > k$, since the dynamics of F_k under Q^i (rather than Q^k) becomes difficult. We are going to derive it now using the change of numeraire toolkit.

The LMM model dynamics in general III

Dynamics of F_k under Q^i.
Consider the forward rate $F_k(t) = F(t, T_{k-1}, T_k)$ and suppose we wish to derive its dynamics first under the T_i-forward measure Q^i with $i < k$. We know that the dynamics under the T_k-forward measure Q^k has null drift. From this dynamics, we propose to recover the dynamics under Q^i. Let us apply the change of numeraire toolkit. The change of numeraire toolkit provides the formula relating Brownian shocks under numeraire 2 (say U) given shocks under Numeraire 1 (say S). See for example Formula (2.13) in Brigo and Mercurio (2001), Chapter 2. We can write

$$dZ_t^S = dZ_t^U - \rho\left(\frac{\mathrm{DC}(S)}{S_t} - \frac{\mathrm{DC}(U)}{U_t}\right)' dt$$

where we abbreviate "Vector Diffusion Coefficient" by "DC".

The LMM model dynamics in general IV

DC is actually a sort of linear operator for diffusion processes that works as follows. $DC(X_t)$ is the row vector $\mathbf{v}$ in

$$dX_t = (...)dt + \mathbf{v}\, dZ_t$$

for diffusion processes X with Z column vector Brownian motion common to all relevant diffusion processes. This is to say that if for example $dF_1 = \sigma_1 F_1 dZ_1^1$, then

$$DC(F_1) = [\sigma_1 F_1,\ 0,\ 0, \ldots,\ 0] = \sigma_1 F_1 e_1.$$

The correlation matrix ρ is the instantaneous correlation among all shocks (the same under any measure):

$$dZ_i dZ_j = \rho_{i,j} dt$$

The LMM model dynamics in general V

The toolkit

$$dZ_t^S = dZ_t^U - \rho \left(\frac{\mathrm{DC}(S)}{S_t} - \frac{\mathrm{DC}(U)}{U_t} \right)' dt$$

can also be written as

$$\boxed{dZ_t^S = dZ_t^U - \rho\, (\mathrm{DC}(\ln(S/U)))'\ dt}$$

This alternative toolkit expression (which we shall use) is obtained by noticing that

$$\frac{\mathrm{DC}(S)}{S_t} - \frac{\mathrm{DC}(U)}{U_t} = \mathrm{DC}(\ln(S)) - \mathrm{DC}(\ln(U))$$
$$= \mathrm{DC}(\ln(S) - \ln(U)) = \mathrm{DC}(\ln(S/U))$$

The LMM model dynamics in general VI

Let us apply the toolkit: $S = P(\cdot, T_k)$ and $U = P(\cdot, T_i)$

$$\boxed{dZ_t^k = dZ_t^i - \rho \mathrm{DC}(\ln(P(\cdot, T_k)/P(\cdot, T_i)))' \, dt}$$

Now notice that

$$\ln \frac{P(t, T_k)}{P(t, T_i)} = \ln \left(\frac{P(t, T_k)}{P(t, T_{k-1})} \frac{P(t, T_{k-1})}{P(t, T_{k-2})} \cdots \frac{P(t, T_{i+1})}{P(t, T_i)} \right) =$$

$$= \ln \left(\frac{1}{1 + \tau_k F_k(t)} \cdot \frac{1}{1 + \tau_{k-1} F_{k-1}(t)} \cdots \frac{1}{1 + \tau_{i+1} F_{i+1}(t)} \right) =$$

$$= \ln \left(1 / \left[\prod_{j=i+1}^{k} (1 + \tau_j F_j(t)) \right] \right) = - \sum_{j=i+1}^{k} \ln \left(1 + \tau_j F_j(t) \right)$$

The LMM model dynamics in general VII

$$\ln \frac{P(t,T_k)}{P(t,T_i)} = -\sum_{j=i+1}^{k} \ln\left(1+\tau_j F_j(t)\right)$$

so that from linearity

$$\mathrm{DC} \ln \frac{P(t,T_k)}{P(t,T_i)} = -\sum_{j=i+1}^{k} \mathrm{DC} \ln\left(1+\tau_j F_j(t)\right)$$

$$= -\sum_{j=i+1}^{k} \frac{\mathrm{DC}(1+\tau_j F_j(t))}{1+\tau_j F_j(t)} = -\sum_{j=i+1}^{k} \tau_j \frac{\mathrm{DC}(F_j(t))}{1+\tau_j F_j(t)} =$$

$$= -\sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) e_j}{1+\tau_j F_j(t)}$$

The LMM model dynamics in general VIII

where e_j is a zero row vector except in the j-th position, where we have 1 (vector diffusion coefficient for dF_j is $\sigma_j F_j e_j$). Recalling

$$\boxed{dZ_t^k = dZ_t^i - \rho \mathrm{DC}(\ln(P(\cdot, T_k)/P(\cdot, T_i)))' \, dt}$$

we may now write

$$\boxed{dZ_t^k = dZ_t^i + \rho \sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) e_j'}{1 + \tau_j F_j(t)} \, dt}$$

Pre-multiply both sides by e_k. We obtain

$$dZ_k^k = dZ_k^i + [\rho_{k,1} \; \rho_{k,2} \ldots \rho_{k,n}] \sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) e_j'}{1 + \tau_j F_j(t)} \, dt$$

The LMM model dynamics in general IX

$$= dZ_k^i + \sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) \rho_{k,j}}{1 + \tau_j F_j(t)} \, dt$$

Substitute this in our usual equation $dF_k = \sigma_k F_k dZ_k^k$ to obtain

$$dF_k = \sigma_k F_k \left(dZ_k^i + \sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) \rho_{k,j}}{1 + \tau_j F_j(t)} \, dt \right)$$

that is finally the equation showing the dynamics of a forward rate with maturity k under the forward measure with maturity i when $i < k$. The case $i > k$ is analogous.

The LMM model dynamics in general X

$$dF_k(t) = \mu_i^k(t, F(t))\sigma_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t),$$
$$dF_k(t) = \sigma_k(t)F_k(t)dZ_k^k(t)$$
$$dF_k(t) = -\mu_k^i(t, F(t))\sigma_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t),$$

for $i < k.i = k$ and $i > k$ respectively, where we have set

$$\mu_l^m = \sum_{j=l+1}^{m} \tau_j \frac{\sigma_j(t)F_j(t)\rho_{m,j}}{1 + \tau_j F_j(t)}$$

As for existence and uniqueness of the solution, the case $i = k$ is trivial. In the case $i < k$, use Ito's formula:

$$d \ln F_k(t) = \sigma_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1 + \tau_j F_j(t)} dt - \frac{\sigma_k(t)^2}{2} dt + \sigma_k(t)dZ_k(t).$$

The LMM model dynamics in general XI

The diffusion coefficient is deterministic and bounded. Moreover, since $0 < \tau_j F_j(t)/(1 + \tau_j F_j(t)) < 1$, also the drift is bounded, besides being smooth in the F's (that are positive). This ensures existence and uniqueness of a strong solution for the above SDE. The case $i > k$ is analogous.

LIBOR model under the Spot Measure I

It may happen that in simulating forward rates F_k under numeraires Q^i with i much larger or smaller than k, the effect of the discretization procedure worsens the approximation with respect to cases where i is closer to k.
A remedy to situations where we may need to simulate F_k very far away from the numeraire Q^i is to adopt the spot measure.
Consider a discretely rebalanced bank-account numeraire as an alternative to the continuously rebalanced bank account $B(t)$ (whose value, at any time t, changes according to $dB(t) = r_t B(t)dt$). We introduce a bank account that is rebalanced only on the times in our discrete-tenor structure. To this end, consider the numeraire asset

$$B_d(t) = \frac{P(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P(T_{j-1}, T_j)} = \prod_{j=0}^{\beta(t)-1} (1 + \tau_j F_j(T_{j-1}))\, P(t, T_{\beta(t)-1}) \,.$$

LIBOR model under the Spot Measure II

Here in general $T_{\beta(u)-2} < u \leq T_{\beta(u)-1}$.

$$B_d(t) = \frac{P(t, T_{\beta(t)-1})}{\prod_{j=0}^{\beta(t)-1} P(T_{j-1}, T_j)} = \prod_{j=0}^{\beta(t)-1} (1 + \tau_j F_j(T_{j-1}))\, P(t, T_{\beta(t)-1})\,.$$

The interpretation of $B_d(t)$ is that of the value at time t of a portfolio defined as follows. The portfolio starts with one unit of currency at $t = 0$, exactly as in the continuous-bank-account case ($B(0)$=1), but this unit amount is now invested in a quantity X_0 of T_0 zero-coupon bonds. Such X_0 is readily found by noticing that, since we invested one unit of currency, the present value of the bonds needs to be one, so that $X_0 P(0, T_0) = 1$, and hence $X_0 = 1/P(0, T_0)$. At T_0, we cash the bonds payoff X_0 and invest it in a quantity $X_1 = X_0/P(T_0, T_1) = 1/(P(0, T_0)P(T_0, T_1))$ of T_1 zero-coupon bonds. We continue this procedure until we reach the last $T_{\beta(t)-2}$ preceding

LIBOR model under the Spot Measure III

the current time t, where we invest $X_{\beta(t)-1} = 1/\prod_{j=1}^{\beta(t)-1} P(T_{j-1}, T_j)$ in $T_{\beta(t)-1}$ zero-coupon bonds. The present value at the current time t of this investment is $X_{\beta(t)-1}P(t, T_{\beta(t)-1})$, i.e. our $B_d(t)$ above. Thus, $B_d(t)$ is obtained by starting from one unit of currency and reinvesting at each tenor date in zero-coupon bonds for the next tenor. This gives a discrete-tenor counterpart of B, and the subscript "d" in B_d stands for "discrete". B_d is also called **discretely rebalanced bank account**
Now choose B_d as numeraire and apply the change-of-numeraire technique starting from the dynamics $dF_k = \sigma_k F_k dZ_k$ under Q^k, to obtain the dynamics under B_d. Calculations are analogous to those given for the Q^i case.

LIBOR model under the Spot Measure IV

The measure Q^d associated with B_d is called *spot LIBOR measure*. We then have the following **(Spot-LIBOR-measure dynamics in the LMM)**

$$dF_k(t) = \sigma_k(t)\, F_k(t) \sum_{j=\beta(t)}^{k} \frac{\tau_j \rho_{j,k}\, \sigma_j(t)\, F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t)\, F_k(t)\, dZ_k^d(t).$$

Both the spot-measure dynamics and the risk-neutral dynamics admit no known transition densities, so that the related equations need to be discretized in order to perform simulations.

LIBOR model under the Spot Measure: Benefits I

Assume we are in need to value a payoff involving rates $F_1, \ldots, F_{10}$ from time 0 to time T_9.
Consider two possible measures under which we can do pricing.
First Q^{10}. Under this measure, consider each rate F_j in each interval with the number of terms in the drift summation of each rate shown between square brakets:

$$\begin{aligned} 0 - T_0 :&\ F_1[9], F_2[8], \quad F_3[7] \quad , \ldots, F_9[1], F_{10}[0] \\ T_0 - T_1 :&\ \qquad\quad\ F_2[8], \quad F_3[7] \quad , \ldots, F_9[1], F_{10}[0] \\ T_1 - T_2 :&\ \qquad\qquad\qquad\quad\ F_3[7] \quad , \ldots, F_9[1], F_{10}[0] \end{aligned}$$

LIBOR model under the Spot Measure: Benefits II

etc. Notice that if we discretize some rates will be more biased than others. Instead, with the spot LIBOR measure

$$
\begin{array}{llll}
0 - T_0: & F_1[1], F_2[2], & F_3[3] & , \ldots, F_9[9], F_{10}[10] \\
T_0 - T_1 & F_2[1], & F_3[2] & , \ldots, F_9[8], F_{10}[9] \\
T_1 - T_2 & & F_3[1] & , \ldots, F_9[7], F_{10}[8]
\end{array}
$$

etc. Now the bias, if any, is more distributed.

Theoretical incompatibility SMM / LMM I

Recall **LMM**: $dF_i(t) = \sigma_i(t)F_i(t)dZ_i(t), \quad Q^i,$

SMM: $$(*) d\, S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t)\, dW_t, \quad Q^{\alpha,\beta}\ . \qquad (35)$$

Precisely: Can each F_i be lognormal under Q^i **and** $S_{\alpha,\beta}$ be lognormal under $Q^{\alpha,\beta}$, given that

$$(**) \quad S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j\, F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j\, F_j(t)}} \quad ? \qquad (36)$$

Theoretical incompatibility SMM / LMM II

Check distributions of $S_{\alpha,\beta}$ under $Q^{\alpha,\beta}$ for both LMM and SMM. Derive the LMM model under the SMM numeraire $Q^{\alpha,\beta}$:

$$(***) \quad dF_k(t) = \sigma_k(t)F_k(t)\left(\mu_k^{\alpha,\beta}(t)dt + dZ_k^{\alpha,\beta}(t)\right), \qquad (37)$$

$$\mu_k^{\alpha,\beta} = \sum_{j=\alpha+1}^{\beta}(2_{(j\le k)} - 1)\tau_j \frac{P(t,T_j)}{C_{\alpha,\beta}(t)} \sum_{i=\min(k+1,j+1)}^{\max(k,j)} \frac{\tau_i \rho_{k,i}\sigma_i F_i}{1+\tau_i F_i}.$$

When computing the swaption price as the $Q^{\alpha,\beta}$ expectation

$$C_{\alpha,\beta}(0)E^{\alpha,\beta}(S_{\alpha,\beta}(T_\alpha) - K)^+$$

we can use either LMM (**,***) or SMM (*). In general, $S_{\alpha,\beta}$ coming from SMM (*) is LOGNORMAL, whereas $S_{\alpha,\beta}$ coming from LMM (**,***) is NOT. But in practice it is very close. Hence LMM works well also as a substitute for the SMM

Going back to HJM: Proving the drift condition I

We now go back for a minute to the HJM framework to prove the drift condition.

We have seen earlier the HJM framework for the choice of variables $f(t,T)$. While we argued that most useful models coming out of this framework are r models or $F_i,\ S_{\alpha,\beta}$ models, HJM is still quite important historically and in a number of areas (commodities, etc). It is then important to grasp the essentials of the proof for the drift condition. Since we now have all the tools needed to prove the drift condition, we proceed to do so.

Under the risk neutral measure $\mathbb{Q}$ with numeraire bank account B, we mentioned that

$$df(t,T) = \boxed{\sigma(t,T)\left(\int_t^T \sigma'(t,s)ds\right)} dt + \sigma(t,T)dW^B(t),$$

Going back to HJM: Proving the drift condition II

We are now going to prove this equation needs to have precisely the specified drift above, which is completely determined by volatilities. We will assume σ to be a row vector, and W to be a standard multivariate column-vector Brownian motion of the same dimension as σ. Intantaneous correlation will be implicit in the inner product $\sigma\ \sigma'$ and we will not model it explicitly across Brownian motions. That is why we assume the Brownian components of the vector W to be independent of each other.

We are now going to sketch a proof of the drift condition in the above equation using the change of numeraire technique.

Recall that

$$f(t,T) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} \approx \frac{P(t,T) - P(t,T+\Delta T)}{P(t,T)}$$

Going back to HJM: Proving the drift condition III

for small ΔT. Hence this is a tradable asset (difference of two bonds) divided by a second asset (the bond $P(t, T)$), and by FACT ONE of the change of numeraire technique it is a martingale under the $P(\cdot, T)$ numeraire measure $\mathbb{Q}^T$, which we called T-forward measure.

Since "martingale" for regular diffusions means zero drift, we can write

$$df(t, T) = \sigma(t, T) dW_t^T$$

under the T forward measure. We now apply the change of mumeraire toolkit formula we have seen earlier,

$$\boxed{dZ_t^S = dZ_t^U - \rho\,(\mathrm{DC}(\ln(S/U)))'\ dt}$$

Going back to HJM: Proving the drift condition IV

Recall that now Z is W with independent components, so that for the Brownians W the matrix ρ is the identity matrix. We choose $S = B$ (bank account) and $U = P(\cdot, T)$. Then we can write

$$\boxed{dW_t^B = dW_t^T - (\mathrm{DC}(\ln(B/P(\cdot,T))))' \; dt}$$

As usual

$$\mathrm{DC}(\ln(B/P(\cdot,T))) = \mathrm{DC}(\ln(B)) - \mathrm{DC}(\ln P(\cdot,T)) = 0 - \mathrm{DC}(\ln P(\cdot,T)).$$

Our last task is now computing $\mathrm{DC}(\ln P(\cdot,T))$.
By inverting the definition

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$$

Going back to HJM: Proving the drift condition V

we get

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \quad \text{or} \quad \ln P(t,T) = -\int_t^T f(t,u)du$$

We differentiate wrt t both sides:

$$d_t \ln P(t,T) = f(t,t)dt - \int_t^T d_t f(t,u)du = -\int_t^T \left[(\ldots)dt + \sigma(t,u)dW_t\right] du$$

whichever measure we are in, provided $\sigma(t,u)$ is the vector volatility for $df(t,u)$. This last SDE allows us to read the diffusion coefficient of $d \ln P(t,T)$ as

Going back to HJM: Proving the drift condition VI

$$\mathrm{DC}(\ln P(\cdot, T)) = -\int_t^T \sigma(t,u)du.$$

Substituting above yields

$$\mathrm{DC}(\ln(B/P(\cdot, T))) = \int_t^T \sigma(t,u)du$$

and hence

$$dW_t^B = dW_t^T - \int_t^T \sigma(t,u)du\, dt$$

or

$$dW_t^T = dW_t^B + \int_t^T \sigma(t,u)du\, dt$$

Going back to HJM: Proving the drift condition VII

Substituing this into our initial equation

$$df(t,T) = \sigma(t,T)dW_t^T$$

leads to

$$df(t,T) = \sigma(t,T)\left[dW_t^B + \int_t^T \sigma'(t,u)du\, dt\right]$$

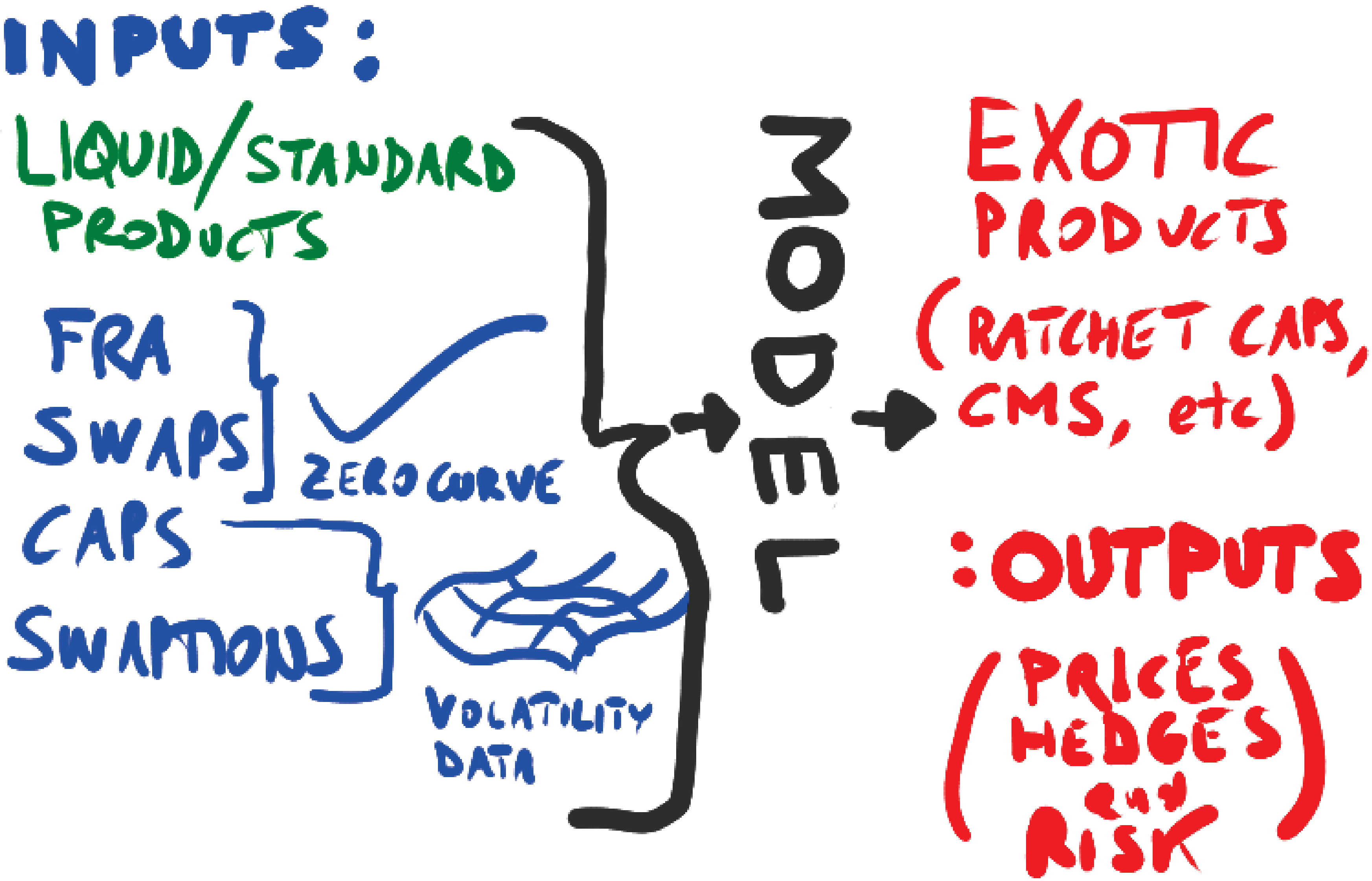
or

$$df(t,T) = \sigma(t,T)\left(\int_t^T \sigma'(t,u)du\right)dt + \sigma(t,T)dW_t^B$$

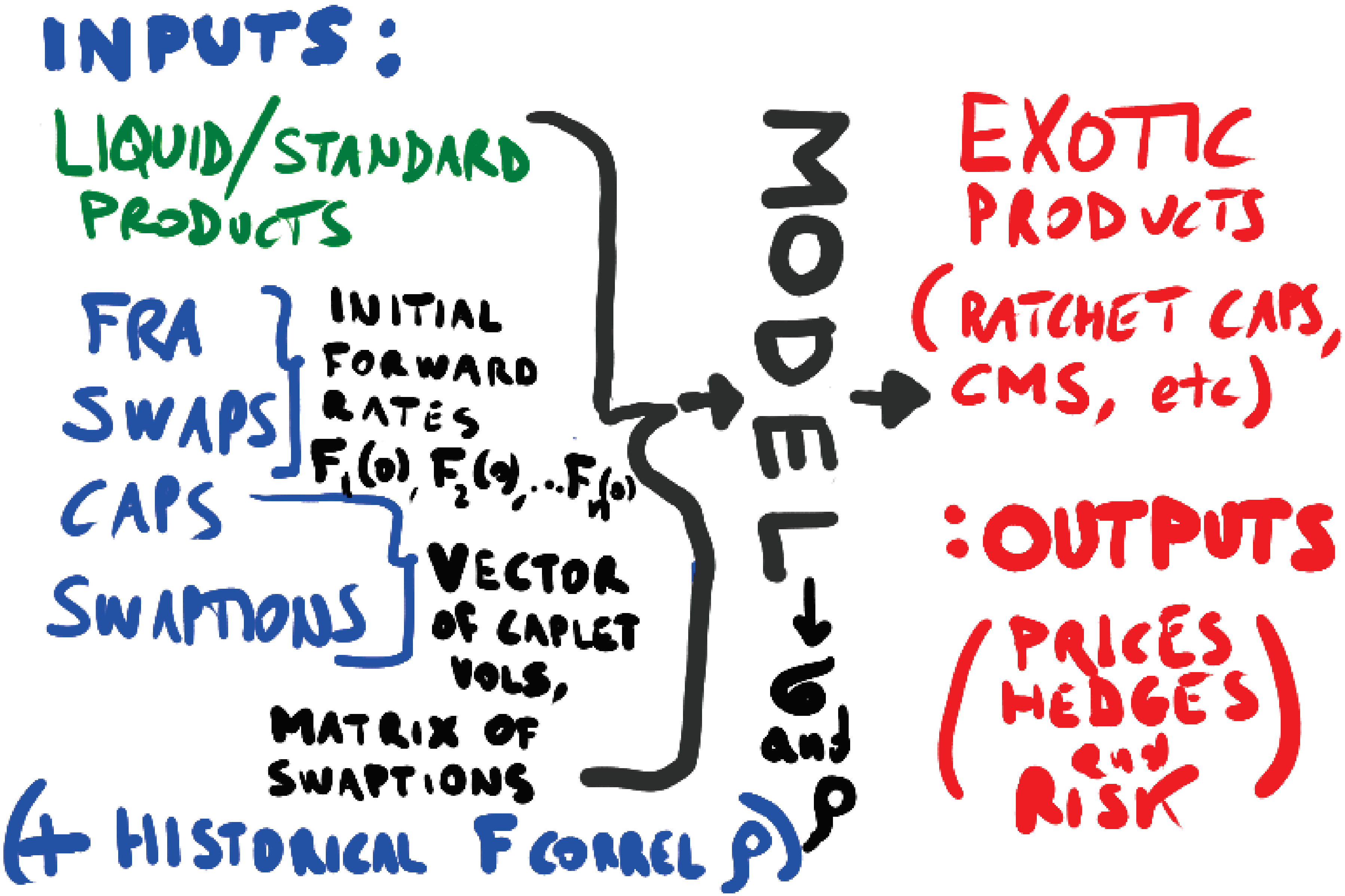
which completes our proof.

We now go back to the LIBOR market model and discuss its calibration to market data.

THE MODEL CALIBRATION



THE MODEL CALIBRATION



LMM instantaneous covariance structures I

LMM is natural for caps and SMM is natural for swaptions. **Choose.**
We choose LMM and adapt it to price swaptions.
Recall: Under numeraire $P(\cdot, T_i) \neq P(\cdot, T_k)$:

$$dF_k(t) = \mu_k^i(t)\, F_k(t)\, dt + \boxed{\sigma_k(t)}\, F_k(t)\, dZ_k, \quad dZ\, dZ' = \boxed{\rho}\, dt$$

Model specification: Choice of $\sigma_k(t)$ and of ρ.

LMM instantaneous covariance structures II

- General Piecewise constant **(GPC)** vols, $\sigma_k(t) = \sigma_{k,\beta(t)}$, $T_{\beta(t)-2} < t \leq T_{\beta(t)-1}$.

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\ldots$	$(T_{M-2}, T_{M-1}]$
Fwd: $F_1(t)$	$\sigma_{1,1}$	Expired	Expired	$\ldots$	Expired
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	Expired	$\ldots$	Expired
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\ldots$	$\sigma_{M,M}$

  Separable Piecewise const **(SPC)**, $\sigma_k(t) = \Phi_k \psi_{k-(\beta(t)-1)}$

- Parametric Linear-Exponential **(LE)** vols

$$\sigma_i(t) = \Phi_i \, \psi(T_{i-1} - t; a, b, c, d)$$
$$:= \Phi_i \left([a(T_{i-1} - t) + d] e^{-b(T_{i-1}-t)} + c \right).$$

Caplet volatilities I

Recall that under numeraire $P(\cdot, T_i)$:

$$dF_i(t) = \sigma_i(t) F_i(t)\, dZ_i, \quad dZ dZ' = \rho\, dt$$

Caplet: Strike rate K, Reset T_{i-1}, Payment T_i:
Payoff: $\tau_i(F_i(T_{i-1}) - K)^+$ at T_i.
"Call option" on F_i, $F_i \sim$ lognormal under Q^i
$\Rightarrow$ Black's formula, with Black vol. parameter

$$v^2_{T_{i-1}-\text{caplet}} := \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i(t)^2 dt.$$

$v_{T_{i-1}-\text{caplet}}$ is T_{i-1}-*caplet volatility*
Only the σ's have impact on caplet (and cap) prices, the ρ's having no influence.

Caplet volatilities II

$$dF_i(t) = \sigma_i(t)F_i(t)\,dZ_i, \quad v^2_{T_{i-1}-\text{caplet}} := \frac{1}{T_{i-1}}\int_0^{T_{i-1}} \sigma_i(t)^2 dt.$$

Under GPC vols, $\sigma_k(t) = \sigma_{k,\beta(t)}$

$$v^2_{T_{i-1}-\text{caplet}} = \frac{1}{T_{i-1}}\sum_{j=1}^{i}(T_{j-1} - T_{j-2})\,\sigma^2_{i,j}$$

Under LE vols, $\sigma_i(t) = \Phi_i\,\psi(T_{i-1} - t; a, b, c, d)$,

$$T_{i-1}v^2_{T_{i-1}-\text{caplet}} = \Phi_i^2\; I^2(T_{i-1}; a, b, c, d)$$

$$:= \Phi_i^2 \int_0^{T_{i-1}} \left([a(T_{i-1} - t) + d]e^{-b(T_{i-1}-t)} + c\right)^2 dt\ .$$

Caplet Vols with GPC LMM volatilities

Important: In the GPC case, caplet volatilties can be computed very simply as follows. The GPC volatilities matrix has a ziggurat shape.

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$\dots$	$(T_{M-2}, T_{M-1}]$
Fwd: $F_1(t)$	$\sigma_{1,1}$				$\dots$	
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$			$\dots$	
$F_3(t)$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,3}$		$\dots$	
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\sigma_{M,4}$	$\dots$	$\sigma_{M,M}$

- square each entry in the table
- for each row, add up all the squared terms, each multiplied by the corresponding year fraction expiry-maturity τ for that volatility.
- Take the total in the previous point and divide it by the caplet reset time (or the sum of all τ's used in that row)
- Take the square root.

Term Structure of Caplet Volatilities I

The term structure of volatility at time T_j is a graph of expiry times T_{h-1} against average volatilities $V(T_j, T_{h-1})$ of the related forward rates $F_h(t)$ up to that expiry time itself, i.e. for $t \in (T_j, T_{h-1})$.
Formally, at time $t = T_j$, graph of points

$$\{(T_{j+1}, V(T_j, T_{j+1})), (T_{j+2}, V(T_j, T_{j+2})), \ldots, (T_{M-1}, V(T_j, T_{M-1}))\}$$

$$V^2(T_j, T_{h-1}) = \frac{1}{T_{h-1} - T_j} \int_{T_j}^{T_{h-1}} \sigma_h^2(t)dt, \quad h > j + 1.$$

The term structure of vols at time 0 is given simply by caplets vols plotted against their expiries.
Different assumptions on the behaviour of instantaneous volatilities (SPC, LE, etc.) imply different evolutions for the term structure of volatilities in time as $t = T_0$, $t = T_1$, $t = T_2$...

Simple calculation for TSOV with GPC LMM

IMPORTANT. In the GPC parameterization, under the Ziggurat matrix, computing the future term structure of caplet volatilities (TSOV) at time T_j is very easy:

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$\dots$	$(T_{M-2}, T_{M-1}]$
Fwd: $F_1(t)$	$\sigma_{1,1}$				$\dots$	
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$			$\dots$	
$F_3(t)$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,3}$		$\dots$	
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\sigma_{M,4}$	$\dots$	$\sigma_{M,M}$

- square each entry in the table
- Starting from the column corresponding to the desired future time, in each row add up all the squares up to the diagonal, each multiplied by the corresponding year fraction τ.
- Take the total in the previous point and divide it by the sum of the τ's you have used.
- Take the square root.

Final time: $[\sigma_{M,M}]$

Final time − 1:

$$\begin{bmatrix} \sigma_{M-1,M-1} \\ \sqrt{\frac{1}{2\tau}\left(\tau\sigma^2_{M,M-1}+\tau\sigma^2_{M,M}\right)} \end{bmatrix}$$

TERM STRUCTURE OF FUTURE CAPLET VOLATILITIES

Final time − 2:

$$\begin{bmatrix} \sigma_{M-2,M-2} \\ \sqrt{\frac{1}{2\tau}\left(\tau\sigma^2_{M-1,M-2}+\tau\sigma^2_{M-1,M-1}\right)} \\ \sqrt{\frac{1}{3\tau}\left(\tau\sigma^2_{M,M-2}+\tau\sigma^2_{M,M-1}+\tau\sigma^2_{M,M}\right)} \end{bmatrix}$$

Final time − 3:

$$\begin{bmatrix} \sigma_{M-3,M-3} \\ \sqrt{\frac{1}{2\tau}\left(\tau\sigma^2_{M-2,M-3}+\tau\sigma^2_{M-2,M-2}\right)} \\ \sqrt{\frac{1}{3\tau}\left(\tau\sigma^2_{M-1,M-3}+\tau\sigma^2_{M-1,M-2}+\tau\sigma^2_{M-1,M-1}\right)} \\ \sqrt{\frac{1}{4\tau}\left(\tau\sigma^2_{M,M-3}+\tau\sigma^2_{M,M-2}+\tau\sigma^2_{M,M-1}+\tau\sigma^2_{M,M}\right)} \end{bmatrix}$$

Cap calibration: Some possible choices I

We implemented a version with:

- Semi-annual tenors, $T_i - T_{i-1} = 6m$.
- Instantaneous correlation estimated historically, first fitted on the full rank parametric form in ρ_∞, α:

$$\rho_\infty + (1 - \rho_\infty) \exp(-\alpha |i - j|)$$

  and then possibly fitted to a reduced rank correlation (no impact on caps but need for ratchets etc., more on this later)

Cap calibration: Some possible choices II

- Vol. parameterization $\sigma_k(t) = \sigma_{k,\beta(t)} := \Phi_k \psi_{k-(\beta(t)-1)}$,

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\dots$	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	$\Phi_1\psi_1$	Dead	Dead	$\dots$	Dead
$F_2(t)$	$\Phi_2\psi_2$	$\Phi_2\psi_1$	Dead	$\dots$	Dead
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$F_M(t)$	$\Phi_M\psi_M$	$\Phi_M\psi_{M-1}$	$\Phi_M\psi_{M-2}$	$\dots$	$\Phi_M\psi_1$

Note: $\Phi = 1$ (use only ψ) leads to "stationary vol term structure" as in the top figure, next page;
$\psi = 1$ (use only Φ) leads to constant volatilities and is the easiest calibration possible, since then $\Phi_i = v_{T_{i-1}-\text{caplet}}$, but leads also to bad term-structure evolution, middle figure next page.

Cap calibration: Some possible choices III

Vol. parameterization: HOMOGENEOUS IN THE TIME-TO-EXPIRY (constancy along the DIAGONALS of the "ziggurat"): $\sigma_k(t) = \psi_{k-(\beta(t)-1)}$, and in particular $\sigma_k(T_j-) = \psi_{k-j}$;

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	ψ_1	Dead	Dead	...	Dead
$F_2(t)$	ψ_2	ψ_1	Dead	...	Dead
⋮	...	...	...	...	...
$F_M(t)$	ψ_M	ψ_{M-1}	ψ_{M-2}	...	ψ_1

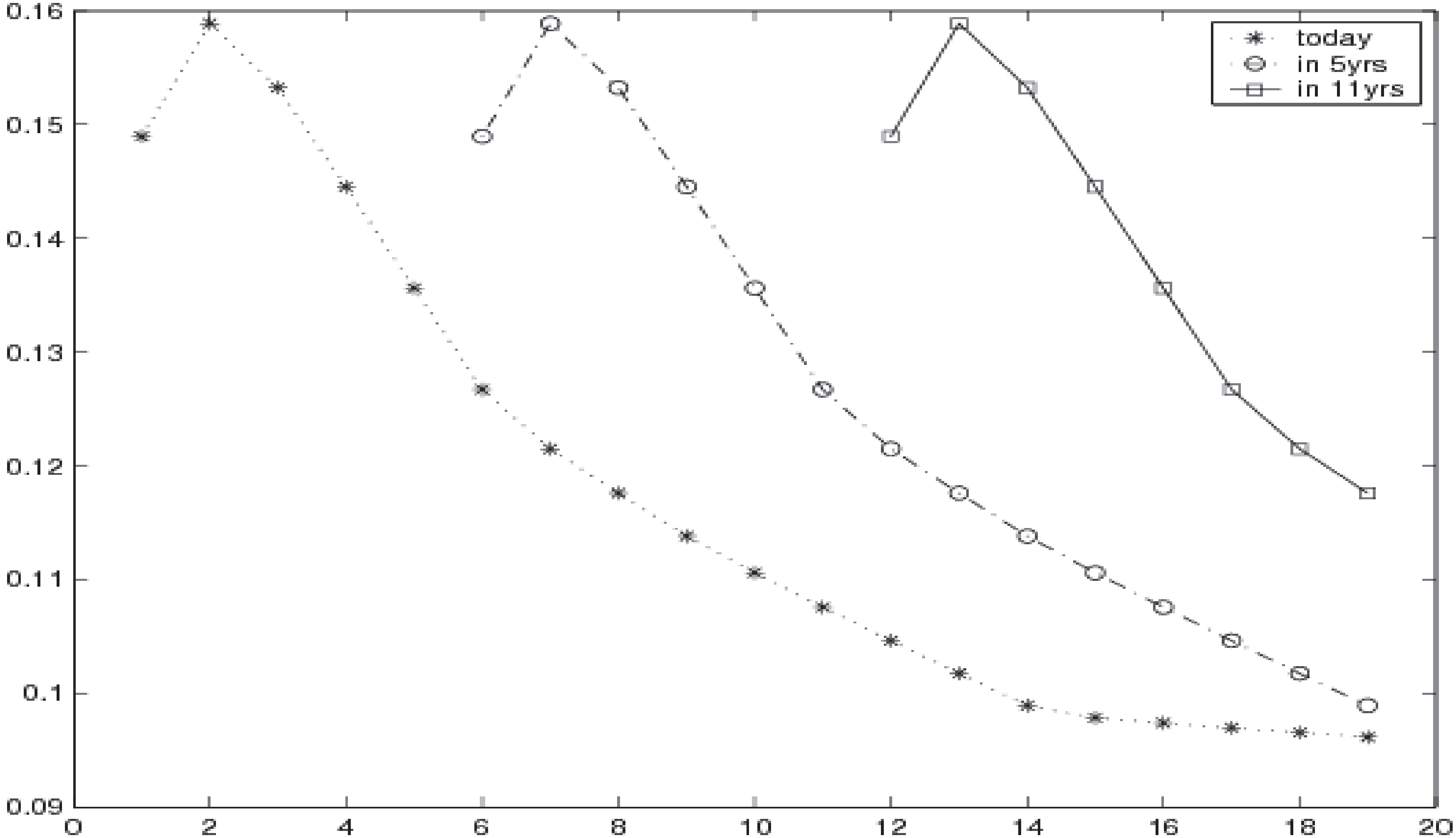
Vol. parameterization: HOMOGENEOUS IN TIME (constancy along the ROWS of the "ziggurat"): $\sigma_k(t) = \Phi_k$

Inst. Vols	$t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	...	$(T_{M-2}, T_{M-1}]$
Fwd : $F_1(t)$	Φ_1	Dead	Dead	...	Dead
$F_2(t)$	Φ_2	Φ_2	Dead	...	Dead
⋮	...	...	...	...	...
$F_M(t)$	Φ_M	Φ_M	Φ_M	...	Φ_M

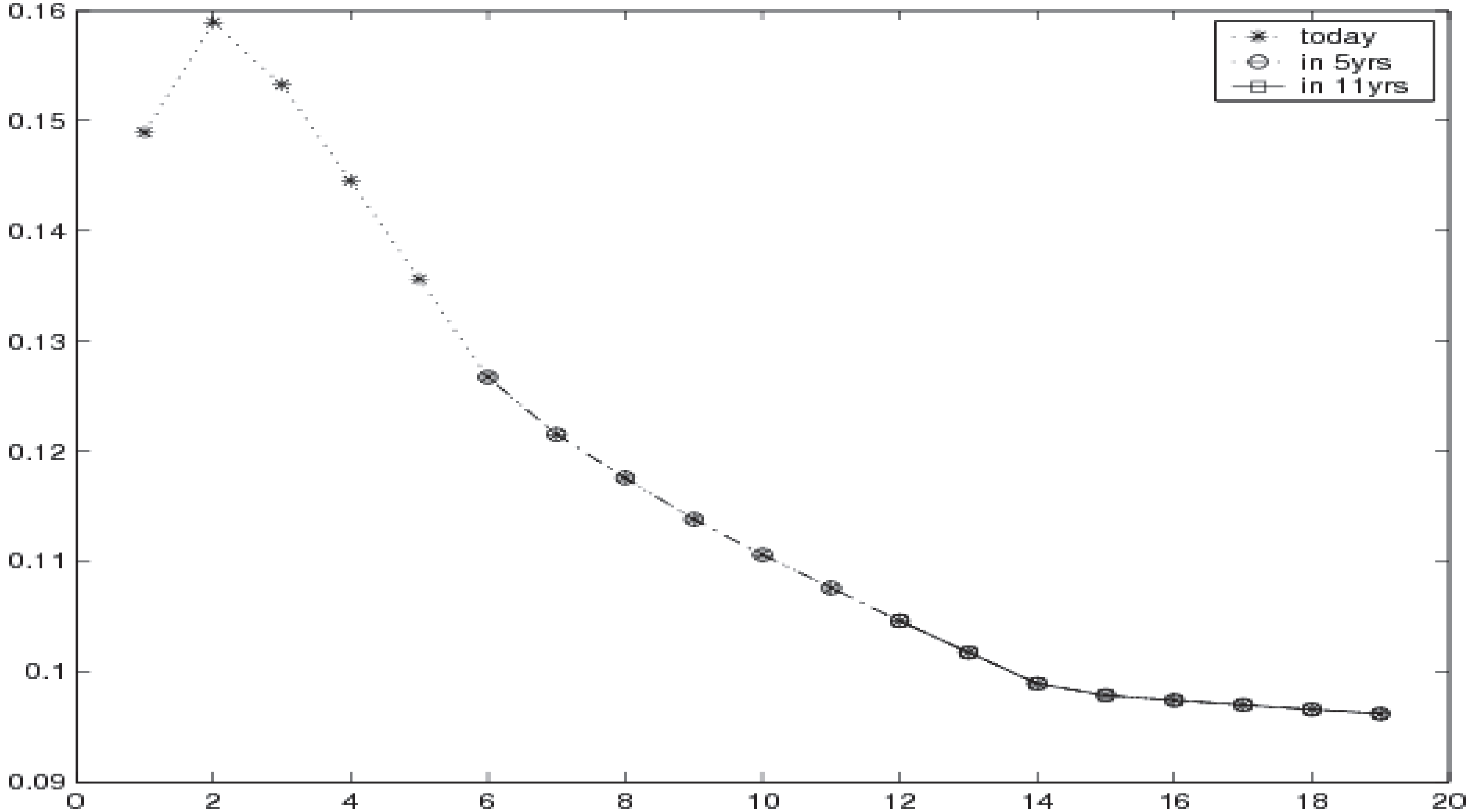
Cap calibration: Some possible choices IV

Let's see the evolution of the term structure of volatilities in the three cases: $\Phi = 1$ (homogeneous in time-to-expiry), $\psi = 1$ (homogeneous in time), and intermediate (neither Φ nor ψ set to one).

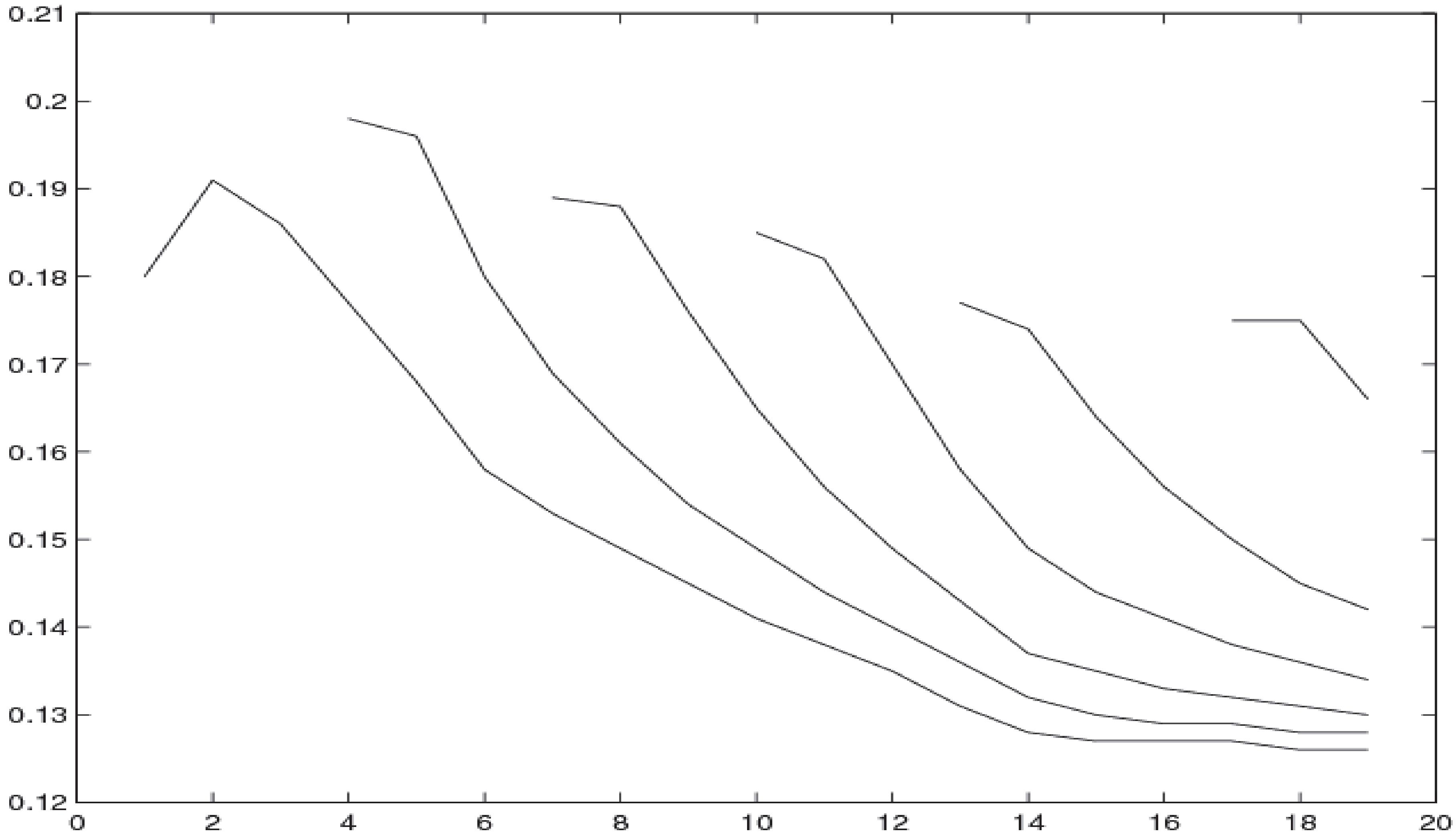
Cap calibration: Some possible choices V



Cap calibration: Some possible choices VI



Cap calibration: Some possible choices VII



Terminal and Instantaneous correlation I

Swaptions depend on **terminal** correlations among fwd rates. E.g., the swaption whose underlying is $S_{1,3}$ depends on

$$\text{corr}(F_2(T_1), F_3(T_1)).$$

This terminal corr. depends both on **inst.** corr. $\rho_{2,3}$ **and** and on the way the $T_1 - T_2$ and $T_2 - T_3$ caplet vols $v_1 = v_{T_1-\text{caplet}}$ and $v_2 = v_{T_2-\text{caplet}}$ are decomposed in instantaneous vols $\sigma_2(t)$ and $\sigma_3(t)$ for t in 0, T_1. We'll show later that (here $\tau = T_i - T_{i-1}$)

$$\text{corr}(F_2(T_1), F_3(T_1)) \approx \frac{\int_0^{T_1} \sigma_2(t)\sigma_3(t)\rho_{2,3}dt}{\sqrt{\int_0^{T_1} \sigma_2^2(t)dt}\,\sqrt{\int_0^{T_1} \sigma_3^2(t)dt}} = \text{under GPC vols}$$

$$= \rho_{2,3}\frac{\tau\sigma_{2,1}\sigma_{3,1} + \tau\sigma_{2,2}\sigma_{3,2}}{\sqrt{\tau\sigma_{2,1}^2 + \tau\sigma_{2,2}^2}\,\sqrt{\tau\sigma_{3,1}^2 + \tau\sigma_{3,2}^2}} = \rho_{2,3}\frac{\sigma_{2,1}\sigma_{3,1} + \sigma_{2,2}\sigma_{3,2}}{v_1\sqrt{T_1}\,\sqrt{\sigma_{3,1}^2 + \sigma_{3,2}^2}}.$$

Terminal and Instantaneous correlation II

No such formula for general short-rate models

$$\mathrm{corr}(F_2(T_1), F_3(T_1)) \approx \rho_{2,3} \frac{\sigma_{2,1}\sigma_{3,1} + \sigma_{2,2}\sigma_{3,2}}{v_1\sqrt{T_1}\ \sqrt{\sigma_{3,1}^2 + \sigma_{3,2}^2}} \ .$$

Fix $\rho_{2,3} = 1$, $\tau_i = 1$ and caplet vols:

$$v_1^2 T_1 = \sigma_{2,1}^2 + \sigma_{2,2}^2; \qquad v_2^2 T_2 = \sigma_{3,1}^2 + \sigma_{3,2}^2 + \sigma_{3,3}^2 \ .$$

Decompose v_1 and v_2 in two different ways: **First case**

$$\sigma_{2,1} = v_1\sqrt{T_1},\ \sigma_{2,2} = 0; \qquad \sigma_{3,1} = v_2\sqrt{T_2}, \sigma_{3,2} = \sigma_{3,3} = 0.$$

In this case the above fomula yields easily

$$\mathrm{corr}(F_2(T_1), F_3(T_1)) = \rho_{2,3} = 1 \ .$$

Terminal and Instantaneous correlation III

The **second case** is obtained as

$$\sigma_{2,1} = 0,\ \sigma_{2,2} = v_1\sqrt{T_1}; \qquad \sigma_{3,1} = v_2\sqrt{T_2}, \sigma_{3,2} = \sigma_{3,3} = 0.$$

In this second case the above fomula yields immediately

$$\mathrm{corr}(F_2(T_1), F_3(T_1)) = 0\rho_{2,3} = 0\ .$$

Instantaneous correlation: Parametric forms I

Swaptions depend on **terminal** correlation among forward rates (ρ's **and** σ's). How do we model ρ?

Full Rank Parametric forms for instant. correl. ρ
Schoenmakers and Coffey (2000) propose a finite sequence

$$1 = c_1 < c_2 < \ldots < c_M, \quad \frac{c_1}{c_2} < \frac{c_2}{c_3} < \ldots < \frac{c_{M-1}}{c_M},$$

and they set ("F" stands for "Full" (Rank))

$$\rho^F(c)_{i,j} := c_i/c_j, \quad i \leq j, \quad i,j = 1,\ldots,M.$$

Notice that the correlation between changes in adjacent rates is $\rho^F_{i+1,i} = c_i/c_{i+1}$.

Instantaneous correlation: Parametric forms II

The above requirements on c's translate into the requirement that *the sub-diagonal of the resulting correlation matrix* $\rho^F(c)$ *be increasing when moving from NW to SE.*
This bears the interpretation that when we move along the yield curve, the larger the tenor, the more correlated changes in adjacent forward rates become. This corresponds to the experienced fact that the forward curve tends to flatten and to move in a more "correlated" way for large maturities than for small ones. This holds also for lower levels below the diagonal.
The number of parameters needed in this formulation is M, versus the $M(M-1)/2$ number of entries in the general correlation matrix. One can prove that $\rho^F(c)$ is always a viable correlation matrix if defined as above (symmetric, positive semidefinite and with ones in the diagonal).

Instantaneous correlation: Parametric forms III

Schoenmakers and Coffey (2000) observe also that this parameterization can be always characterized in terms of a finite sequence of non-negative numbers $\Delta_2, \ldots, \Delta_M$:

$$c_i = \exp\left[\sum_{j=2}^{i} j\Delta_j + \sum_{j=i+1}^{M} (i-1)\Delta_j\right].$$

Some particular cases in this class of parameterizations that Schoenmakers and Coffey (2000) consider to be promising can be formulated through suitable changes of variables as follows. The first is the case where all Δ's are zero except the last two: by a change of variable one has

Instantaneous correlation: Parametric forms IV

Stable, full rank, two-parameters, "increasing along sub-diagonals" parameterization for instantaneous correlation:

$$\rho_{i,j} = \exp\left[-\frac{|i-j|}{M-1}\left(-\ln\rho_\infty + \eta\frac{M-1-i-j}{M-2}\right)\right].$$

Stability here is meant to point out that relatively small movements in the c-parameters connected to this form cause relatively small changes in ρ_∞ and η.

Notice that $\rho_\infty = \rho_{1,M}$ is the correlation between the farthest forward rates in the family considered, whereas η is related to the first non-zero Δ, i.e. $\eta = \Delta_{M-1}(M-1)(M-2)/2$.

A 3-parameters form is obtained with Δ_i's following a straight line (two parameters) for $i = 2, 3, \ldots, M-1$ and set to a third parameter for $i = M$.

Instantaneous correlation: Parametric forms V

Stable, full rank, 3-parameters, “increasing along sub-diagonals” parameterization S&C3:

$$\rho_{i,j} = \exp\left[-|i-j|\left(\beta - \frac{\alpha_2}{6M-18}\left(i^2+j^2+ij-6i-6j-3M^2+15M-7\right)\right.\right.$$
$$\left.\left.+\frac{\alpha_1}{6M-18}\left(i^2+j^2+ij-3Mi-3Mj+3i+3j+3M^2-6M+2\right)\right)\right]. \quad (38)$$

where the parameters should be constrained to be non-negative, if one wants to be sure all the typical desirable properties are indeed present.
In order to get parameter stability, Schoenmakers and Coffey introduce a change of variables, thus obtaining a laborious expression generalizing the earlier two-parameters one. The calibration experiments pointed out, however, that the parameter associated with the final point Δ_{M-1} of our straight line in the Δ's is practically always

Instantaneous correlation: Parametric forms VI

close to zero. Setting thus $\Delta_{M-1} = 0$ and maintaining the other characteristics of the last parameterization leads to the following **Improved, stable, full rank, two-parameters, "increasing along sub-diagonals" parameterization for instantaneous correlations (S&C2):**

$$\rho_{i,j} = \exp\left[-\frac{|i-j|}{M-1}\left(-\ln \rho_\infty + \eta \frac{i^2 + j^2 + ij - 3Mi - 3Mj + 3i + 3j + 2M^2 - M - 4}{(M-2)(M-3)}\right)\right]. \quad (39)$$

As before, $\rho_\infty = \rho_{1,M}$, whereas η is related to the steepness of the straight line in the Δ's.

Instantaneous correlation: Parametric forms VII

Full Rank, Classical, two-parameters, exponentially decreasing parameterization

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty) \exp[-\beta |i - j|], \quad \beta \geq 0.$$

where now ρ_∞ is only asymptotically representing the correlation between the farthest rates in the family.
Schoenmakers and Coffey (2000) point out that Rebonato's (1999c,d) full-rank parameterization, consisting in the following perturbation of the classical structure:
Full Rank, Rebonato's three parameters form

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty) \exp[-|i - j|(\beta - \alpha (\max(i,j) - 1))], \qquad (40)$$

has still the desirable property of being increasing along sub-diagonals. However, the domain of positivity for the resulting matrix is not specified "off-line" in terms of $\alpha, \beta, \rho_\infty$.

Instantaneous correlation: Reducing the rank I

Instant. correl: Approximate ρ ($M \times M$, Rank M) with a n-rank

$\rho^B = B \times B'$, with B an $M \times n$ matrix, $n << M$.

$$dZ\, dZ' = \rho\, dt \quad \longrightarrow \quad B\, dW (B\, dW)' = BB' dt\ .$$

$\rho^B = B \times B'$, with B an $M \times n$ matrix, $n << M$.

Eigenvalues zeroing and rescaling.
We know that, being ρ a positive definite symmetric matrix, it can be written as

$$\rho = PHP',$$

where P is a real orthogonal matrix, $P'P = PP' = I_M$, and H is a diagonal matrix of the positive eigenvalues of ρ.
The columns of P are the eigenvectors of ρ, associated to the eigenvalues located in the corresponding position in H.

Instantaneous correlation: Reducing the rank II

Let Λ be the diagonal matrix whose entries are the square roots of the corresponding entries of H, so that if we set $A := P\Lambda$ we have both

$$AA' = \rho, \quad A'A = H \ .$$

$$\rho = PHP', \quad \text{“}\Lambda = \sqrt{H}\text{”}, \quad A := P\Lambda, \quad AA' = \rho, \quad A'A = H.$$

We can try and mimic the decomposition $\rho = AA'$ by means of a suitable n-rank $M \times n$ matrix B such that BB' is an n-rank correlation matrix, with typically $n << M$.
Consider the diagonal matrix $\bar{\Lambda}^{(n)}$ defined as the matrix Λ with the $M - n$ smallest diagonal terms set to zero.
Define then $\bar{B}^{(n)} := P\bar{\Lambda}^{(n)}$, and the related candidate correlation matrix $\bar{\rho}^{(n)} := \bar{B}^{(n)}(\bar{B}^{(n)})'$.

Instantaneous correlation: Reducing the rank III

We can also equivalently define $\Lambda^{(n)}$ as the $n \times n$ diagonal matrix obtained from Λ by taking away (instead of zeroing) the $M - n$ smallest diagonal elements and shrinking the matrix correspondingly. Analogously, we can define the $M \times n$ matrix $P^{(n)}$ as the matrix P from which we take away the columns corresponding to the diagonal elements we took away from Λ. The result does not change, in that if we define the $M \times n$ matrix $B^{(n)} = P^{(n)}\Lambda^{(n)}$ we have

$$\bar{\rho}^{(n)} = \bar{B}^{(n)}(\bar{B}^{(n)})' = B^{(n)}(B^{(n)})'.$$

We keep the $B^{(n)}$ formulation.

Instantaneous correlation: Reducing the rank IV

$$\bar{\rho}^{(n)} = B^{(n)}(B^{(n)})', \quad B^{(n)} = P^{(n)}\Lambda^{(n)}.$$

Now the problem is that, in general, while $\bar{\rho}^{(n)}$ is positive semidefinite, it does not feature ones in the diagonal. Throwing away some eigenvalues from Λ has altered the diagonal. The solution is to interpret $\bar{\rho}^{(n)}$ as a *covariance* matrix, and to derive the correlation matrix associated with it. We can do this immediately by defining

$$\rho_{i,j}^{(n)} := \bar{\rho}_{i,j}^{(n)} / (\sqrt{\bar{\rho}_{i,i}^{(n)} \ \bar{\rho}_{j,j}^{(n)}}).$$

Now $\rho_{i,j}^{(n)}$ is an n-rank approximation of the original matrix ρ. But how good is the approximation, and are there more precise methods to approximate a full rank correlation matrix with a n-rank matrix? Can we find, in a sense, the **best** reduced rank correlation matrix approximating a given full rank one?

Instantaneous correlation: Reducing the rank. Angles parameterization and optimization I

An angles parametric form for B. Rebonato:

$$\begin{aligned} b_{i,1} &= \cos\theta_{i,1} \\ b_{i,k} &= \cos\theta_{i,k}\sin\theta_{i,1}\cdots\sin\theta_{i,k-1}, \quad 1<k<n, \\ b_{i,n} &= \sin\theta_{i,1}\cdots\sin\theta_{i,n-1}, \quad \text{for } \ i=1,2,\ldots,M. \end{aligned}$$

Angles are redundant: one can assume with no loss of generality that $\theta_{i,k}=0$ for $i \le k$ (“trapezoidal” angles matrix)

$$\text{For} \quad n=2, \quad \rho^B_{i,j} = b_{i,1}b_{j,1} + b_{i,2}b_{j,2} = \cos(\theta_i - \theta_j).$$

(redendancy: can assume $\theta_1 = 0$ with no loss of generality.) This structure consists of M parameters $\theta_1,\ldots,\theta_M$ obtained either by forcing the LMM model to recover market swaptions prices (market

Instantaneous correlation: Reducing the rank. Angles parameterization and optimization II

implied data), or through historical estimation (time-series/econometrics). More on this later.
Given full rank ρ^F, can find optimal θ by minimizing numerically

$$\theta^* = \text{argmin}_\theta \left(\sum_{i,j=1}^{M} (\rho^F_{i,j} - \rho_{i,j}(\theta))^2 \right).$$

Instantaneous correlation: Reducing the rank. Angles parameterization and optimization III

Example: full rank ρ

1	0.9756	0.9524	0.9304	0.9094	0.8894	0.8704	0.8523	0.8352	0.8188
0.9756	1	0.9756	0.9524	0.9304	0.9094	0.8894	0.8704	0.8523	0.8352
0.9524	0.9756	1	0.9756	0.9524	0.9304	0.9094	0.8894	0.8704	0.8523
0.9304	0.9524	0.9756	1	0.9756	0.9524	0.9304	0.9094	0.8894	0.8704
0.9094	0.9304	0.9524	0.9756	1	0.9756	0.9524	0.9304	0.9094	0.8894
0.8894	0.9094	0.9304	0.9524	0.9756	1	0.9756	0.9524	0.9304	0.9094
0.8704	0.8894	0.9094	0.9304	0.9524	0.9756	1	0.9756	0.9524	0.9304
0.8523	0.8704	0.8894	0.9094	0.9304	0.9524	0.9756	1	0.9756	0.9524
0.8352	0.8523	0.8704	0.8894	0.9094	0.9304	0.9524	0.9756	1	0.9756
0.8188	0.8352	0.8523	0.8704	0.8894	0.9094	0.9304	0.9524	0.9756	1

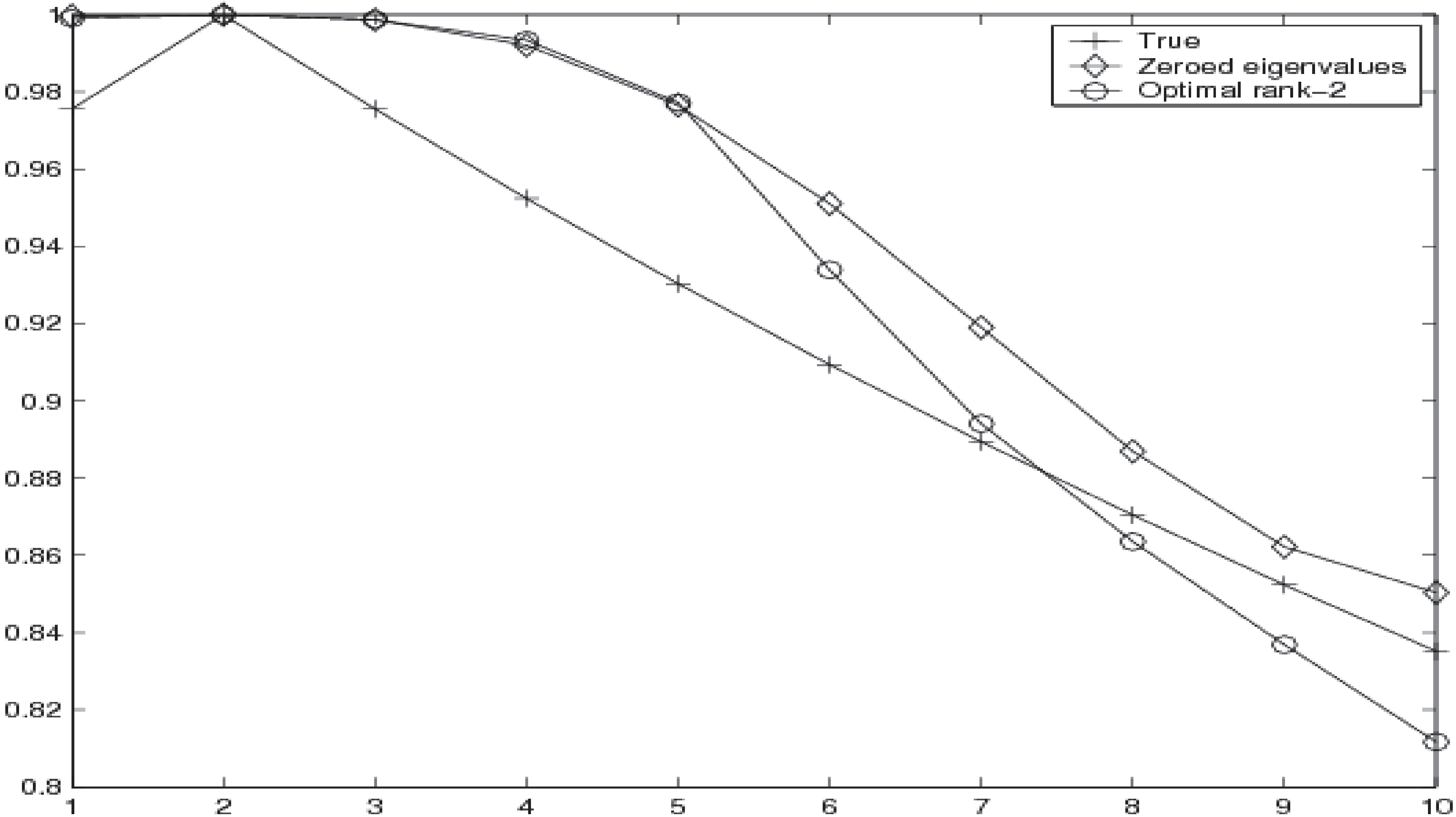
Rank-2 optimal approximation:

$$= [1.2367 \;\; 1.2812 \;\; 1.3319 \;\; 1.3961 \;\; 1.4947 \;\; 1.6469 \;\; 1.7455 \;\; 1.8097 \;\; 1.8604 \;\; 1.9049].$$

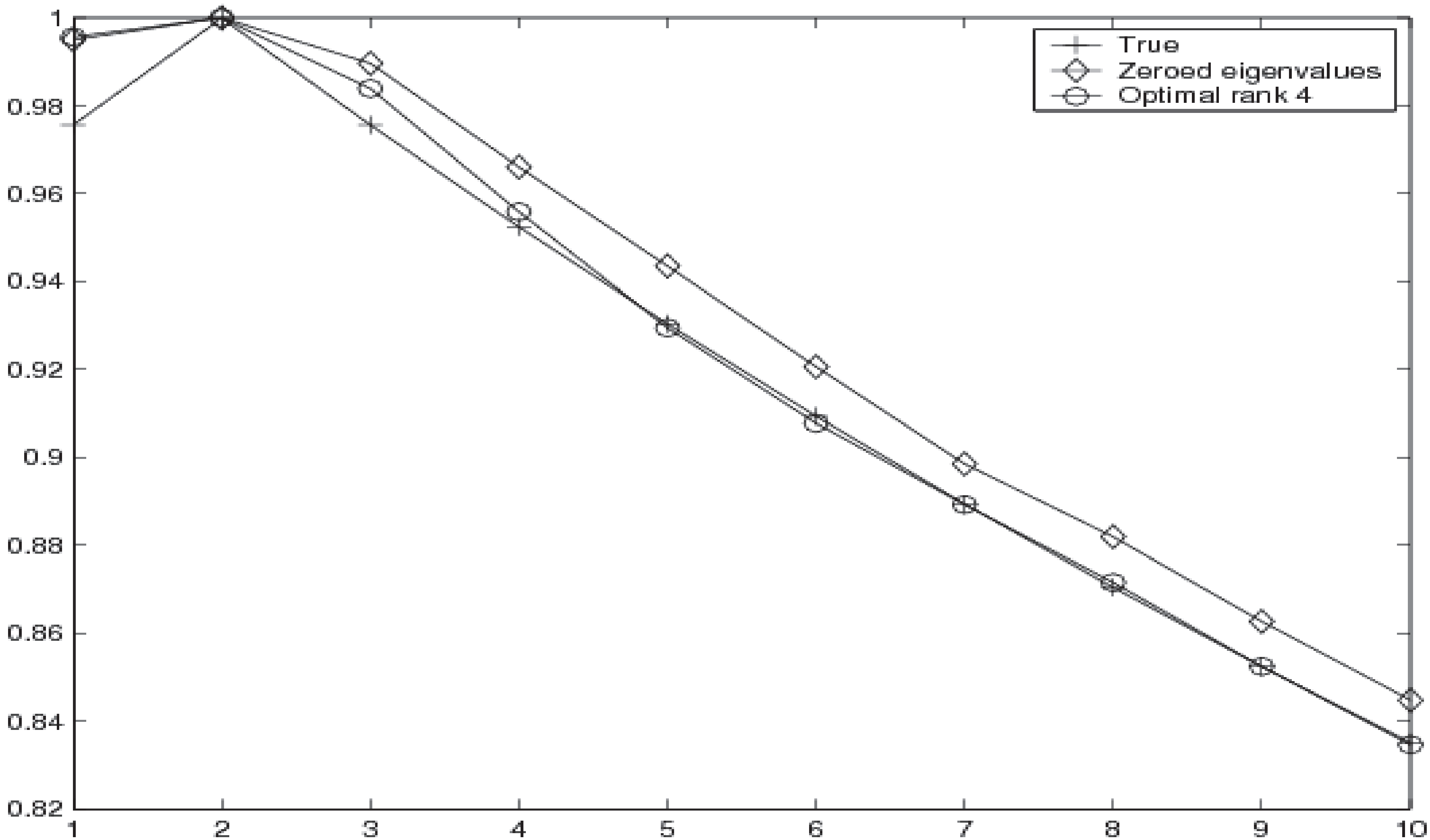
The resulting optimal rank-2 matrix $\rho(\theta^{*(2)})$ is

1	0.999	0.9955	0.9873	0.9669	0.917	0.8733	0.8403	0.8117	0.7849
0.999	1	0.9987	0.9934	0.9773	0.9339	0.8941	0.8636	0.8369	0.8117
0.9955	0.9987	1	0.9979	0.9868	0.9508	0.9157	0.888	0.8636	0.8403
0.9873	0.9934	0.9979	1	0.9951	0.9687	0.9396	0.9157	0.8941	0.8733
0.9669	0.9773	0.9868	0.9951	1	0.9885	0.9687	0.9508	0.9339	0.917
0.917	0.9339	0.9508	0.9687	0.9885	1	0.9951	0.9868	0.9773	0.9669
0.8733	0.8941	0.9157	0.9396	0.9687	0.9951	1	0.9979	0.9934	0.9873
0.8403	0.8636	0.888	0.9157	0.9508	0.9868	0.9979	1	0.9987	0.9955
0.8117	0.8369	0.8636	0.8941	0.9339	0.9773	0.9934	0.9987	1	0.999
0.7849	0.8117	0.8403	0.8733	0.917	0.9669	0.9873	0.9955	0.999	1

Problems of low rank correlation: sigmoid shape



Problems of low rank correlation: sigmoid shape



Higher rank correlation

Another example: Consider the rapidly decreasing 10×10 full-rank $\hat{\rho}_{i,j} = \exp[-|i-j|]$.
rank-4 approximation: the zeroed-eigenvalues procedure yields a matrix $\rho^{(4)}$ given by

1	0.9474	0.5343	-0.0116	-0.1967	-0.0427	0.1425	0.1378	-0.042	-0.1511
0.9474	1	0.775	0.2884	0.0164	-0.03	0.0316	0.0538	0	-0.042
0.5343	0.775	1	0.8137	0.4993	0.0979	-0.1229	-0.1035	0.0538	0.1378
-0.0116	0.2884	0.8137	1	0.8583	0.3725	-0.0336	-0.1229	0.0316	0.1425
-0.1967	0.0164	0.4993	0.8583	1	0.7658	0.3725	0.0979	-0.03	-0.0427
-0.0427	-0.03	0.0979	0.3725	0.7658	1	0.8583	0.4993	0.0164	-0.1967
0.1425	0.0316	-0.1229	-0.0336	0.3725	0.8583	1	0.8137	0.2884	-0.0116
0.1378	0.0538	-0.1035	-0.1229	0.0979	0.4993	0.8137	1	0.775	0.5343
-0.042	0	0.0538	0.0316	-0.03	0.0164	0.2884	0.775	1	0.9474
-0.1511	-0.042	0.1378	0.1425	-0.0427	-0.1967	-0.0116	0.5343	0.9474	1

optimal angle-parameterized rank-4 matrix $\rho(\theta^{*(4)})$:

1	0.9399	0.4826	-0.0863	-0.2715	-0.0437	0.1861	0.1808	-0.077	-0.2189
0.9399	1	0.7515	0.234	-0.0587	-0.0572	0.0496	0.0843	-0.0135	-0.077
0.4826	0.7515	1	0.7935	0.4329	0.015	-0.1745	-0.1195	0.0843	0.1808
-0.0863	0.234	0.7935	1	0.8432	0.3222	-0.0872	-0.1745	0.0496	0.1861
-0.2715	-0.0587	0.4329	0.8432	1	0.7421	0.3222	0.015	-0.0572	-0.0437
-0.0437	-0.0572	0.015	0.3222	0.7421	1	0.8432	0.4329	-0.0587	-0.2715
0.1861	0.0496	-0.1745	-0.0872	0.3222	0.8432	1	0.7935	0.234	-0.0863
0.1808	0.0843	-0.1195	-0.1745	0.015	0.4329	0.7935	1	0.7515	0.4826
-0.077	-0.0135	0.0843	0.0496	-0.0572	-0.0587	0.234	0.7515	1	0.9399
-0.2189	-0.077	0.1808	0.1861	-0.0437	-0.2715	-0.0863	0.4826	0.9399	1

Higher rank

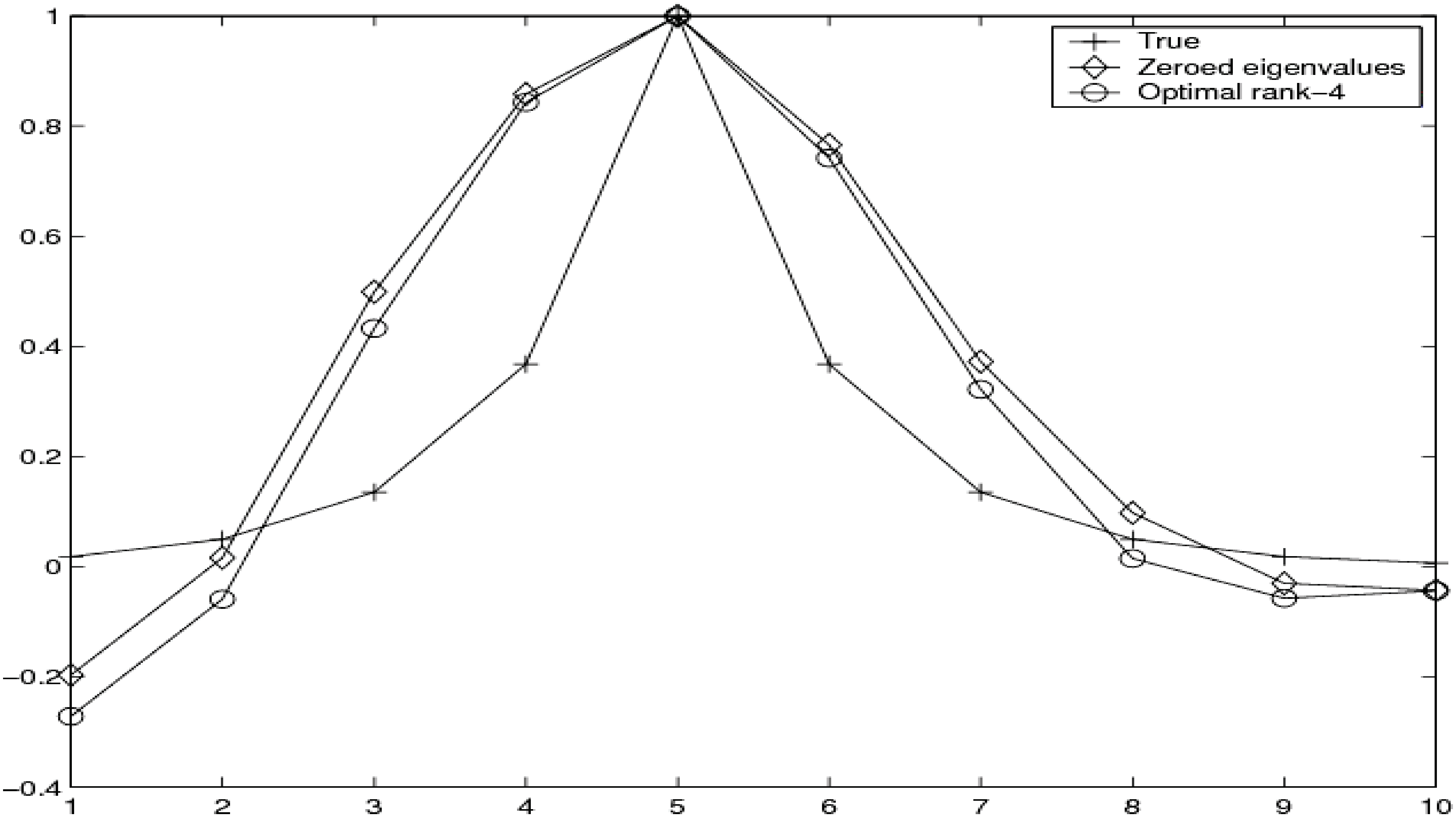
again 10×10 full-rank $\hat{\rho}_{i,j} = \exp[-|i-j|]$.
If we resort to a rank-7 approximation, the zeroed-eigenvalues approach yields the following matrix $\rho^{(7)}$:

1	0.5481	0.0465	0.0944	0.0507	-0.0493	0.034	0.0169	-0.0441	0.0284
0.5481	1	0.6737	0.0647	0.0312	0.112	-0.0477	-0.0162	0.0691	-0.0441
0.0465	0.6737	1	0.579	0.1227	0.0353	0.0562	0.0012	-0.0162	0.0169
0.0944	0.0647	0.579	1	0.5822	0.0674	0.0806	0.0562	-0.0477	0.034
0.0507	0.0312	0.1227	0.5822	1	0.6472	0.0674	0.0353	0.112	-0.0493
-0.0493	0.112	0.0353	0.0674	0.6472	1	0.5822	0.1227	0.0312	0.0507
0.034	-0.0477	0.0562	0.0806	0.0674	0.5822	1	0.579	0.0647	0.0944
0.0169	-0.0162	0.0012	0.0562	0.0353	0.1227	0.579	1	0.6737	0.0465
-0.0441	0.0691	-0.0162	-0.0477	0.112	0.0312	0.0647	0.6737	1	0.5481
0.0284	-0.0441	0.0169	0.034	-0.0493	0.0507	0.0944	0.0465	0.5481	1

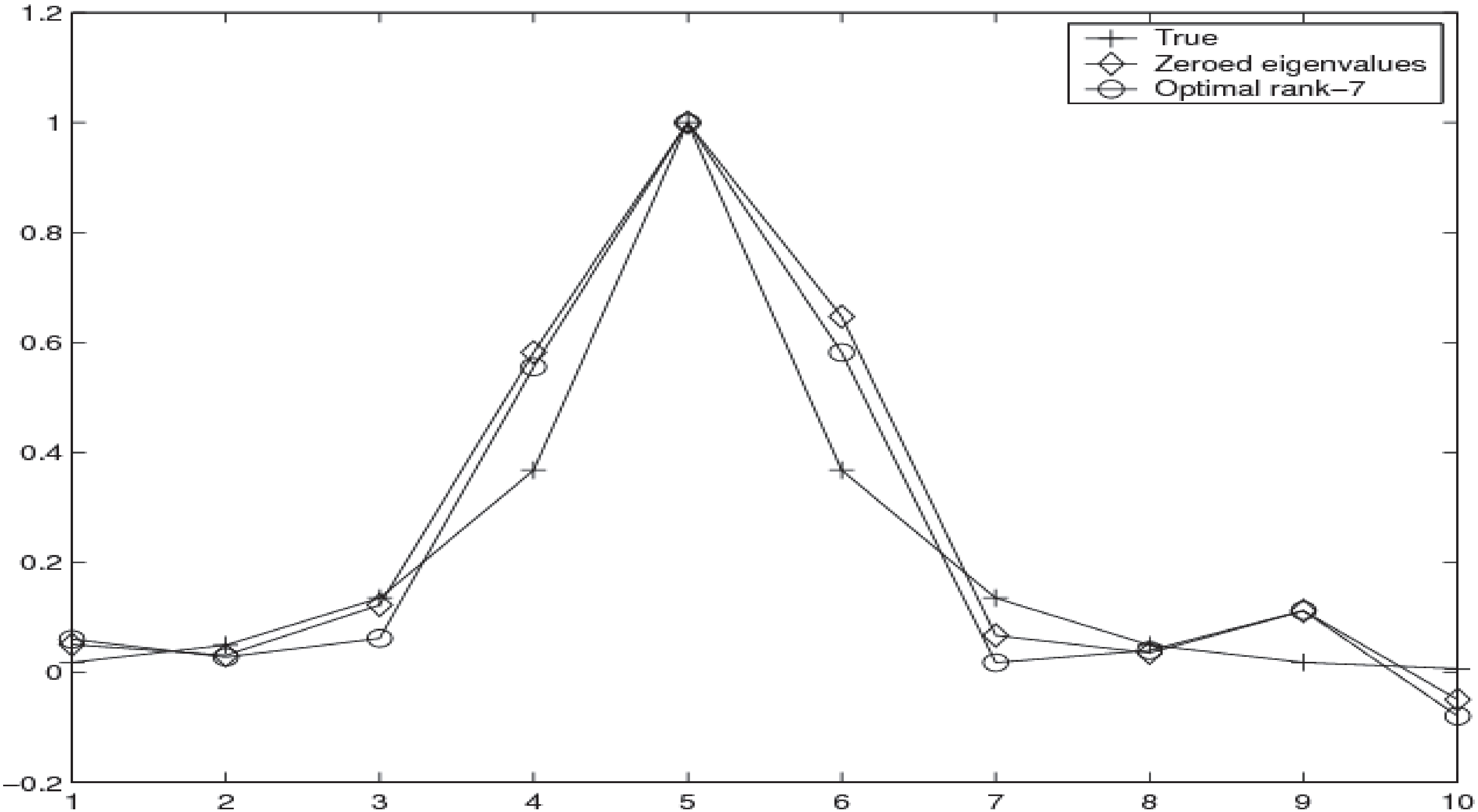
Optimization on an angle-parameterized rank-7 matrix yields the following output matrix $\rho(\theta^{*(7)})$:

1	0.5592	-0.0177	0.1085	0.0602	-0.0795	0.0589	0.018	-0.0734	0.0667
0.5592	1	0.5992	0.0202	0.0277	0.1123	-0.0652	-0.008	0.0797	-0.0734
-0.0177	0.5992	1	0.5464	0.0618	0.0401	0.0561	-0.012	-0.008	0.018
0.1085	0.0202	0.5464	1	0.5556	0.018	0.0834	0.0561	-0.0652	0.0589
0.0602	0.0277	0.0618	0.5556	1	0.5819	0.018	0.0401	0.1123	-0.0795
-0.0795	0.1123	0.0401	0.018	0.5819	1	0.5556	0.0618	0.0277	0.0602
0.0589	-0.0652	0.0561	0.0834	0.018	0.5556	1	0.5464	0.0202	0.1085
0.018	-0.008	-0.012	0.0561	0.0401	0.0618	0.5464	1	0.5992	-0.0177
-0.0734	0.0797	-0.008	-0.0652	0.1123	0.0277	0.0202	0.5992	1	0.5592
0.0667	-0.0734	0.018	0.0589	-0.0795	0.0602	0.1085	-0.0177	0.5592	1

Higher rank


True
Zeroed eigenvalues
Optimal rank−4
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
1
2
3
4
5
6
7
8
9
10

Higher rank


True
Zeroed eigenvalues
Optimal rank−7
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
1
2
3
4
5
6
7
8
9
10

Monte Carlo pricing swaptions with LMM I

$$E^{\boxed{B}}\left(\frac{\boxed{B(0)}}{\boxed{B(T_\alpha)}}(S_{\alpha,\beta}(T_\alpha)-K)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)\right)=$$

$$=E^{\boxed{\alpha}}\left[\frac{\boxed{P(0,T_\alpha)}}{\boxed{P(T_\alpha,T_\alpha)}}(S_{\alpha,\beta}(T_\alpha)-K)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)\right].$$

$$=P(0,T_\alpha)E^{\alpha}\left[(S_{\alpha,\beta}(T_\alpha)-K)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)\right].$$

Since $\quad S_{\alpha,\beta}(T_\alpha)=\dfrac{1-\prod_{j=\alpha+1}^{\beta}\frac{1}{1+\tau_j F_j(T_\alpha)}}{\sum_{i=\alpha+1}^{\beta}\tau_i\prod_{j=\alpha+1}^{i}\frac{1}{1+\tau_j F_j(T_\alpha)}}$

Monte Carlo pricing swaptions with LMM II

the above expectation depends on the *joint* distrib. under Q^α of

$$F_{\alpha+1}(T_\alpha), F_{\alpha+2}(T_\alpha), \ldots, F_\beta(T_\alpha)$$

Recall the dynamics of forward rates under Q^α:

$$dF_k(t) = \sigma_k(t) F_k(t) \sum_{j=\alpha+1}^{k} \frac{\rho_{k,j}\, \tau_j\, \sigma_j\, F_j}{1 + \tau_j F_j(t)}\, dt + \sigma_k(t) F_k(t) dZ_k,$$

Monte Carlo pricing swaptions with LMM III

$$E^B\left(D(0,T_\alpha)\,(S_{\alpha,\beta}(T_\alpha)-K)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)\right)=$$

$$=P(0,T_\alpha)E^\alpha\left[(S_{\alpha,\beta}(T_\alpha)-K)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)\right].$$

Since $S_{\alpha,\beta}(T_\alpha)=\dfrac{1-\prod_{j=\alpha+1}^{\beta}\frac{1}{1+\tau_j\,F_j(T_\alpha)}}{\sum_{i=\alpha+1}^{\beta}\tau_i\prod_{j=\alpha+1}^{i}\frac{1}{1+\tau_j\,F_j(T_\alpha)}}$

Milstein scheme for $\ln F$:

$$\ln F_k^{\Delta t}(t+\Delta t)=\ln F_k^{\Delta t}(t)+\sigma_k(t)\sum_{j=\alpha+1}^{k}\frac{\rho_{k,j}\,\tau_j\,\sigma_j(t)\,F_j^{\Delta t}}{1+\tau_j F_j^{\Delta t}}\Delta t+$$

Monte Carlo pricing swaptions with LMM IV

$$-\frac{\sigma_k^2(t)}{2}\Delta t + \sigma_k(t)(Z_k(t+\Delta t) - Z_k(t))$$

leads to an approximation such that there exists a δ_0 with

$$E^\alpha\{|\ln F_k^{\Delta t}(T_\alpha) - \ln F_k(T_\alpha)|\} \le C(T_\alpha)(\Delta t)^1 \text{ for all } \Delta t \le \delta_0$$

where $C(T_\alpha) > 0$ is a constant (strong convergence of order 1).
$(Z_k(t+\Delta t) - Z_k(t))$ is GAUSSIAN and KNOWN, easy to simulate.

Monte Carlo pricing with LMM I

A refined variance for simulating the shocks: Notice that in integrating **exactly** the dF equation between t and $t + \Delta t$, the resulting Brownian-motion part, in vector notation, is

$$\Delta\zeta_t := \int_t^{t+\Delta t} \underline{\sigma}(s) dZ(s) \sim \mathcal{N}(0, \mathrm{COV}_t)$$

(here the product of vectors acts component by component), where the matrix COV_t is given by

$$(\mathrm{COV}_t)_{h,k} = \int_t^{t+\Delta t} \rho_{h,k}\sigma_h(s)\sigma_k(s)\, ds.$$

Therefore, in principle we have no need to approximate this term by

$$\underline{\sigma}(t)(Z(t+\Delta t) - Z(t)) \sim \mathcal{N}(0,\ \Delta t\, \underline{\sigma}(t)\, \rho\, \underline{\sigma}(t)')$$

Monte Carlo pricing with LMM II

as is done in the classical general MC scheme given earlier. Indeed, we may consider a more refined scheme where the following substitution occurs:

$$\underline{\sigma}(t)(Z(t+\Delta t) - Z(t)) \longrightarrow \Delta\zeta_t.$$

The new shocks vector $\Delta\zeta_t$ can be simulated easily through its Gaussian distribution given above.

Monte Carlo pricing with LMM: Standard error I

Assume we need to value a payoff $\Pi(T)$ depending on the realization of different forward LIBOR rates

$$F(t) = [F_{\alpha+1}(t), \ldots, F_\beta(t)]'$$

in a time interval $t \in [0, T]$, where typically $T \le T_\alpha$.
We have seen a particular case of $\Pi(T) = \Pi(T_\alpha)$ as the swaption payoff. The earlier simulation scheme for the rates entering the payoff provides us with the F's needed to form scenarios on $\Pi(T)$. Denote by a superscript the scenario (or path) under which a quantity is considered, $n_p = \#$ paths.
The Monte Carlo price of our payoff is computed, based on the simulated paths, as $E[D(0,T)\Pi(T)] =$
$= P(0,T)E^T(\Pi(T)) = P(0,T)\sum_{j=1}^{n_p} \Pi^j(T)/n_p,$

Monte Carlo pricing with LMM: Standard error II

where the forward rates F^j entering $\Pi^j(T)$ have been simulated under the T-forward measure. We omit the T-argument in $\Pi(T)$, E^T and Std^T to contain notation: all distributions, expectations and statistics are under the T-forward measure. However, the reasoning is general and holds under any other measure.

We wish to have an estimate of the error me have when estimating the true expectation $E(\Pi)$ by its Monte Carlo estimate $\sum_{j=1}^{n_p} \Pi^j / n_p$. To do so, the classic reasoning is as follows.

Let us view $(\Pi^j)_j$ as a sequence of independent identically distributed (iid) random variables, distributed as Π. By the central limit theorem, we know that under suitable assumptions one has

$$\frac{\sum_{j=1}^{n_p} (\Pi^j - E(\Pi))}{\sqrt{n_p}\,\mathrm{Std}(\Pi)} \to \mathcal{N}(0,1),$$

Monte Carlo pricing with LMM: Standard error III

in law, as $n_p \to \infty$, from which we have that we may write, approximately and for large n_p:

$$\frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} - E(\Pi) \sim \frac{\mathrm{Std}(\Pi)}{\sqrt{n_p}} \mathcal{N}(0,1).$$

It follows that

$$Q^T \left\{ \left| \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} - E(\Pi) \right| < \epsilon \right\} = Q^T \left\{ |\mathcal{N}(0,1)| < \epsilon \, \frac{\sqrt{n_p}}{\mathrm{Std}(\Pi)} \right\}$$

$$= 2\Phi\left(\epsilon \, \frac{\sqrt{n_p}}{\mathrm{Std}(\Pi)} \right) - 1,$$

where as usual Φ denotes the cumulative distribution function of the standard Gaussian random variable.

Monte Carlo pricing with LMM: Standard error IV

$$Q^T\left\{\left|\frac{\sum_{j=1}^{n_p}\Pi^j}{n_p}-E(\Pi)\right|<\epsilon\right\}=2\Phi\left(\epsilon\,\frac{\sqrt{n_p}}{\mathrm{Std}(\Pi)}\right)-1,$$

The above equation gives the probability that our Monte Carlo estimate $\sum_{j=1}^{n_p}\Pi^j/n_p$ is not farther than ϵ from the true expectation $E(\Pi)$ we wish to estimate. Typically, one sets a desired value for this probability, say 0.98, and derives ϵ by solving

$$2\Phi\left(\epsilon\,\frac{\sqrt{n_p}}{\mathrm{Std}(\Pi)}\right)-1=0.98.$$

For example, since we know from the Φ tables that

$$2\Phi(z)-1=0.98 \iff \Phi(z)=0.99 \iff z\approx 2.33,$$

Monte Carlo pricing with LMM: Standard error V

we have that

$$\epsilon = 2.33 \, \frac{\mathrm{Std}(\Pi)}{\sqrt{n_p}}.$$

The true value of $E(\Pi)$ is thus inside the "window"

$$\left[\frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} - 2.33 \, \frac{\mathrm{Std}(\Pi)}{\sqrt{n_p}}, \quad \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + 2.33 \, \frac{\mathrm{Std}(\Pi)}{\sqrt{n_p}} \right]$$

with a 98% probability. This is called a 98% confidence interval for $E(\Pi)$. Other typical confidence levels are given in Table 1.

Monte Carlo pricing with LMM: Standard error VI

$2\Phi(z) - 1$	$z \approx$
99%	2.58
98%	2.33
95.45%	2
95%	1.96
90%	1.65
68.27%	1

Table: Confidence levels

We can see that, ceteris paribus, as n_p increases, the window shrinks as $1/\sqrt{n_p}$, which is worse than $1/n_p$. If we need to reduce the window size to one tenth, we have to increase the number of scenarios by a factor 100. Sometimes, to reach a chosen accuracy (a small enough window), we need to take a huge number of scenarios n_p. When this is

Monte Carlo pricing with LMM: Standard error VII

too time-consuming, there are “variance-reduction” techniques that may be used to reduce the above window size.

A more fundamental problem with the above window is that the true standard deviation Std(Π) of the payoff is usually unknown. This is typically replaced by the known sample standard deviation obtained by the simulated paths,

$$(\widehat{\mathrm{Std}}(\Pi; n_p))^2 := \sum_{j=1}^{n_p} (\Pi^j)^2/n_p - (\sum_{j=1}^{n_p} \Pi^j/n_p)^2$$

and the actual 98% Monte Carlo window we compute is

$$\left[\frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} - 2.33\,\frac{\widehat{\mathrm{Std}}(\Pi; n_p)}{\sqrt{n_p}},\quad \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + 2.33\,\frac{\widehat{\mathrm{Std}}(\Pi; n_p)}{\sqrt{n_p}}\right]. \qquad (41)$$

Monte Carlo pricing with LMM: Standard error VIII

To obtain a 95% (narrower) window it is enough to replace 2.33 by 1.96, and to obtain a (still narrower) 90% window it is enough to replace 2.33 by 1.65. All other sizes may be derived by the Φ tables.
We know that in some cases, to obtain a 98% window whose (half-) width $2.33\,\widehat{\mathrm{Std}}(\Pi; n_p)/\sqrt{n_p}$ is small enough, we are forced to take a huge number of paths n_p. This can be a problem for computational time. A way to reduce the impact of this problem is, for a given n_p that we deem to be large enough, to find alternatives that reduce the variance $(\widehat{\mathrm{Std}}(\Pi; n_p))^2$, thus narrowing the above window without increasing n_p.
One of the most effective methods to do this is the control variate technique.

Monte Carlo pricing with LMM: Control Variate I

We begin by selecting an alternative payoff Π^{an} which we know how to evaluate analytically, in that

$$E(\Pi^{\text{an}}) = \pi^{\text{an}}$$

is known. When we simulate our original payoff Π we now simulate also the analytical payoff Π^{an} as a function of the same scenarios for the underlying variables F. We define a new control-variate estimator for $E\Pi$ as

$$\widehat{\Pi}_c(\gamma; n_p) := \frac{\sum_{j=1}^{n_p} \Pi^j}{n_p} + \gamma \left(\frac{\sum_{j=1}^{n_p} \Pi^{\text{an},j}}{n_p} - \pi^{\text{an}} \right),$$

with γ a constant to be determined. When viewing Π^j as iid copies of Π and $\Pi^{\text{an},j}$ as iid copies of Π^{an}, the above estimator remains unbiased, since we are subtracting the true known mean π^{an} from the correction

Monte Carlo pricing with LMM: Control Variate II

term in γ. So, once we have found that the estimator has not been biased by our correction, we may wonder whether our correction can be used to lower the variance.
Consider the random variable

$$\Pi_c(\gamma) := \Pi + \gamma(\Pi^{\text{an}} - \pi^{\text{an}})$$

whose expectation is the $E(\Pi)$ we are estimating, and compute

$$\text{Var}(\Pi_c(\gamma)) = \text{Var}(\Pi) + \gamma^2 \text{Var}(\Pi^{\text{an}}) + 2\gamma \text{Corr}(\Pi, \Pi^{\text{an}}) \text{Std}(\Pi) \text{Std}(\Pi^{\text{an}}),$$

We may minimize this function of γ by differentiating and setting the first derivative to zero.

Monte Carlo pricing with LMM: Control Variate III

We obtain easily that the variance is minimized by the following value of γ: $\gamma^* := -\text{Corr}(\Pi, \Pi^{\text{an}})\text{Std}(\Pi) \;/\text{Std}(\Pi^{\text{an}})$. By plugging $\gamma = \gamma^*$ into the above expression, we obtain easily

$$\text{Var}(\Pi_c(\gamma^*)) = \text{Var}(\Pi)(1 - \text{Corr}(\Pi, \Pi^{\text{an}})^2),$$

from which we see that $\Pi_c(\gamma^*)$ has a smaller variance than our original Π, the smaller this variance the larger (in absolute value) the correlation between Π and Π^{an}. Accordingly, when moving to simulated quantities, we set

$$\widehat{\text{Std}}(\Pi_c(\gamma^*); n_p) = \widehat{\text{Std}}(\Pi; n_p)(1 - \widehat{\text{Corr}}(\Pi, \Pi^{\text{an}}; n_p)^2)^{1/2},$$

where $\widehat{\text{Corr}}(\Pi, \Pi^{\text{an}}; n_p)$ is the sample correlation

$$\widehat{\text{Corr}}(\Pi, \Pi^{\text{an}}; n_p) = \frac{\widehat{\text{Cov}}(\Pi, \Pi^{\text{an}}; n_p)}{\widehat{\text{Std}}(\Pi; n_p)\,\widehat{\text{Std}}(\Pi^{\text{an}}; n_p)}$$

Monte Carlo pricing with LMM: Control Variate IV

and the sample covariance is

$$\widehat{\text{Cov}}(\Pi, \Pi^{\text{an}}; n_p) = \sum_{j=1}^{n_p} \Pi^j \Pi^{\text{an},j} / n_p - (\sum_{j=1}^{n_p} \Pi^j)(\sum_{j=1}^{n_p} \Pi^{\text{an},j})/(n_p^2)$$

and

$$(\widehat{\text{Std}}(\Pi^{\text{an}}; n_p))^2 := \sum_{j=1}^{n_p} (\Pi^{\text{an},j})^2 / n_p - (\sum_{j=1}^{n_p} \Pi^{\text{an},j} / n_p)^2.$$

One may include the correction factor $n_p/(n_p - 1)$ to correct for the bias of the variance estimator, although the correction is irrelevant for large n_p.

We see from

$$\widehat{\text{Std}}(\Pi_c(\gamma^*); n_p) = \widehat{\text{Std}}(\Pi; n_p)(1 - \widehat{\text{Corr}}(\Pi, \Pi^{\text{an}}; n_p)^2)^{1/2},$$

Monte Carlo pricing with LMM: Control Variate V

that for the variance reduction to be relevant, we need to choose the analytical payoff Π^{an} to be as (positively or negatively) correlated as possible with the original payoff Π. Notice that in the limit case of correlation equal to one the variance shrinks to zero.
The window for our control-variate Monte Carlo estimate $\widehat{\Pi}_c(\gamma; n_p)$ of $E(\Pi)$ is now:

$$\left[\widehat{\Pi}_c(\gamma; n_p) - 2.33\,\frac{\widehat{\mathrm{Std}}(\Pi_c(\gamma^*); n_p)}{\sqrt{n_p}},\quad \widehat{\Pi}_c(\gamma; n_p) + 2.33\,\frac{\widehat{\mathrm{Std}}(\Pi_c(\gamma^*); n_p)}{\sqrt{n_p}}\right],$$

This window is narrower than the corresponding simple Monte Carlo one by a factor $(1 - \widehat{\mathrm{Corr}}(\Pi, \Pi^{an}; n_p)^2)^{1/2}$.
We may wonder about a good possible Π^{an}. We may select as Π^{an} the simplest payoff depending on the underlying rates

$$F(t) = [F_{\alpha+1}(t), \dots, F_\beta(t)]'.$$

Monte Carlo pricing with LMM: Control Variate VI

This is given by the Forward Rate Agreement (FRA) contract seen earlier. We consider the sum of at-the-money FRA payoffs, each on a single forward rate included in our family.
In other terms, if we are simulating under the T_j forward measure a payoff paying at T_α, with , the payoff we consider is

$$\Pi^{\text{an}}(T_\alpha) = \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)(F_i(T_\alpha) - F_i(0))/P(T_\alpha, T_j)$$

whose expected value under the Q^j measure is easily seen to be 0 by remembering that quantities featuring $P(\cdot, T_j)$ as denominator are martingales. Thus in our case $\pi^{\text{an}} = 0$ and we may use the related control-variate estimator. Somehow surprisingly, this simple correction has allowed us to reduce the number of paths of up to a factor 10 in several cases, including for example Monte Carlo evaluation of ratchet caps.

Analytical swaption prices with LMM I

Approximated method to compute swaption prices with the LMM LIBOR MODEL without resorting to Monte Carlo simulation.
This method is rather simple and its quality has been tested in Brace, Dun, and Barton (1999) and by ourselves.
Recall the SWAP MODEL SMM leading to Black's formula for swaptions:

$$d\ S_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t)\ dW_t^{\alpha,\beta}, \quad Q^{\alpha,\beta}\ .$$

A crucial role is played by the Black swap volatility component

$$\int_0^{T_\alpha} \sigma^2_{\alpha,\beta}(t)dt = \int_0^{T_\alpha} \sigma_{\alpha,\beta}(t) dW^{\alpha,\beta}(t) \sigma_{\alpha,\beta}(t) dW^{\alpha,\beta}(t)$$

$$= \int_0^{T_\alpha} (d \ln S_{\alpha,\beta}(t))(d \ln S_{\alpha,\beta}(t))$$

Analytical swaption prices with LMM II

We compute an analogous approximated quantity in the LMM.

$$\begin{aligned} S_{\alpha,\beta}(t) &= \sum_{i=\alpha+1}^{\beta} w_i(t)\, F_i(t), \\ w_i(t) &= w_i(F_{\alpha+1}(t), F_{\alpha+2}(t), \ldots, F_\beta(t)) = \\ &= \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}} . \end{aligned}$$

Freeze the w's at time 0:

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t)\, F_i(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0)\, F_i(t) .$$

Analytical swaption prices with LMM III

(variability of the w's is much smaller than variability of F's)

$$dS_{\alpha,\beta} \approx \sum_{i=\alpha+1}^{\beta} w_i(0)\ dF_i = (\ldots)dt + \sum_{i=\alpha+1}^{\beta} w_i(0)\sigma_i(t)F_i(t)dZ_i(t)\ ,$$

under any of the forward adjusted measures. Compute

$$dS_{\alpha,\beta}(t)dS_{\alpha,\beta}(t) \approx \sum_{i,j=\alpha+1}^{\beta} w_i(0)\sigma_i(t)F_i(t)dZ_i w_j(0)F_j(t)\sigma_j(t)\ dZ_j\ =$$

$$= \sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(t)F_j(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)\ dt\ .$$

Analytical swaption prices with LMM IV

The percentage quadratic covariation is

$$(d\ln S_{\alpha,\beta}(t))(d\ln S_{\alpha,\beta}(t)) = \frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)}\frac{dS_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} =$$

$$\approx \frac{\sum_{i,j=\alpha+1}^{\beta} w_i(0)w_j(0)F_i(t)F_j(t)\rho_{i,j}\sigma_i(t)\sigma_j(t)}{S_{\alpha,\beta}(t)^2}\, dt\ .$$

Introduce a further approx by freezing again all F's (as was done earlier for the w's) to time zero: $(d\ln S_{\alpha,\beta})(d\ln S_{\alpha,\beta}) \approx$

$$\approx \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{\alpha,\beta}(0)^2}\,\sigma_i(t)\sigma_j(t)\, dt\ .$$

Analytical swaption prices with LMM V

Now compute the time-averaged percentage variance of S as

(Rebonato's Formula)

$$(v_{\alpha,\beta}^{\text{LMM}})^2 = \frac{1}{T_\alpha}\int_0^{T_\alpha} (d\ln S_{\alpha,\beta}(t))(d\ln S_{\alpha,\beta}(t))$$

$$= \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{T_\alpha\ S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\ dt\ .$$

$v_{\alpha,\beta}^{\text{LMM}}$ can be used as a proxy for the Black volatility $v_{\alpha,\beta}(T_\alpha)$.
Use Black's formula for swaptions with volatility $v_{\alpha,\beta}^{\text{LMM}}$ to price swaptions **analytically** with the LMM.
It turns out that the approximation is not at all bad, as pointed out by Brace, Dun and Barton (1999) and by ourselves.

Analytical swaption prices with LMM VI

A slightly more sophisticated version of this procedure has been pointed out for example by Hull and White (1999).

This pricing formula is ALGEBRAIC and very quick (compare with short-rate models)

H–W refine this formula by differentiating $S_{\alpha,\beta}(t)$ without immediately freezing the w. Same accuracy in practice.

Analytical terminal correlation I

By similar arguments (freezing the drift and collapsing all measures) we may find a formula for terminal correlation.

$\mathrm{Corr}(F_i(T_\alpha), F_j(T_\alpha))$ should be computed with MC simulation and depends on the chosen numeraire

Useful to have a first idea on the stability of the model correlation at future times.

Traders need to check this quickly, no time for MC

In Brigo and Mercurio (2001), we obtain easily

$$\frac{\exp\left(\int_0^{T_\alpha}\sigma_i(t)\sigma_j(t)\rho_{i,j}dt\right)-1}{\sqrt{\exp\left(\int_0^{T_\alpha}\sigma_i^2(t)dt\right)-1}\sqrt{\exp\left(\int_0^{T_\alpha}\sigma_j^2(t)dt\right)-1}}$$

$$\approx \rho_{i,j}\frac{\int_0^{T_\alpha}\sigma_i(t)\sigma_j(t)\,dt}{\sqrt{\int_0^{T_\alpha}\sigma_i^2(t)dt}\sqrt{\int_0^{T_\alpha}\sigma_j^2(t)dt}},$$

Analytical terminal correlation II

the second approximation as from Rebonato (1999). Schwartz's inequality: *terminal correlations are always smaller, in absolute value, than instantaneous correlations*.

Calibration to swaptions prices I

Swaption calibration: Find σ and ρ in LMM such that the LMM reproduces market swaption vols (the first column is T_α and the first row is the underlying swap length $T_\beta - T_\alpha$)

$v_{\alpha,\beta}^{MKT}$	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	16.4	15.8	14.6	13.8	13.3	12.9	12.6	12.3	12.0	11.7
2y	17.7	15.6	14.1	13.1	12.7	12.4	12.2	11.9	11.7	11.4
3y	17.6	15.5	13.9	12.7	12.3	12.1	11.9	11.7	11.5	11.3
4y	16.9	14.6	12.9	11.9	11.6	11.4	11.3	11.1	11.0	10.8
5y	15.8	13.9	12.4	11.5	11.1	10.9	10.8	10.7	10.5	10.4
7y	14.5	12.9	11.6	10.8	10.4	10.3	10.1	9.9	9.8	9.6
10y	13.5	11.5	10.4	9.8	9.4	9.3	9.1	8.8	8.6	8.4

Table: Black vols of EURO ATM swaptions May 16, 2000

Calibration to swaptions prices II

Table (brokers) not updated uniformly. Some entries may refer to older market situations.
"Temporal misalignment/Stale data"
Calibrated parameters σ or ρ might reflect this by weird configurations.
If so:
Trust the model $\Rightarrow$ detect misalignments
Trust the data $\Rightarrow$ need a better parameterization.

Instantaneous Correlations: Inputs or Outputs? I

Swaptions: Fit "Market prices" to "model prices(σ, ρ)".
Should we infer ρ itself from swaption market quotes or should we estimate ρ exogenously and impose it, leaving the calibration only to σ? Are the parameters in ρ inputs or outputs to the calibration?
Inputs? We might consider a time series of past interest-rate curves data, which are observed under the real world probability measure. This would allow us, through interpolation, to obtain a corresponding time series for the particular forward LIBOR rates being modelled in our LIBOR model. These series would be observed under the objective or real-world measure. Thanks to the Girsanov theorem this is not a problem, since instantaneous correlations, considered as instantaneous covariations between driving Brownian motions in forward rate dynamics, do not depend on the probability measure. Then, by using historical estimation, we obtain an historical estimate of the instantaneous correlation matrix. This ρ, or a stylized version of it,

Instantaneous Correlations: Inputs or Outputs? II

can be considered as a given ρ for our LIBOR model, and the remaining free parameters σ are to be used to calibrate market derivatives data. In this case calibration will consist in finding the σ's such that the model (caps and) swaptions prices match the corresponding market prices. In this "matching" procedure (often an optimization) ρ is fixed from the start to the found historical estimate and we play on the volatility parameters σ to achieve our matching.

Instantaneous Correlations: Inputs or Outputs? I

Outputs? This second possibility considers instantaneous correlations as fitting parameters. The model swaptions prices are functions of ρ^B, and possibly of some remaining instantaneous volatility parameters, that are forced to match as much as possible the corresponding market swaptions prices, so that the parameters values implied by the market, $\rho^B = \rho^B_{\text{MKT}}$, are found. In the two-factor angles case for example, one obtains the values of $\theta_1, \ldots, \theta_M$ (and of the volatility parameters not determined by the calibration to caps) that are implied by the market.

INPUTS? OUTPUTS? Which of the two methods is preferable? We will consider again this question later on. Now we try and address the issue of determining a decent historical ρ in case we are to decide later for the “inputs” approach.

Inst Corrs as Inputs: The historical matrix I

Since European swaptions turn out to be relatively insensitive to instantaneous (rather than terminal) correlation details (e.g. Jäckel and Rebonato (2000)), we may impose a good exogenous instantaneous correlation matrix and subsequently play on volatilities to calibrate swaptions.

Smoothing the rough historically estimated matrix through a parsimonious “pivot” form enjoying desirable properties may guarantee a smooth and regular behaviour of terminal correlations, and slightly more regular σ’s when calibrating. This also avoids problems related to outliers, non-synchronous data and discontinuities in correlation surfaces. These and further problems are recalled by Rebonato e Jäckel (1999), that consequently propose to fit a parametric form onto the estimate.

Secondly, the chosen parametric forms may enjoy particularly interesting properties typical of forward rates correlations.

Inst Corrs as Inputs: The historical matrix II

Thirdly, “pivot” forms depend on a low number of parameters, so that we can more easily control the main features of the matrix, detecting those that provoke undesirable anomalous outputs so as to avoid them. Incorporating personal views or recent changes in the market is also easier with pivot forms.

Inst Corrs as Inputs: The historical matrix I

"Reduced rank pivot historical correlation matrix":

1. A market historical correlation matrix is estimated;
2. The parameters of a parsimonious form are determined by keeping the historical estimate as a reference;
3. An angles form of the desired rank is fitted to the resulting parsimonious matrix;

Historical Estimation: In estimating correlations, we take into account the particular nature of forward rates in the LMM, characterized by a fixed maturity, contrary to market quotations, where a fixed *time-to-maturity* is usually considered as time passes. We observe from the market, at different times t

$$P(t, t+Z), P(t+1, t+1+Z), \ldots, P(t+n, t+n+Z),$$

Inst Corrs as Inputs: The historical matrix II

where Z is ranging in a standard set of time-to-maturities. We need instead

$$P(t, T), P(t+1, T), \ldots, P(t+n, T),$$

for the maturities T included in the tenor structure of the chosen LMM. Accordingly, a log-interpolation between discount factors has been carried out and only one year of data has been used, since the first forward rate in the family expires in one year from the starting date. These data span from February 1, 2001 to February 1, 2002.

Inst Corrs as Inputs: The historical matrix I

From these daily quotations of notional zero-coupon bonds, whose maturities range from one to twenty years from today, we extracted daily log-returns of the annual forward rates involved in the model. Starting from the following usual gaussian approximation

$$\left[\ln\left(\frac{F_1(t+\Delta t)}{F_1(t)}\right), \ldots, \ln\left(\frac{F_{19}(t+\Delta t)}{F_{19}(t)}\right)\right] \sim MN(\mu, V),$$

where $\Delta t = 1$ day, our estimations of the parameters are based on sample mean and covariance for gaussian variables, and are

$$\hat{\mu}_i = \frac{1}{m}\sum_{k=0}^{m-1} \ln\left(\frac{F_i(t_{k+1})}{F_i(t_k)}\right),$$

$$\hat{V}_{i,j} = \frac{1}{m}\sum_{k=0}^{m-1}\left[\left(\ln\left(\frac{F_i(t_{k+1})}{F_i(t_k)}\right) - \hat{\mu}_i\right)\left(\ln\left(\frac{F_j(t_{k+1})}{F_j(t_k)}\right) - \hat{\mu}_j\right)\right],$$

Inst Corrs as Inputs: The historical matrix II

where m is the number of observed log-returns for each rate, so that our estimation of the general correlation element $\rho_{i,j}$ is

$$\hat{\rho}_{i,j} = \frac{\hat{V}_{i,j}}{\sqrt{\hat{V}_{i,i}}\sqrt{\hat{V}_{j,j}}}.$$

Resulting matrix:

	1	2	3	4	5	6	7	8	9	10
1	1.00	.823	.693	.652	.584	.467	.290	.235	.434	.473
2	.823	1.00	.798	.730	.682	.546	.447	.398	.529	.566
3	.693	.798	1.00	.764	.722	.629	.472	.557	.671	.610
4	.652	.730	.764	1.00	.777	.674	.577	.561	.681	.701
5	.584	.682	.722	.777	1.00	.842	.661	.667	.711	.734
6	.467	.546	.629	.674	.842	1.00	.774	.682	.729	.688
7	.290	.447	.472	.577	.661	.774	1.00	.718	.709	.647
8	.235	.398	.557	.561	.667	.682	.718	1.00	.735	.659
9	.434	.529	.671	.681	.711	.729	.709	.735	1.00	.748
10	.473	.566	.610	.701	.734	.688	.647	.659	.748	1.00
11	.331	.418	.484	.562	.696	.770	.648	.639	.591	.632
12	.432	.453	.519	.593	.669	.694	.619	.561	.665	.675
13	.288	.476	.483	.581	.640	.659	.714	.610	.688	.704
14	.230	.343	.542	.498	.590	.634	.619	.720	.693	.634
15	.259	.346	.462	.499	.581	.615	.628	.588	.690	.636
16	.206	.321	.422	.478	.649	.677	.663	.645	.634	.651
17	.227	.323	.450	.488	.653	.702	.638	.642	.644	.625
18	.293	.312	.420	.439	.534	.569	.524	.492	.518	.524
19	.245	.322	.352	.354	.422	.447	.375	.459	.402	.399

	11	12	13	14	15	16	17	18	19
1	0.331	0.432	0.288	0.230	0.259	0.206	0.227	0.293	0.245
2	0.418	0.453	0.476	0.343	0.346	0.321	0.323	0.312	0.322
3	0.484	0.519	0.483	0.542	0.462	0.422	0.450	0.420	0.352
4	0.562	0.593	0.581	0.498	0.499	0.478	0.488	0.439	0.354
5	0.696	0.669	0.640	0.590	0.581	0.649	0.653	0.534	0.422
6	0.770	0.694	0.659	0.634	0.615	0.677	0.702	0.569	0.447
7	0.648	0.619	0.714	0.619	0.628	0.663	0.638	0.524	0.375
8	0.639	0.561	0.610	0.720	0.588	0.645	0.642	0.492	0.459
9	0.591	0.665	0.688	0.693	0.690	0.634	0.644	0.518	0.402
10	0.632	0.675	0.704	0.634	0.636	0.651	0.625	0.524	0.399
11	1.000	0.832	0.722	0.642	0.581	0.679	0.727	0.566	0.448
12	0.832	1.000	0.819	0.687	0.675	0.704	0.686	0.654	0.426
13	0.722	0.819	1.000	0.785	0.776	0.785	0.715	0.594	0.425
14	0.642	0.687	0.785	1.000	0.820	0.830	0.788	0.599	0.453
15	0.581	0.675	0.776	0.820	1.000	0.901	0.796	0.501	0.222
16	0.679	0.704	0.785	0.830	0.901	1.000	0.939	0.707	0.464
17	0.727	0.686	0.715	0.788	0.796	0.939	1.000	0.818	0.657
18	0.566	0.654	0.594	0.599	0.501	0.707	0.818	1.000	0.836
19	0.448	0.426	0.425	0.453	0.222	0.464	0.657	0.836	1.000

Inst Corrs as Inputs: The historical matrix I

Examining the matrix, we see a pronounced and approximately monotonic decorrelation along the columns, when moving away from the diagonal. We see also a relevant initial steepness of the decorrelation pattern. The upward trend along the sub-diagonals is not remarkable. That might be due to the smaller extent of such a phenomenon, more likely to be hidden by noise or differences in liquidity amongst longer rates. Not very different features are visible also in the previous similar estimate showed in Brace, Gatarek and Musiela (1997).

We did some tests on the stability of the estimates, finding out that the values remain rather constant even if we change the sample size or its time positioning.

Inst Corrs as Inputs: The historical matrix I

Principal component analysis reveals that 7 factors are required to explain 90% of the overall variability.

Inst Corrs as Inputs: The historical matrix II

1	11,6992	61,575%	61,575%
2	2,1478	11,304%	72,879%
3	1,1803	6,212%	79,091%
4	0,7166	3,772%	82,863%
5	0,6413	3,375%	86,238%
6	0,4273	2,249%	88,487%
7	0,386	2,032%	90,519%
8	0,3389	1,784%	92,303%
9	0,2805	1,476%	93,779%
10	0,2542	1,338%	95,117%
11	0,1995	1,050%	96,167%
12	0,1692	0,891%	97,057%
13	0,1611	0,848%	97,905%
14	0,1503	0,791%	98,696%
15	0,0877	0,462%	99,158%
16	0,0601	0,316%	99,474%
17	0,0515	0,271%	99,745%
18	0,0333	0,175%	99,921%
19	0,0151	0,079%	100,000%

Inst Corrs as Inputs: Pivot matrices I

Here we concentrate on the full rank parameterizations seen earlier (S&C3, Classical Exponential, Rebonato exponential). The classic methodology is fitting the chosen parametric form to the historically estimated matrix by minimizing some loss function of the difference between the two matrices.

Morini (2002) proposes instead to invert directly the functional structure of the parametric forms. Parameters are expressed as functions of key elements of the target historical matrix, so that such elements will be exactly reproduced. We dub such key elements "pivot points" of the historical matrix, and the resulting parametric matrices "pivot matrices". The Pivot approach:

1. does not need any optimization routine;
2. If the pivot points are chosen appropriately, it generates a matrix with the same typical monotonicity and positivity properties as the original one.

Inst Corrs as Inputs: Pivot matrices II

3. parameters have a clear, intuitive meaning, since they are expressed in terms of correlation entries considered to be particularly significant. This allows us to easily alter and deform the matrix playing with the parameters in a controlled way, as might be needed in the market practice.
4. It keeps out the negative effects of irregularities and clear outliers typical of historical estimations.
5. In our examples the fitting error with the Pivot method is not so far from the error in a complete, optimal fitting.

Inst Corrs as Inputs: Pivot matrices I

Pivot points must be chosen carefully. We will start by considering three-parameters structures. We consider the entries $\rho_{1,2}$, $\rho_{1,M}$ and $\rho_{M-1,M}$. Such elements embed basic monotonicity information of the historical correlation matrix.
Morini (2002) computes, starting with Rebonato's exponential form,

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\exp[-|i-j|(\beta - \alpha(\max(i,j) - 1))], \quad \beta \geq 0.$$

the equations

$$\left(\frac{\rho_{1,M} - \rho_\infty}{1 - \rho_\infty}\right) = \left(\frac{\rho_{M-1,M} - \rho_\infty}{1 - \rho_\infty}\right)^{(M-1)},$$

Inst Corrs as Inputs: Pivot matrices II

for ρ_∞, and

$$\alpha = \frac{\ln\left(\frac{\rho_{1,2}-\rho_\infty}{\rho_{M-1,M}-\rho_\infty}\right)}{2-M}, \qquad \beta = \alpha - \ln\left(\frac{\rho_{1,2}-\rho_\infty}{1-\rho_\infty}\right).$$

The results are

$$\rho_\infty = 0.23551, \quad \alpha = 0.00126, \quad \beta = 0.26388.$$

Inst Corrs as Inputs: Pivot matrices III

Let us now move on to form SC3,

$$\rho_{i,j} = \exp\left[-|i-j|\left(\beta - \frac{\alpha_2}{6M-18}\left(i^2+j^2+ij-6i-6j-3M^2+15M-7\right)\right.\right.$$
$$\left.\left.+\frac{\alpha_1}{6M-18}\left(i^2+j^2+ij-3Mi-3Mj+3i+3j+3M^2-6M+2\right)\right)\right]. \tag{42}$$

Morini computes

$$\beta = -\ln\left(\rho_{M-1,M}\right).$$

and

$$\alpha_1 = \frac{6\ln\rho_{1,M}}{(M-1)(M-2)} - \frac{2\ln\rho_{M-1,M}}{(M-2)} - \frac{4\ln\rho_{1,2}}{(M-2)},$$
$$\alpha_2 = -\frac{6\ln\rho_{1,M}}{(M-1)(M-2)} + \frac{4\ln\rho_{M-1,M}}{(M-2)} + \frac{2\ln\rho_{1,2}}{(M-2)},$$

Inst Corrs as Inputs: Pivot matrices IV

leading to

$$\alpha_1 = 0.03923, \quad \alpha_2 = -0.03743, \quad \beta = 0.17897.$$

Consider also the pivot version of S&C2:

$$\rho_{i,j} = \exp\Bigg[- \frac{|i-j|}{M-1}\Bigg(-\ln \rho_\infty + \eta \frac{i^2 + j^2 + ij - 3Mi - 3Mj + 3i + 3j + 2M^2 - M - 4}{(M-2)(M-3)}\Bigg)\Bigg].$$

Use as pivot points $\rho_{1,M}$ and $\rho_{1,2}$. $\rho_{1,2}$ is selected for reasons that will be clear later on. We have

$$\rho_\infty = \rho_{1,M}, \quad \eta = \frac{(-\ln \rho_{1,2})(M-1) + \ln \rho_\infty}{2},$$

and obtain $\rho_\infty = 0.24545, \quad \eta = 1.04617.$

Inst Corrs as Inputs: Pivot matrices V

We compared the two three-parameters pivot forms with respect to the goodness of fit (to the historical matrix). S&C3 pivot is superior when we take as loss function the simple average squared difference (denoted by MSE), whilst Rebonato pivot is better if considering the average squared *relative* difference with respect to the estimated matrix (denoted by MSE%). This is shown in the following table.

	MSE	MSE%	$\sqrt{\text{MSE}}$	$\sqrt{\text{MSE\%}}$
Reb. 3 pivot	0.030121	0.09542	0.173554	0.30890
S&C3 pivot	0.024127	0.10277	0.155327	0.32058

Inst Corrs as Inputs: Pivot matrices I

Some reasons for considering Rebonato pivot form preferable in this context arise from the graphical observation of the behaviour of these matrices. As visible in the first figure below, showing the plot of the first columns, such matrix seems a better approximation of the estimated tendency, whereas S&C3 pivot tends to keep higher than the historical matrix. Moreover, in matching the estimated values selected, the parameter α_2 in S&C3 has turned out to be negative. This has led to a non-monotonic trend for sub-diagonals, see in fact the humped shape for the first sub-diagonal.

Inst Corrs as Inputs: Pivot matrices I

Inst Corrs as Inputs: Pivot matrices II

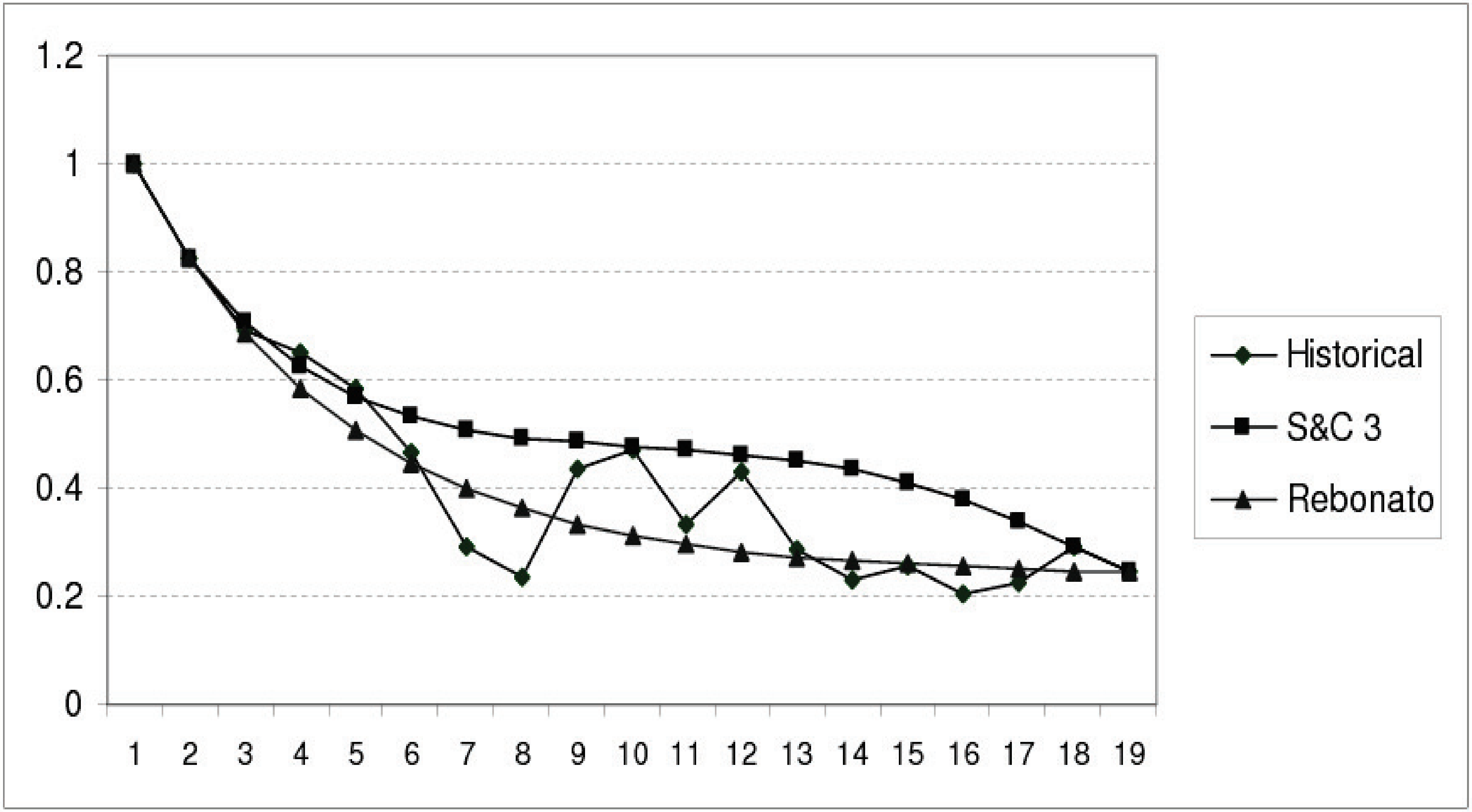


Figure: First columns of the historical and fitted “pivot” matrices

Inst Corrs as Inputs: Pivot matrices

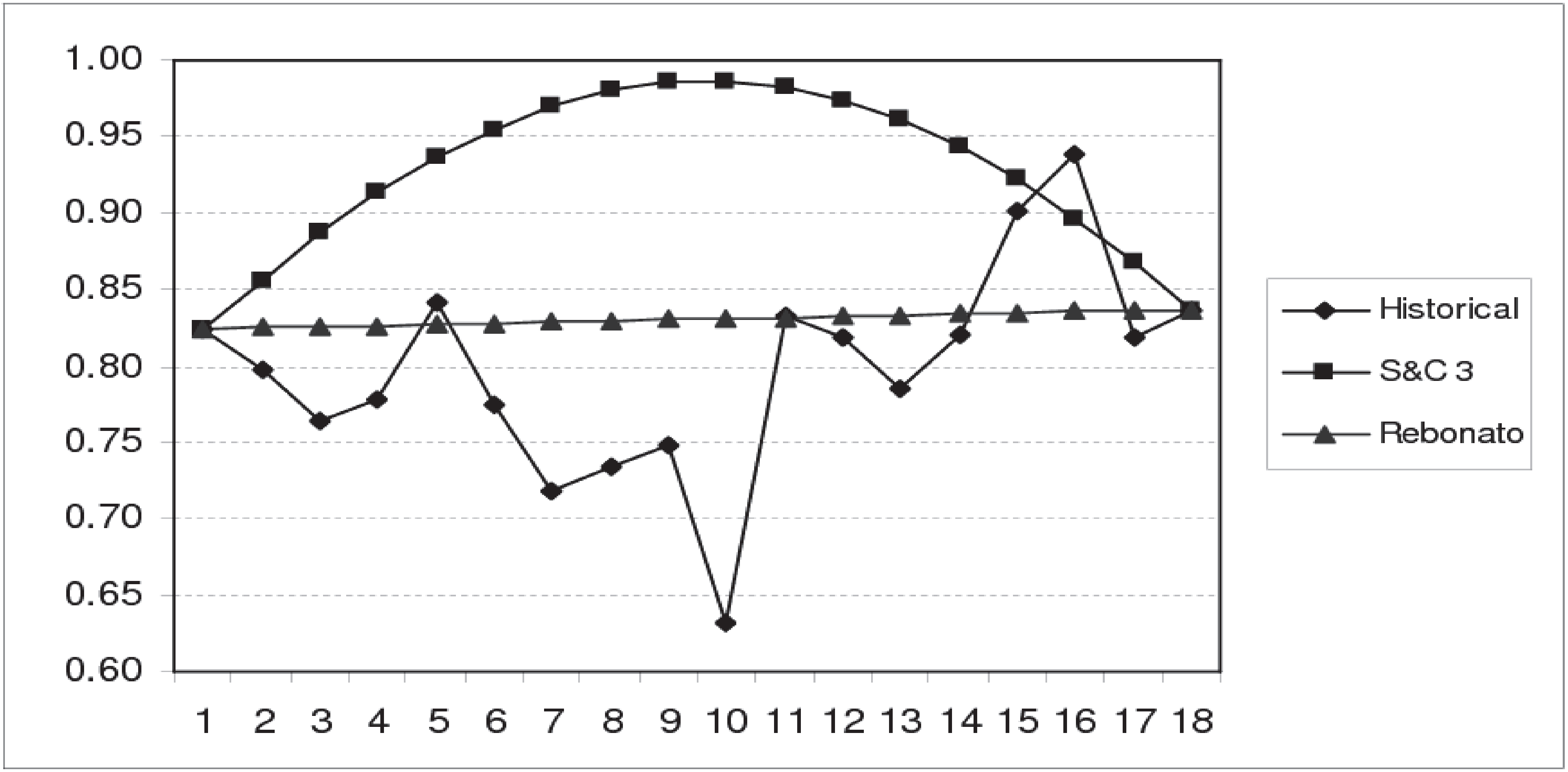


Figure: Corresponding sub-diagonals

Inst Corrs as Inputs: Pivot matrices I

A similar problem is hinted at also by Schoenmakers and Coffey (2000). In their constrained tests α_2 tends to assume always the minimum value allowed, namely zero. They propose the form S&C2. Our results with S&C2 pivot suggest that this very faint increasing tendency along sub-diagonals, joined with the level of decorrelation along the columns seen in the historical estimate, represent a configuration very hard to replicate with S&C parameterizations. Indeed, by building a pivot S&C2 keeping out information upon the sub-diagonal behaviour, one gets a matrix spontaneously featuring a *strong* increase along such sub-diagonals. On the other hand, including information on this estimated behaviour, a far larger decorrelation is implied than in the historically estimated matrix. More elements and details on such tests are given in Morini (2002).
No such problem has emerged for Rebonato's 3 parameters pivot form, that seems to allow for an easier separation of the tendency

Inst Corrs as Inputs: Pivot matrices II

along sub-diagonal from the one along the columns. Moreover, notice that Rebonato pivot form, with our data, turns out to be positive definite, so that its main theoretical limitation does not represent a problem in practice.
Our preferred choice is Rebonato-exponential 3-parameters.

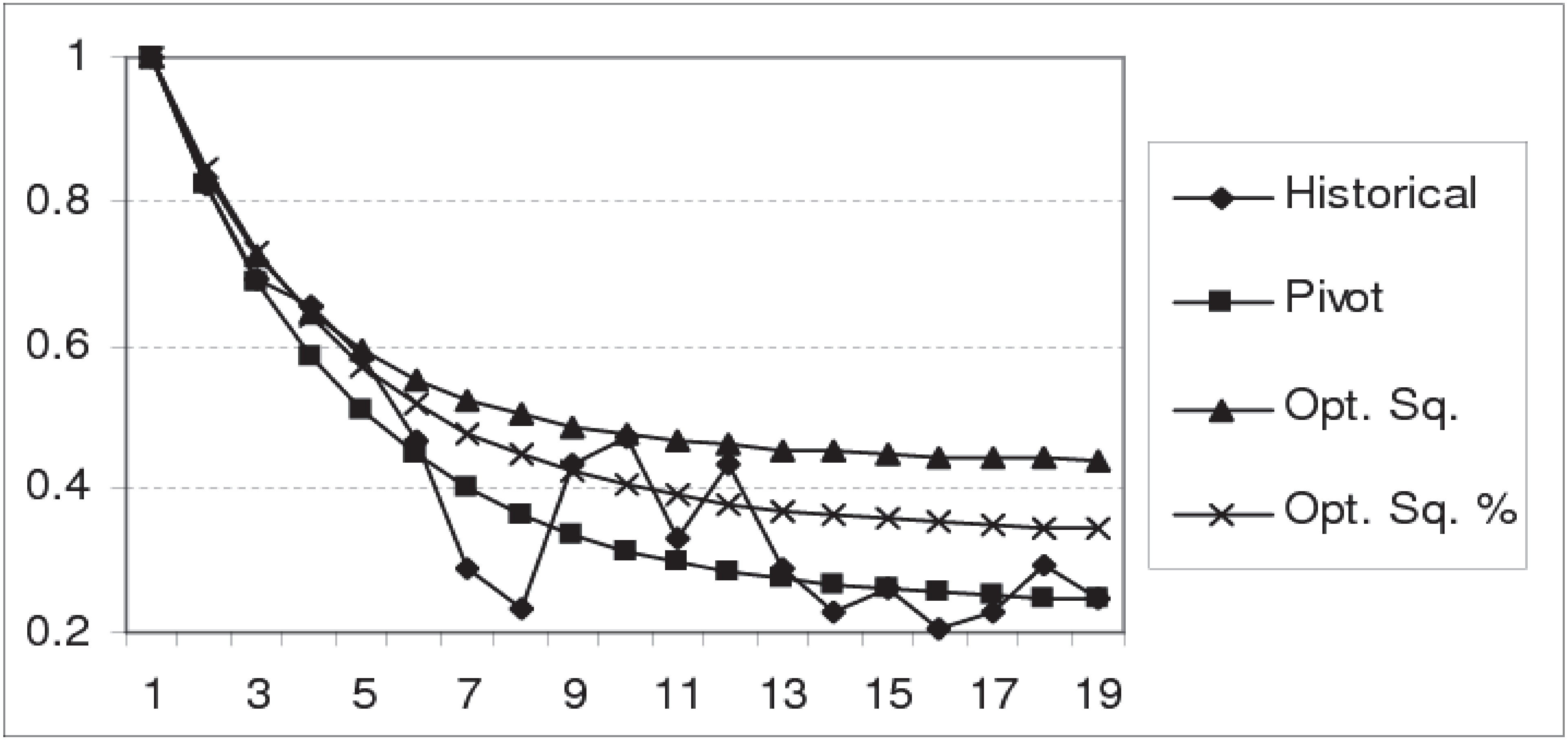
Inst Corrs as Inputs: Pivot matrices I

Now we have still to check the divergence between pivot matrices and matrices optimally fitted to the entire target matrix.
We compare the pivot version of Rebonato's parameterization with two optimal specifications of the same form obtained by minimizing the aforementioned loss functions. In the following table we present for each optimal form the square root of the corresponding error, besides the value obtained, for the same measure, when considering the pivot form.

	$\sqrt{\text{MSE}}$	$\sqrt{\text{MSE\%}}$
Fitted vs Historical	0.108434	0.25949
Pivot vs Historical	0.173554	0.30890

Differences are relatively small. First columns are plotted below.

Inst Corrs as Inputs: Pivot matrices I



First columns of correlation matrices

Inst Corrs as Inputs: Pivot matrices II

Conclusion: The pivot approach can be helpful when trying to describe the essential stylized feature of the historical correlation matrix. The related matrix, or a reduced rank version of it, can be considered as a reasonable exogenous correlation matrix to be used as input for calibration to (caps and) swaptions.

Inst Corrs as Outputs: Joint calibration to caps and swaptions I

We start with ρ as calibration outputs.
CALIBRATION: Need to find $\sigma(t)$ and ρ such that the market prices of caps and swaptions are recovered by LMM(σ, ρ).
caplet-volat-LMM(σ)= market-caplet-volat (Almost automatic).
swaptions-LMM(σ, ρ)= market-swaptions.
Caplets: Algebraic formula; Immediate calibration, almost automatic.
Swaptions: In principle Monte Carlo pricing. But MC pricing at each optimization step is too computationally intensive.
Use Rebonato's approximation and at each optimization step evaluate swaptions analytically with the LMM model.

ρ as outputs. Joint calibration: Market cases I

SPC vols, $\sigma_k(t) = \sigma_{k,\beta(t)} := \Phi_k \psi_{k-(\beta(t)-1)}$.
ρ rank-2 with angles $-\pi/2 < \theta_i - \theta_{i-1} < \pi/2$
Data below as of May 16, 2000, $F(0;0,1y) = 0.0469$, plus swaptions matrix as in the earlier slide.

ρ as outputs. Joint calibration: Market cases II

Index	initial $F0$	v_{caplet}
1	0.050114	0.180253
2	0.055973	0.191478
3	0.058387	0.186154
4	0.060027	0.177294
5	0.061315	0.167887
6	0.062779	0.158123
7	0.062747	0.152688
8	0.062926	0.148709
9	0.062286	0.144703
10	0.063009	0.141259
11	0.063554	0.137982
12	0.064257	0.134708
13	0.064784	0.131428
14	0.065312	0.128148
15	0.063976	0.127100
16	0.062997	0.126822
17	0.061840	0.126539
18	0.060682	0.126257
19	0.059360	0.125970

Index	ψ	Φ	θ
1	2.5114	0.0718	1.7864
2	1.5530	0.0917	2.0767
3	1.2238	0.1009	1.5122
4	1.0413	0.1055	1.6088
5	0.9597	0.1074	2.3713
6	1.1523	0.1052	1.6031
7	1.2030	0.1043	1.1241
8	0.9516	0.1055	1.8323
9	1.3539	0.1031	2.3955
10	1.1912	0.1021	2.5439
11	0	0.1046	1.6118
12	3.3778	0.0844	1.3172
13	0	0.0857	1.2225
14	1.2223	0.0847	1.0995
15	0	0.0869	1.2602
16	0	0.0896	1.0905
17	0	0.0921	0.8006
18	0.1156	0.0946	0.8739
19	0.5753	0.0965	1.7096

ρ as outputs. Joint calibration: Market cases (cont'd) I

Quality of calibration: Caplets are fitted exactly, whereas we calibrated the whole swaptions volatility matrix except for the first column.
Matrix: 100(Mkt swaptions vol - LMM swaption vol)/Mkt swaptions vol:

	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	-0.71	0.90	1.67	4.93	3.00	3.25	2.81	0.83	0.11
2y	-2.43	-3.48	-1.54	-0.70	0.70	0.01	-0.22	-0.45	0.49
3y	-3.84	1.28	-2.44	-0.69	-1.18	0.21	1.51	1.57	-0.01
4y	1.87	-2.52	-2.65	-3.34	-2.17	-0.44	-0.11	-0.63	-0.38
5y	1.80	4.15	-1.40	-1.89	-1.74	-0.79	-0.34	-0.07	1.28
7y	-0.33	2.27	1.47	-0.97	-0.77	-0.65	-0.57	-0.15	0.19
10y	-0.02	0.61	0.45	-0.31	0.02	-0.03	0.01	0.23	-0.30

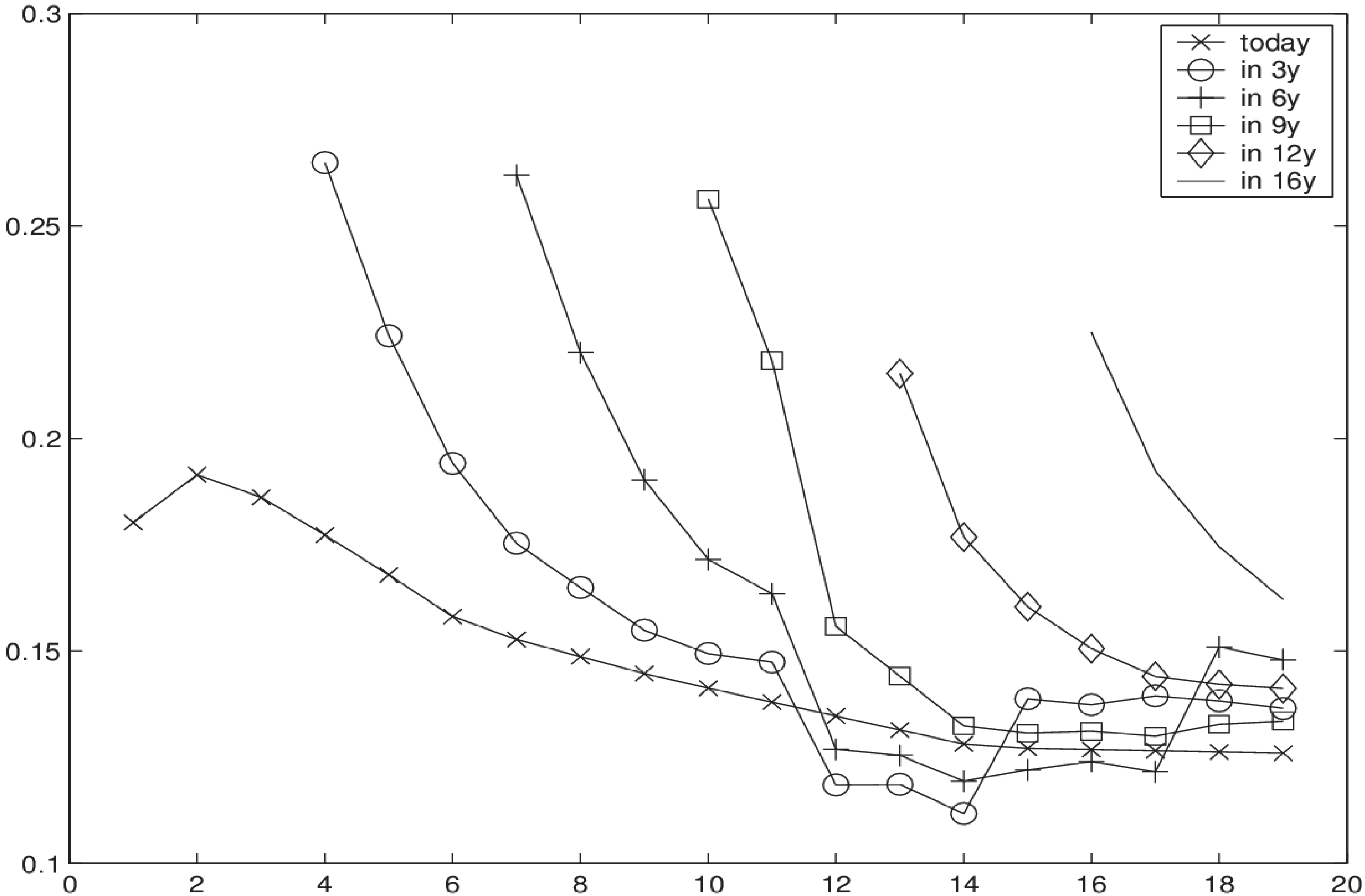
Calibr error OK for 19 caplets and 63 swaptions, but... calibrated θ's imply erratic, oscillating (+/-) ρ's and 10y terminal correlations:

ρ as outputs. Joint calibration: Market cases (cont'd) II

	10y	11y	12y	13y	14y	15y	16y	17y	18y	19
10y	1.00	0.56	0.27	0.19	0.09	0.21	0.08	-0.10	-0.06	0.37
11y	0.56	1.00	0.61	0.75	0.67	0.68	0.64	0.44	0.42	0.50
12y	0.27	0.61	1.00	0.42	0.71	0.53	0.48	0.43	0.40	0.42
13y	0.19	0.75	0.42	1.00	0.36	0.71	0.50	0.41	0.43	0.34
14y	0.09	0.67	0.71	0.36	1.00	0.32	0.67	0.43	0.40	0.36
15y	0.21	0.68	0.53	0.71	0.32	1.00	0.28	0.59	0.39	0.33
16y	0.08	0.64	0.48	0.50	0.67	0.28	1.00	0.22	0.62	0.30
17y	-0.10	0.44	0.43	0.41	0.43	0.59	0.22	1.00	0.17	0.36
18y	-0.06	0.42	0.40	0.43	0.40	0.39	0.62	0.17	1.00	0.07
19y	0.37	0.50	0.42	0.34	0.36	0.33	0.30	0.36	0.07	1.00

Joint calibration: Market cases (cont'd) I

Joint calibration: Market cases (cont'd) II



Joint calibration: Market cases (cont'd) I

Tried other calibrations with SPC σ's

Tried: More stringent constraints on the θ

Fixed θ both to typical and atypical values, leaving the calibration only to the vol parameters

Fixed θ so as to have all $\rho = 1$.

Summary: To have good calibration to swaptions need to keep the angles unconstrained and allow for partly oscillating ρ's.

If we force "smooth/monotonic" ρ's and leave calibr to vols, results are essentially the same as in the case of a one-factor LMM with $\rho = 1$.

Maybe inst correlations do not have a strong link with European swaptions prices? (Rebonato)

Maybe permanence of "bad results", no matter the particular "smooth" choice of fixed ρ, reflects an impossibility of a low-rank ρ to decorrelate quickly fwd rates in a steep initial pattern? (Rebonato)

Joint calibration: Market cases (cont'd) II

3-4 factor ρ's does not seem to help. Increase drastically $\#$ factors? But MC... More on this later.

Joint calibration: Market cases (cont'd) I

Calibration with the LE parametric σ's.

Same inputs as before

Rank-2 ρ with $-\pi/3 < \theta_i - \theta_{i-1} < \pi/3, \quad 0 < \theta_i < \pi$

Constraint "$1 - 0.1 \leq \Phi_i(a, b, c, d) \leq 1 + 0.1$"

Calibrated parameters and calibration error (caps exact):

$$a = 0.29342753, \quad b = 1.25080230, \quad c = 0.13145869, \quad d = 0.00,$$

$$\theta_{1-7} = [1.75411\ 0.57781\ 1.68501\ 0.58176\ 1.53824\ 2.43632\ 0.88011],$$

$$\theta_{8-12} = [1.89645\ 0.48605\ 1.28020\ 2.44031\ 0.94480],$$

$$\theta_{13-19} = [1.34053\ 2.91133\ 1.99622\ 0.70042\ 0\ 0.81518\ 2.38376].$$

Calibration error:

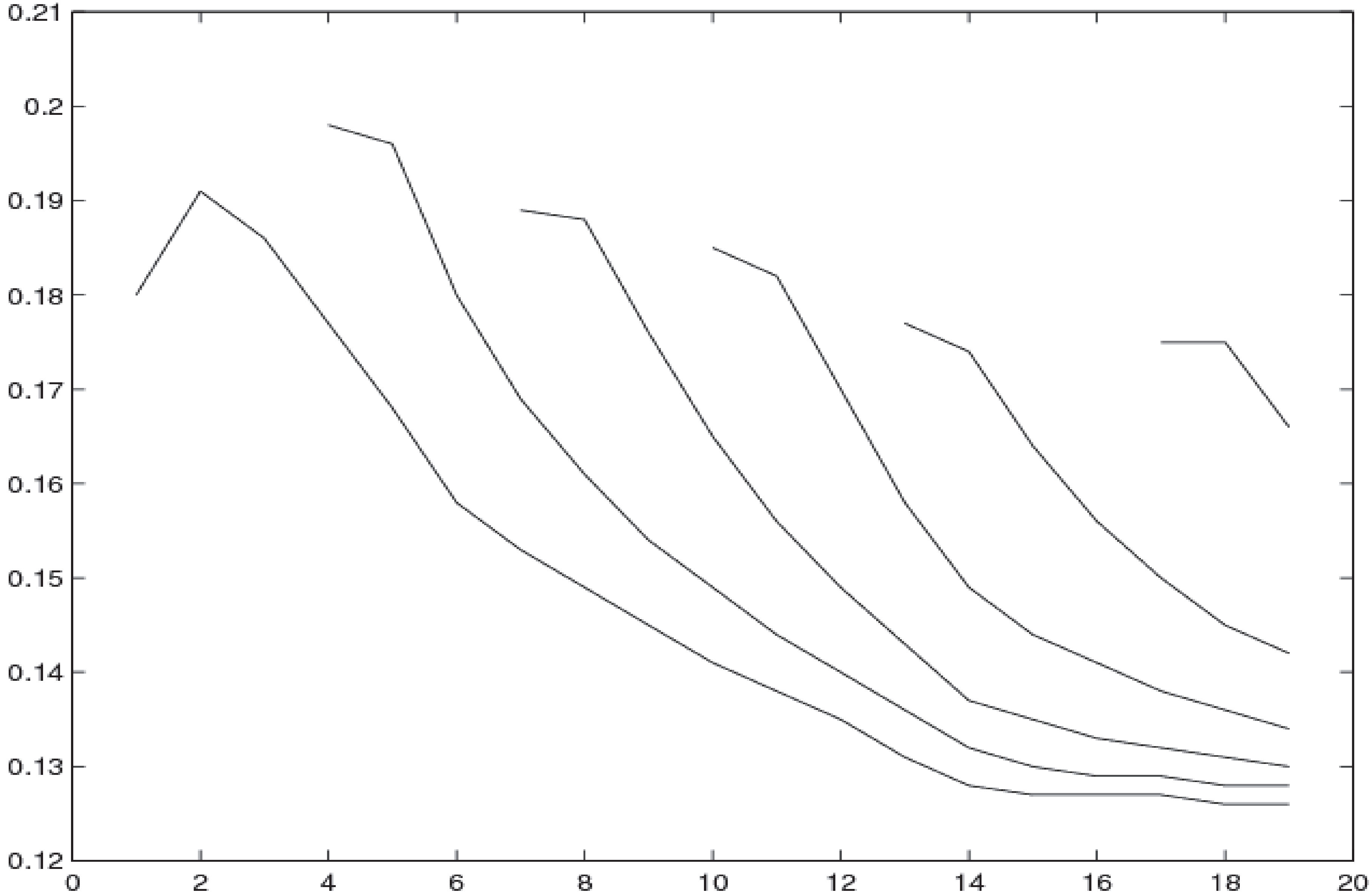
	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	2.28	-3.74	-3.19	-4.68	2.46	1.50	0.72	1.33	-1.42
2y	-1.23	-7.67	-9.97	2.10	0.49	1.33	1.56	-0.44	1.88
3y	2.23	-6.20	-1.30	-1.32	-1.43	1.86	-0.19	2.42	1.17
4y	-2.59	9.02	1.70	0.79	3.22	1.19	4.85	3.75	1.21
5y	-3.26	-0.28	-8.16	-0.81	-3.56	-0.23	-0.08	-2.63	2.62
7y	0.10	-2.59	-10.85	-2.00	-3.67	-6.84	2.15	1.19	0.00
10y	0.29	-3.44	-11.83	-1.31	-4.69	-2.60	4.07	1.11	0.00

Joint calibration: Market cases (cont'd) II

Inst correlations are again oscillating and non-monotonic. Terminal correlations share part of this negative behaviour.

Joint calibration: Market cases (cont'd) I

Joint calibration: Market cases (cont'd) II



Joint calibration: Market cases (cont'd) I

Evolution of term structure of vols looks better

Many more experiments with rank-three correlations, less or more stringent constraints on the angles and on the Φ's.
Fitting to the whole swaption matrix can be improved, but at the cost of an erratic behaviour of both correlations and of the evolution of the term structure of volatilities in time.
3-factor choice does not seem to help that much, as before.
LE σ's allow for an easier control of the evolution of the term structure of vols, but produce more erratic ρ's: most of the "noise" in the swaption data ends up in the angles (we have only 4 vol parameters a, b, c, d for fitting swaptions)

Cascade Calibration with GPC vols I

Cascade calibration is a very fast and accurate calibration procedure, that can be implemented with easy tools such as spreadsheets. However, one needs to be careful to avoid numerical instability and to obtain a robust procedure.

Cascade Calibration with GPC vols II

ρ's as **inputs** to the calibration (e.g. historical estimation)

$$(v_{\alpha,\beta}^{\text{LMM}})^2 \approx \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\, dt,$$

$$T_\alpha \; S_{\alpha,\beta}(0)^2 \; v_{\alpha,\beta}^2 =$$

$$= \sum_{i,j=\alpha+1}^{\beta-1} w_i w_j F_i F_j \rho_{i,j} \sum_{h=0}^{\alpha} (T_h - T_{h-1})\, \sigma_{i,h+1}\, \sigma_{j,h+1}$$

$$+ \; 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta w_j F_\beta F_j \rho_{\beta,j} \sum_{h=0}^{\alpha-1} (T_h - T_{h-1})\, \sigma_{\beta,h+1}\, \sigma_{j,h+1}$$

$$+ \; 2 \sum_{j=\alpha+1}^{\beta-1} w_\beta w_j F_\beta F_j \rho_{\beta,j}\, (T_\alpha - T_{\alpha-1}) \boxed{\sigma_{\beta,\alpha+1}}\, \sigma_{j,\alpha+1}$$

$$+ \; w_\beta^2 F_\beta^2 \sum_{h=0}^{\alpha-1} (T_h - T_{h-1})\, \sigma_{\beta,h+1}^2$$

$$+ \; w_\beta^2 F_\beta^2 (T_\alpha - T_{\alpha-1}) \boxed{\sigma_{\beta,\alpha+1}^2}\, .$$

Cascade Calibr with general PC vols: I

One to one corresp with swaption vols (cont'd)

Length / Maturity	1y	2y	3y
$T_0 = 1y$	$v_{0,1}$ $\sigma_{1,1}$	$v_{0,2}$ $\sigma_{1,1}, \sigma_{2,1}$	$v_{0,3}$ $\sigma_{1,1}, \sigma_{2,1}, \sigma_{3,1}$
$T_1 = 2y$	$v_{1,2}$ $\sigma_{2,1}, \sigma_{2,2}$	$v_{1,3}$ $\sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \sigma_{3,2}$	- -
$T_2 = 3y$	$v_{2,3}$ $\sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}$	-	-

Problem: can obtain negative or imaginary σ's.
Possible cause: Illiquidity/stale data on the v's.
Possible remedy: Smooth the input swaption v's matrix with a 17-dimensional parametric form and recalibrate: imaginary and negative vols σ disappear.
Term structure of caplet vols evolves regularly but loses hump

Cascade Calibr with general PC vols: II

One to one corresp with swaption vols (cont'd)

Instantaneous correlations good because chosen exogenously
Terminal correlations positive and monotonically decreasing
This form can help in Vega breakdown analysis (helpful for hedging)

Exact Swaption Cascade Calibration with GPC: Numerical example I

Calibrate σ's to the following swaptions matrix (2000)

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	0.180	0.167	0.154	0.145	0.138	0.134	0.130	0.126	0.124	0.122
2y	0.181	0.162	0.145	0.135	0.127	0.123	0.120	0.117	0.115	0.113
3y	0.178	0.155	0.137	0.125	0.117	0.114	0.111	0.108	0.106	0.104
4y	0.167	0.143	0.126	0.115	0.108	0.105	0.103	0.100	0.098	0.096
5y	0.154	0.132	0.118	0.109	0.104	0.104	0.099	0.096	0.094	0.092
6y	0.147	0.127	0.113	0.104	0.098	0.098	0.094	0.092	0.090	0.089
7y	0.140	0.121	0.107	0.098	0.092	0.091	0.089	0.087	0.086	0.085
8y	0.137	0.117	0.103	0.095	0.089	0.088	0.086	0.084	0.083	0.082
9y	0.133	0.114	0.100	0.091	0.086	0.085	0.083	0.082	0.081	0.080
10y	0.130	0.110	0.096	0.088	0.083	0.082	0.080	0.079	0.078	0.077

added vols for 6y,8y and 9y maturities by linear interpolation. assume nice decreasing positive rank 2 corr given *exogenously*, $\rho_{i,j} = \cos(\theta_i - \theta_j)$, corresponding to the angles

$$\theta_{1-9} = [\,0.0147\ 0.0643\ 0.1032\ 0.1502\ 0.1969\ 0.2239\ 0.2771\ 0.2950\ 0.3630\,],$$
$$\theta_{10-19} = [\,0.3810\ 0.4217\ 0.4836\ 0.5204\ 0.5418\ 0.5791\ 0.6496\ 0.6679\ 0.7126\ 0.7659\,].$$

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd) I

0.1800	-	-	-	-	-	-	-	-	-
0.1548	0.2039	-	-	-	-	-	-	-	-
0.1285	0.1559	0.2329	-	-	-	-	-	-	-
0.1178	0.1042	0.1656	0.2437	-	-	-	-	-	-
0.1091	0.0988	0.0973	0.1606	0.2483	-	-	-	-	-
0.1131	0.0734	0.0781	0.1009	0.1618	0.2627	-	-	-	-
0.1040	0.0984	0.0502	0.0737	0.1128	0.1633	0.2633	-	-	-
0.0940	0.1052	0.0938	0.0319	0.0864	0.0969	0.1684	0.2731	-	-
0.1065	0.0790	0.0857	0.0822	0.0684	0.0536	0.0921	0.1763	0.2848	-
0.1013	0.0916	0.0579	0.1030	0.1514	- 0.0316	0.0389	0.0845	0.1634	0.2777
0.0916	0.0916	0.0787	0.0431	0.0299	0.2088	- 0.0383	0.0746	0.0948	0.1854
0.0827	0.0827	0.0827	0.0709	0.0488	0.0624	0.1561	- 0.0103	0.0731	0.0911
0.0744	0.0744	0.0744	0.0744	0.0801	0.0576	0.0941	0.1231	- 0.0159	0.0610
0.0704	0.0704	0.0704	0.0704	0.0704	0.1009	0.0507	0.0817	0.1203	- 0.0210
0.0725	0.0725	0.0725	0.0725	0.0725	0.0725	0.1002	0.0432	0.0619	0.1179
0.0753	0.0753	0.0753	0.0753	0.0753	0.0753	0.0753	0.0736	0.0551	0.0329
0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0719	0.0708	0.0702
0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0690	0.0680
0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663	0.0663

Calibration shows negative signs in σ's. "Temporal misalignments" caused by illiquidity in the swaption matrix? In some cases one can

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd) II

also have complex volatilities. To avoid this, smooth the market swaption matrix by fitting

$$\mathrm{vol}(S,T) = \gamma(S) + \left(\frac{\exp(f\cdot\ln(T))}{e\cdot S} + D(S)\right)\cdot\exp(-\beta\cdot\exp(p\cdot\ln(T))),$$

where (S is the maturity, T the tenor)

$$\begin{aligned}\gamma(S) &= c + (\exp(h\cdot\ln(S))\cdot a + d)\cdot\exp(-b\cdot\exp(m\cdot\ln(S))),\\ D(S) &= (\exp(g\cdot\ln(S))\cdot q + r)\cdot\exp(-s\cdot\exp(t\cdot\ln(S))) + \delta,\end{aligned}$$

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd) I

a	b	c	d	e	f	del	bet
0.000359	1.432288	2.5269	-1.93552	5.751286	0.065589	0.02871	-5.41842
g	h	m	p	q	r	s	t
-0.02129	17.64259	2.043768	-0.06907	-0.09817	-0.87881	2.017844	0.600784

Difference between the market and the smoothed matrices:

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	-0.46	0.49	0.33	0.16	-0.01	0.01	-0.06	-0.18	-0.14	-0.14
2y	-0.39	0.53	0.18	0.03	-0.17	-0.11	-0.05	-0.05	0.01	0.03
3y	0.03	0.64	0.22	-0.13	-0.32	-0.16	-0.10	-0.10	-0.05	-0.03
4y	0.01	0.43	0.05	-0.23	-0.35	-0.21	-0.06	-0.08	-0.04	-0.03
5y	-0.36	0.12	-0.02	-0.15	-0.10	0.31	0.14	0.11	0.14	0.13
6y	-0.31	0.19	-0.02	-0.18	-0.21	0.13	0.09	0.10	0.16	0.20
7y	-0.27	0.25	-0.01	-0.21	-0.32	-0.05	0.05	0.09	0.19	0.27
8y	-0.13	0.27	-0.04	-0.22	-0.32	-0.06	0.02	0.09	0.18	0.25
9y	0.00	0.30	-0.05	-0.24	-0.32	-0.07	0.00	0.10	0.18	0.25
10y	0.15	0.32	-0.07	-0.25	-0.31	-0.08	-0.02	0.09	0.17	0.23

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd) I

σ's obtained calibrating the smoothed swaption matrix:

18.46	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
14.09	22.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12.84	13.11	24.71	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12.14	11.17	13.00	25.94	0.00	0.00	0.00	0.00	0.00	0.00
11.64	10.11	10.59	12.54	27.10	0.00	0.00	0.00	0.00	0.00
11.19	9.51	9.44	9.87	12.73	28.06	0.00	0.00	0.00	0.00
10.94	8.88	8.47	8.53	9.82	13.01	28.58	0.00	0.00	0.00
10.59	8.61	7.82	7.57	8.58	10.06	12.92	29.62	0.00	0.00
10.37	8.25	7.53	6.81	7.52	8.61	9.74	13.51	30.20	0.00
10.26	7.73	7.21	6.43	7.14	7.65	8.31	10.45	13.56	30.35
8.89	8.89	7.08	6.31	6.39	7.23	7.38	8.73	10.40	13.41
8.07	8.07	8.07	6.23	6.30	6.82	6.79	7.96	8.63	10.10
7.35	7.35	7.35	7.35	6.27	6.43	6.29	7.38	7.96	8.44
7.01	7.01	7.01	7.01	7.01	6.39	5.85	6.89	6.70	7.46
6.53	6.53	6.53	6.53	6.53	6.53	6.29	5.96	6.92	6.68
6.23	6.23	6.23	6.23	6.23	6.23	6.23	6.97	5.58	6.57
6.06	6.06	6.06	6.06	6.06	6.06	6.06	6.06	6.57	5.77
5.76	5.76	5.76	5.76	5.76	5.76	5.76	5.76	5.76	6.35
5.62	5.62	5.62	5.62	5.62	5.62	5.62	5.62	5.62	5.62

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd) II

irregularity and illiquidity in the input swaption matrix can cause negative or even imaginary values in the calibrated σ's. However, by smoothing the input data before calibration, usually this undesirable features can be avoided.

Exact Swaption calibration with GPC

Numerical example (cont'd)

The smoothing procedure also improves terminal correlations. Ten-years terminal correlations for the non-smoothed case:

	10y	11y	12y	13y	14y	15y	16y	17y	18y	19y
10y	1.000	0.677	0.695	0.640	0.544	0.817	0.666	0.762	0.753	0.740
11y	0.677	1.000	0.614	0.617	0.665	0.768	0.696	0.760	0.752	0.740
12y	0.695	0.614	1.000	0.758	0.716	0.938	0.848	0.870	0.862	0.850
13y	0.640	0.617	0.758	1.000	0.740	0.866	0.914	0.894	0.885	0.875
14y	0.544	0.665	0.716	0.740	1.000	0.771	0.919	0.885	0.879	0.868
15y	0.817	0.768	0.938	0.866	0.771	1.000	0.923	0.965	0.960	0.953
16y	0.666	0.696	0.848	0.914	0.919	0.923	1.000	0.983	0.980	0.975
17y	0.762	0.760	0.870	0.894	0.885	0.965	0.983	1.000	0.999	0.995
18y	0.753	0.752	0.862	0.885	0.879	0.960	0.980	0.999	1.000	0.999
19y	0.740	0.740	0.850	0.875	0.868	0.953	0.975	0.995	0.999	1.000

Exact Swaption calibration with GPC I

Numerical example (cont'd)

Compare with the corresponding matrix from smoothed data

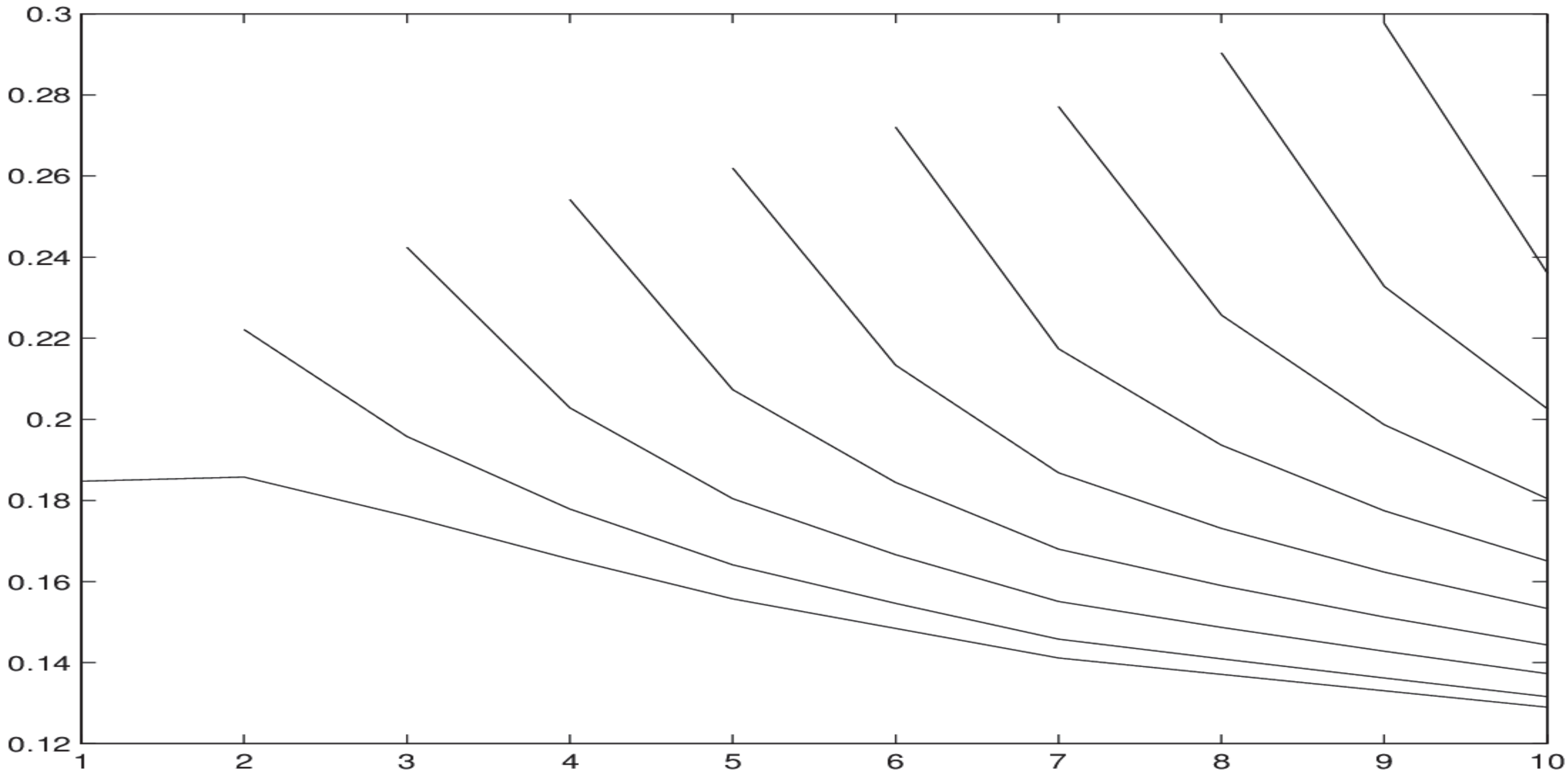
	10y	11y	12y	13y	14y	15y	16y	17y	18y	19y
10y	1.000	0.939	0.898	0.872	0.851	0.838	0.823	0.809	0.817	0.787
11y	0.939	1.000	0.992	0.980	0.969	0.962	0.947	0.941	0.936	0.915
12y	0.898	0.992	1.000	0.996	0.990	0.986	0.975	0.972	0.966	0.950
13y	0.872	0.980	0.996	1.000	0.997	0.995	0.986	0.984	0.979	0.966
14y	0.851	0.969	0.990	0.997	1.000	0.997	0.992	0.989	0.984	0.973
15y	0.838	0.962	0.986	0.995	0.997	1.000	0.994	0.995	0.990	0.982
16y	0.823	0.947	0.975	0.986	0.992	0.994	1.000	0.997	0.997	0.992
17y	0.809	0.941	0.972	0.984	0.989	0.995	0.997	1.000	0.998	0.995
18y	0.817	0.936	0.966	0.979	0.984	0.990	0.997	0.998	1.000	0.998
19y	0.787	0.915	0.950	0.966	0.973	0.982	0.992	0.995	0.998	1.000

Exact Swaption Cascade Calibration with GPC: Numerical example (cont'd)

non-smoothed case is worse: terminal correlations deviate more from monotonicity, roughly corresponding to the portion of instantaneous volatilities that go negative in the calibration. The non-smoothed case shows also a slightly erratic evolution of the term structure of volatilities compared to the smoothed case.

Exact Swaption Cascade Calibration with GPC:

Numerical example (cont'd)



Calibration: A pause for thought and a First summary I

Some desired calibration features:

- A small rank for ρ in view of Monte Carlo
- A small calibration error;
- Positive and decreasing inst. and term. correlations;
- Smooth and stable evolution of the term structure of vols;

Can achieve these targets through a low $\#$ of factors?
Try and combine many of the ideas presented here
The one-to-one formulation is perhaps the most promising: Fitting to swaptions is exact; can fit caps by introducing infra-correlations; instantaneous correlation OK by construction; Terminal correlation not spoiled by the fitted σ's; Terms structure evolution smooth but not fully satisfactory qualitatively.

Calibration: A pause for thought and a First summary II

Requirements hardly checkable with general HJM or short-rate models

More mathematically-advanced issues: **Smile calibration**.

Cascade calibration: further developments I

The examples and considerations given here are based on more recent market data and have appeared earlier Morini (2002) and in Brigo and Morini (2002).

New Data: input swaption matrix, 1 feb 02.

	1	2	3	4	5	6	7	8	9	1
1	**17.90**	**16.50**	**15.30**	**14.40**	**13.70**	**13.20**	**12.80**	**12.50**	**12.30**	**12.0**
2	**15.40**	**14.20**	**13.60**	**13.00**	**12.60**	**12.20**	**12.00**	**11.70**	**11.50**	11.3
3	**14.30**	**13.30**	**12.70**	**12.20**	**11.90**	**11.70**	**11.50**	**11.30**	11.10	10.9
4	**13.60**	**12.70**	**12.10**	**11.70**	**11.40**	**11.30**	**11.10**	10.90	10.80	10.7
5	**12.90**	**12.10**	**11.70**	**11.30**	**11.10**	**10.90**	10.80	10.60	10.50	10.4
6	**12.50**	**11.80**	**11.40**	**10.95**	**10.75**	*10.60*	*10.50*	*10.40*	*10.35*	*10.2*
7	**12.10**	**11.50**	**11.10**	**10.60**	10.40	10.30	10.20	10.20	10.20	10.1
8	**11.80**	**11.20**	**10.83**	*10.40*	*10.23*	*10.17*	*10.10*	*10.10*	*10.07*	*10.0*
9	**11.50**	**10.90**	*10.57*	*10.20*	*10.07*	*10.03*	*10.00*	*10.00*	*9.93*	*9.9*
10	**11.20**	10.60	10.30	10.00	9.90	9.90	9.90	9.90	9.80	9.8

Cascade calibration: further developments II

The annualized forward LIBOR rates from the corresponding zero curve on the same date are

$F(0;0,1)$: 1	0.036712	11	0.058399
$F(0;1,2)$: 2	0.04632	12	0.058458
...: 3	0.050171	13	0.058569
4	0.05222	14	0.058339
5	0.054595	15	0.057951
6	0.056231	16	0.057833
7	0.057006	17	0.057555
8	0.057699	18	0.057297
9	0.05691	19	0.056872
10	0.057746	20	0.056738

Cascade Calibration of Rectangular swaption matrices I

The rows associated with the swaptions maturities of $6, 8$ and 9 years do not refer to market quotations. Considering that the Cascade Calibration Algorithm (CCA) requires a complete swaption matrix, featuring values for each and every maturity (and length) in the range, they have been obtained as before by a simple linear interpolation between the adjacent market values on the same columns, see also Rebonato and Joshi (2001). We discuss the interpolation effects later.

An important point about the basic CCA given earlier is that *results are, in a sense, independent of the matrix size*, in that the output of the calibration to a sub-matrix will be a subset of the output of a calibration to the original matrix.

This implies also that any swaption matrix V can be seen in principle as a sub-matrix of a larger one, say $\bar{V}$, including V itself in its upper

Cascade Calibration of Rectangular swaption matrices II

triangular part, so that all entries of V, including those in its lower triangular part, will be recovered by applying the basic CCA algorithm to the upper part of the larger matrix $\bar{V}$. In other words, this "nested consistency" means that, if all needed market values were available, so that we could *always* embed our given market V in a sufficiently large market $\bar{V}$, the basic "upper part" CCA seen earlier might be considered to be general, with no need for any extension.

Of course this is not usually the case, in that in general there is no larger $\bar{V}$ to be exploited.

If we apply the basic CCA extending it to the elements in the lower triangular part, namely we keep on moving from left to right and top down but now visiting all the boxes in the matrix, in certain positions of the table we will have more than one unknown in the relevant inversion formula.

Cascade Calibration of Rectangular swaption matrices III

However, we can still manage by assuming these unknowns to be equal to each other, as we tacitly did earlier.
Let us sum up the CCA main advantages and typical problems.

1. *The correlation matrix is an exogenous input*;
2. The remaining inputs are *a complete swaption volatilities matrix and the zero coupon curve, so cap data are not involved in the calibration*;
3. *The calibration can be carried out through closed form formulas*;
4. If the industry formula is used for pricing swaptions in combination with Black's formula, *market swaption prices are recovered exactly*;

Cascade Calibration of Rectangular swaption matrices IV

5. The method establishes *a one-to-one correspondence between model volatility parameters and market swaption volatilities*, at least in its basic form.

The last three points clearly represent the main advantages. The first point allows for imposing satisfactory instantaneous correlations. Avoiding any optimization routine, CCA does not allow one to set any constraints on the output, so that there is no guarantee that the calibrated instantaneous volatilities will be real and non-negative. On the contrary, we have seen some cases in where we obtain negative entries in the output. We have solved this problem earlier by a rather drastic and too rough smoothing of the input swaption matrix. Here we try and find different, less drastic ways to get rid of such inconveniences.

Cascade calibration: Further numerical studies

New input data of 1 feb 02, seen earlier. At first we will consider the results of calibration to only the upper (bold-faced) part of the swaption matrix.

The first exogenous correlation matrix we apply is Rebonato 3 parameters pivot, possibly rank-reduced. start with rank 7. The calibrated σ volatilities are

Cascade calibration: Further numerical studies

0.179									
0.153	0.155								
0.144	0.129	0.154							
0.144	0.134	0.105	0.156						
0.140	0.122	0.112	0.112	0.154					
0.143	0.134	0.103	0.101	0.106	0.153				
0.143	0.127	0.143	0.088	0.097	0.086	0.144			
0.146	0.153	0.128	0.078	0.070	0.098	0.093	0.145		
0.157	0.109	0.155	0.160	0.067	0.007	0.101	0.081	0.107	
0.136	0.152	0.126	0.123	0.121	0.108	-0.040	0.120	0.077	0.067

Cascade calibration: Further numerical studies I

So there is a negative volatility, $\sigma_{10,7}$. What can we do to avoid this problem? Let us start by changing the rank of the correlation matrix. A calibration with full rank, equal to 19, gives us not only the same negative volatility, but also a complex one, $\sigma_{10,10}$.
Let us then try and reduce the rank. Down to rank 5 we get the same negative volatility, though reduced in absolute value. At rank 4 the negative entry disappears, and the output is completely acceptable, as visible in the following table.

Cascade calibration: Further numerical studies

0.179									
0.152	0.156								
0.131	0.130	0.165							
0.123	0.132	0.120	0.164						
0.128	0.123	0.120	0.118	0.153					
0.141	0.128	0.098	0.101	0.108	0.162				
0.144	0.115	0.122	0.082	0.102	0.106	0.159			
0.147	0.137	0.106	0.065	0.071	0.110	0.114	0.159		
0.156	0.098	0.136	0.131	0.054	0.031	0.119	0.111	0.139	
0.134	0.147	0.117	0.106	0.095	0.086	0.007	0.138	0.102	0.122

Cascade calibration: Further numerical studies I

The same happens for rank 3 and 2. What might cause a similar behaviour? Recall that lowering the rank of a correlation matrix amounts to impose an oscillating tendency to the columns, that for very low ranks is represented by a sigmoid-like shape.
Some features of the lower rank correlations seem to be better suited to these swaptions data. In particular, we might elicit that correlation matrices characterized by less steep initial decorrelation allow for acceptable results.
More evidence? Further tests with synthetic correlation matrices, whose essential features can be easily modified and controlled. Let us see how the calibrated volatilities change with ρ_∞ and β in $\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\exp[-\beta|i - j|], \;\; \beta \geq 0$. The parameters are modified for the exogenous ρ at each calibration (same swaptions inputs) as follows:

Cascade calibration: Further numerical studies II

a) $\rho_\infty = 0.5$, $\beta = 0.05$;
b) Reduce ρ_∞ to 0;
c) Set β to 0.2;
d) Set ρ_∞ up to 0.5;
e) Set β to 0.4;
f) Take $\beta = 0.2$ and $\rho_\infty = 0.4$;
g) $\rho_\infty = 0$, $\beta = 0.1$.

Cascade calibration: Further numerical studies

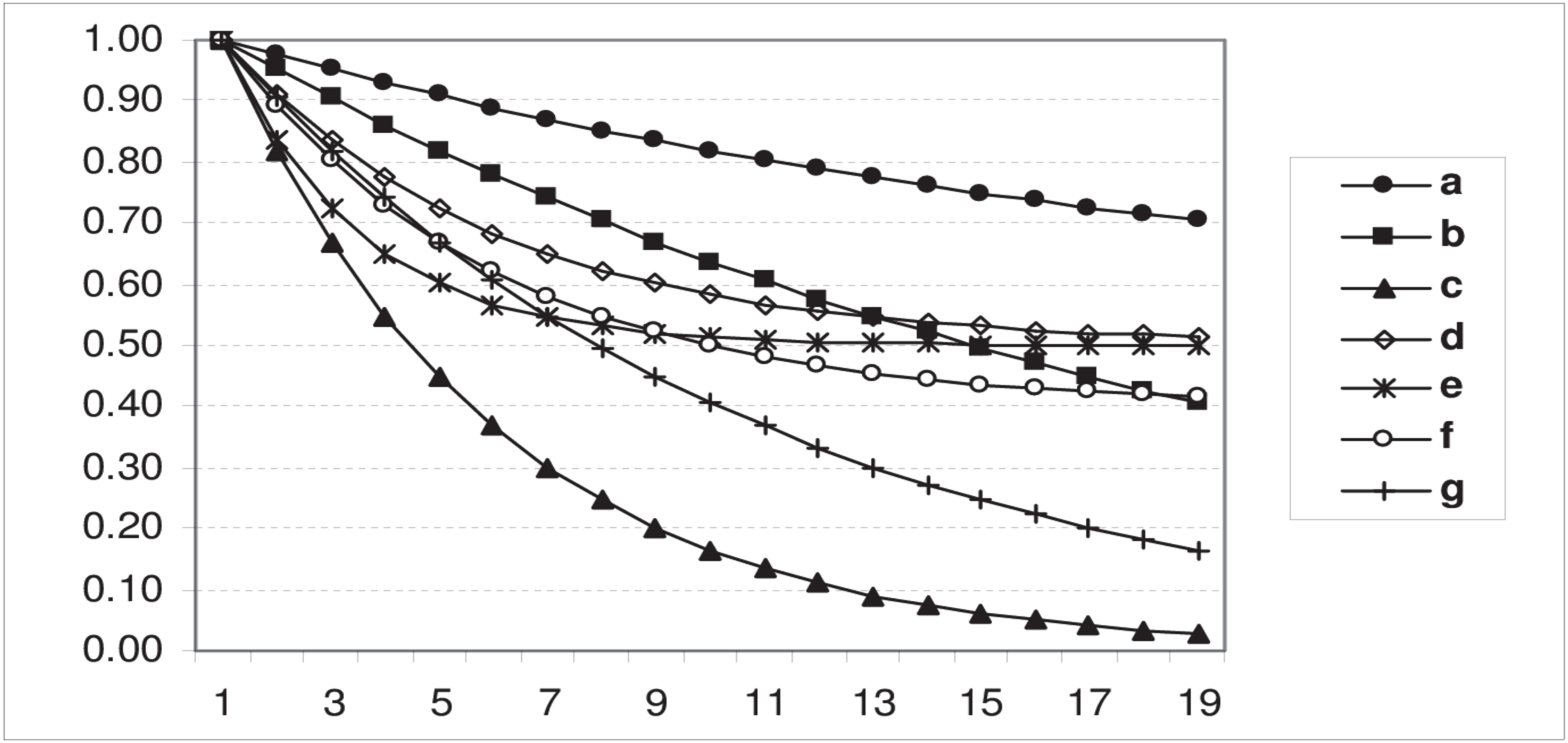


Figure: First columns of classic exponential structure

Cascade calibration: Further numerical studies I

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\exp[-\beta|i-j|], \quad \beta \geq 0.$$

a) $\rho_\infty = 0.5$, $\beta = 0.05$;
b) Reduce ρ_∞ to 0;
c) Set β to 0.2;
d) Set ρ_∞ up to 0.5;
e) Set β to $0,4$;
f) Take $\beta = 0,2$ and $\rho_\infty = 0.4$;
g) $\rho_\infty = 0$, $\beta = 0,1$.

We start with the matrix whose first column is represented by **a**, obtained by setting $\rho_\infty = 0.5$ and $\beta = 0.05$. With such a correlation, at full rank we obtain volatilities all real and positive, even calibrating to the entire swaption matrix. Then we lower the rank, first to 15 and then to 5, a level we keep in the following because representing the first

Cascade calibration: Further numerical studies II

problematic level when increasing the rank of Rebonato three-parameters form. We find acceptable results.
Then we move ρ_∞ and β, producing all the configurations shown, different in terms of extent of the decorrelation, initial steepness, and final level reached by correlation. With the correlations corresponding to **b**, **d** and **g**, we avoid negative or complex vols, whereas **c**, **e** and **f** give again a negative $\sigma_{10,7}$. We find bad results for those correlations featuring columns initially steeper, while the four configurations characterized by less initial steepness result in real and positive volatilities.
Now, let us see if S&C2 pivot can avoid problems for Rebonato 3 parameters correlations. S&C2 pivot is characterized by a more pronounced increase along sub-diagonals and less steep initial decorrelation. This correlation gives us volatilities all real and positive, at full 19 rank and when reducing the rank by optimizing a lower rank

Cascade calibration: Further numerical studies III

angles form onto the S&C2 pivot form. In particular, for rank 2 matrices, we do not have nonsensical correlations even if we calibrate to the entire matrix, as shown in the next table.

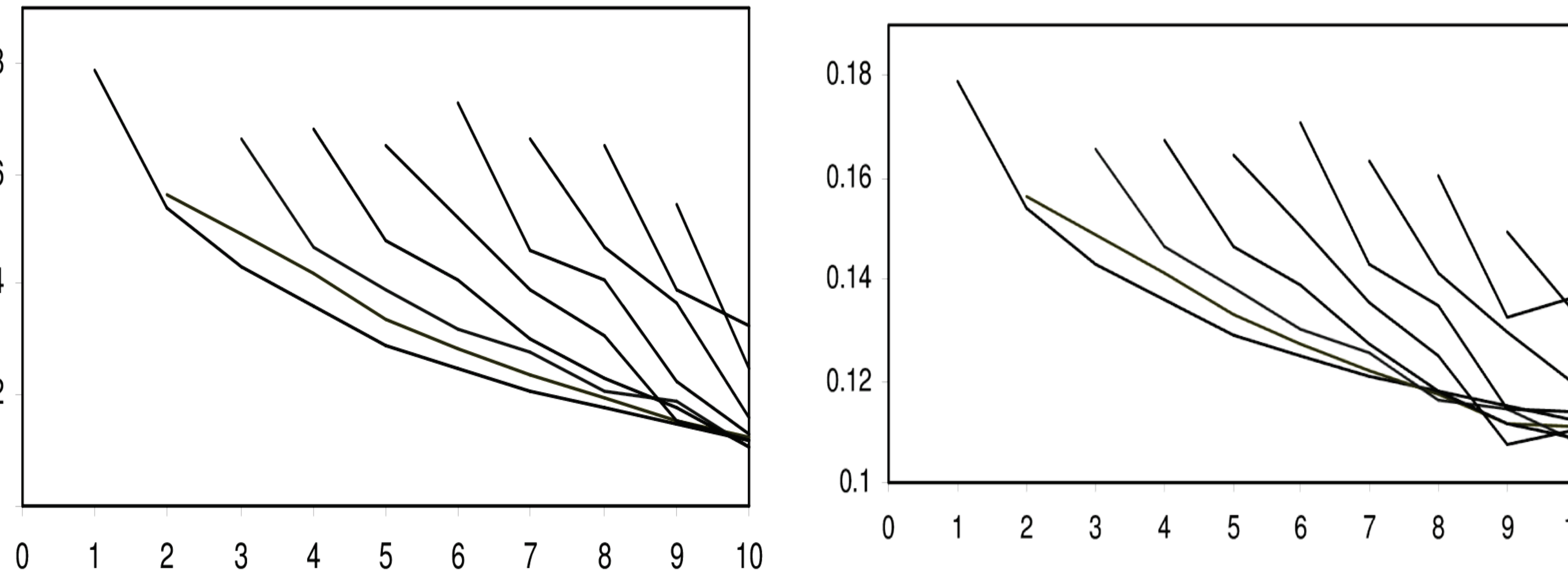
Cascade calibration: Further numerical studies

0.179									
0.152	0.156								
0.130	0.130	0.166							
0.119	0.131	0.122	0.167						
0.112	0.115	0.120	0.126	0.164					
0.112	0.115	0.100	0.113	0.126	0.171				
0.113	0.103	0.119	0.098	0.120	0.119	0.163			
0.122	0.124	0.108	0.082	0.091	0.121	0.119	0.160		
0.138	0.093	0.130	0.129	0.073	0.047	0.123	0.113	0.149	
0.121	0.129	0.106	0.098	0.092	0.090	0.023	0.144	0.118	0.147
0.120	0.120	0.101	0.093	0.134	0.063	0.060	0.045	0.142	0.108
0.107	0.107	0.107	0.142	0.036	0.135	0.078	0.063	0.051	0.143
0.112	0.112	0.112	0.112	0.084	0.084	0.074	0.108	0.062	0.052
0.103	0.103	0.103	0.103	0.103	0.123	0.116	0.043	0.105	0.061
0.097	0.097	0.097	0.097	0.097	0.097	0.169	0.088	0.068	0.108
0.093	0.093	0.093	0.093	0.093	0.093	0.093	0.153	0.117	0.089
0.094	0.094	0.094	0.094	0.094	0.094	0.094	0.094	0.090	0.155
0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.097	0.016
0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099	0.099

Cascade calibration: Further numerical studies I

Diagnostics in these new cases? We examine first the evolution of the term structure of volatilities (TSV). We see below how it appears in case of a calibration with Rebonato three-parameters pivot correlation matrix at rank 2 (left) and with S&C2 at rank 2 (right).

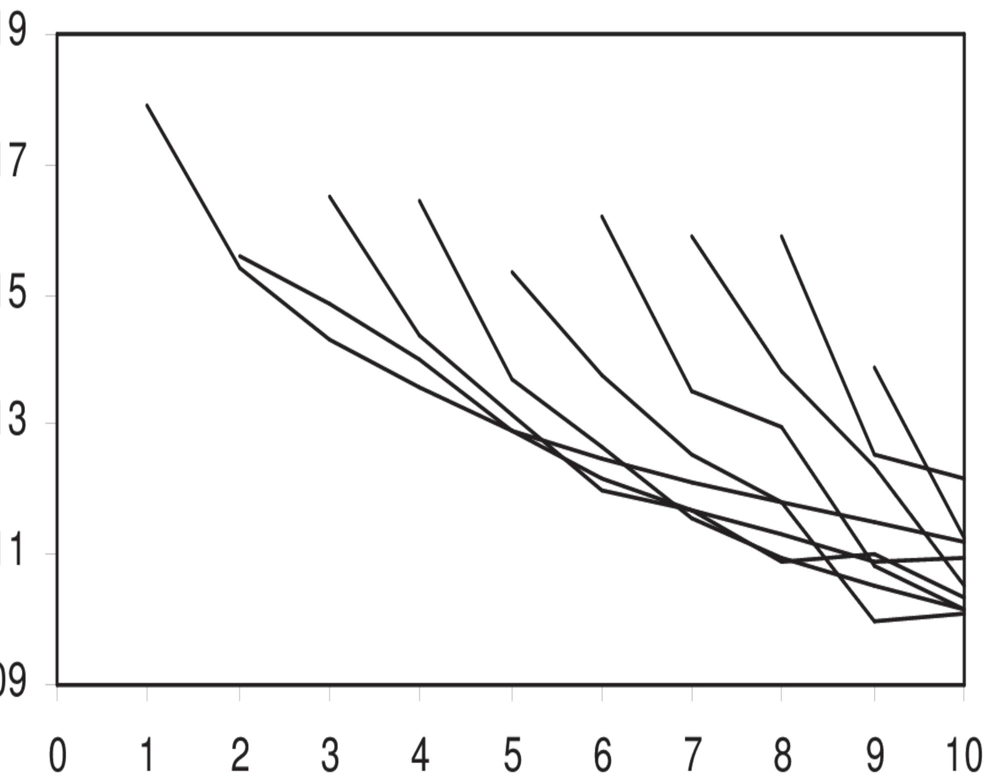
Cascade calibration: Further numerical studies

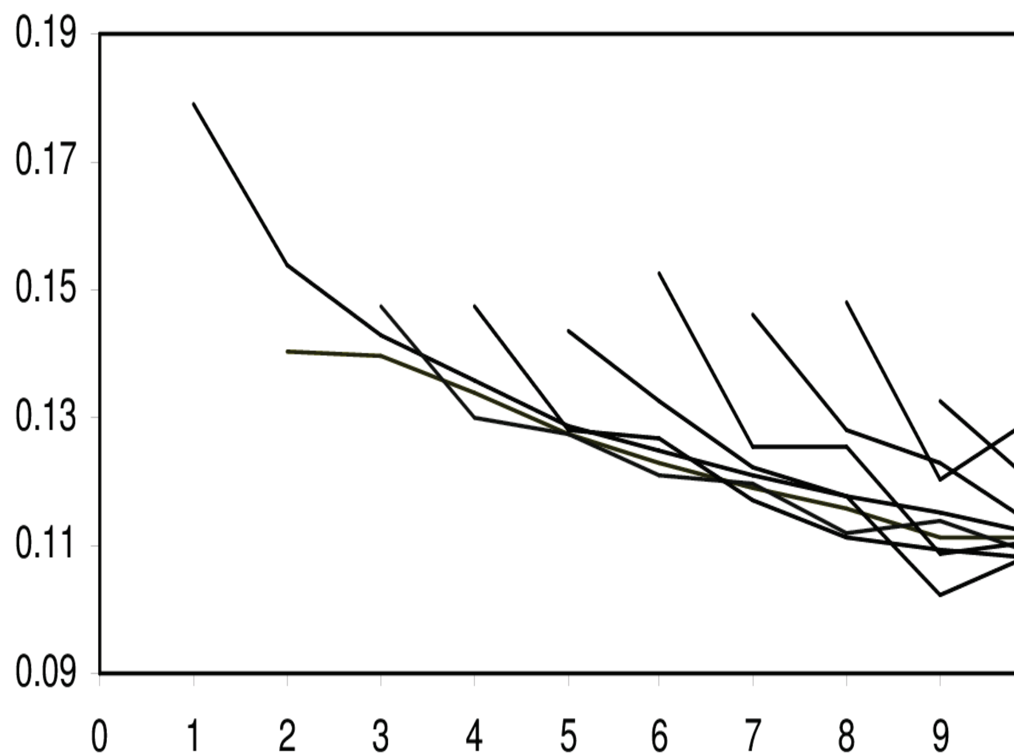


Cascade calibration: Further numerical studies I

The left “Rebo3” evolution appears surprisingly regular, smooth and stable over time, as well as being rather realistic. The right “S&C2” evolution shows the same general features with a little worsening. And when increasing the rank? We plot now the results with Rebonato pivot at rank 4, and S&C2 pivot at rank 10.

Cascade calibration: Further numerical studies





Cascade calibration: Further numerical studies I

Now we examine terminal correlations (TC's). Low rank correlation matrices, through flat initial patterns, may induce oscillating TC patterns. Better with high rank.
10y TC with S&C2 pivot rank 2 and rank 10.

Cascade calibration: Further numerical studies I

	10	11	12	13	14	15	16	17
10	1.000	0.928	0.895	0.920	0.855	0.846	0.928	0.924
11	0.928	1.000	0.863	0.909	0.933	0.881	0.901	0.923
12	0.895	0.863	1.000	0.916	0.908	0.910	0.878	0.939
13	0.920	0.909	0.916	1.000	0.944	0.931	0.956	0.926
14	0.855	0.933	0.908	0.944	1.000	0.954	0.923	0.928
15	0.846	0.881	0.910	0.931	0.954	1.000	0.937	0.958
16	0.928	0.901	0.878	0.956	0.923	0.937	1.000	0.957
17	0.924	0.923	0.939	0.926	0.928	0.958	0.957	1.000

Cascade calibration: Further numerical studies I

	10	11	12	13	14	15	16	17
10	1.000	0.887	0.806	0.792	0.708	0.690	0.757	0.734
11	0.887	1.000	0.822	0.837	0.825	0.746	0.749	0.753
12	0.806	0.822	1.000	0.877	0.841	0.820	0.758	0.801
13	0.792	0.837	0.877	1.000	0.919	0.877	0.881	0.806
14	0.708	0.825	0.841	0.919	1.000	0.932	0.878	0.840
15	0.690	0.746	0.820	0.877	0.932	1.000	0.915	0.914
16	0.757	0.749	0.758	0.881	0.878	0.915	1.000	0.934
17	0.734	0.753	0.801	0.806	0.840	0.914	0.934	1.000

Cascade calibration: Further numerical studies II

Low rank corr is OK for TSV; High rank corr is OK for TC;
However, using particularly smooth and stylized corr it is possible to attain a regular evolution even at full rank.

Cascade calibration: Further numerical studies I

Although one may find comfort in the existence of typical correlation features avoiding the common problems of cascade algorithms, it is worthwhile to keep in mind that such results depend on the particular market quotations we had available, and similar analysis should be carried out again for markedly different market situations. Moreover, we remark that intermediate configurations, with respect to the features we considered to be decisive, might give rise to less clear results, possibly due to the influence of some different, less evident factors. Finally, these findings depend also on the interpolation used for missing market quotations. We address this issue now.

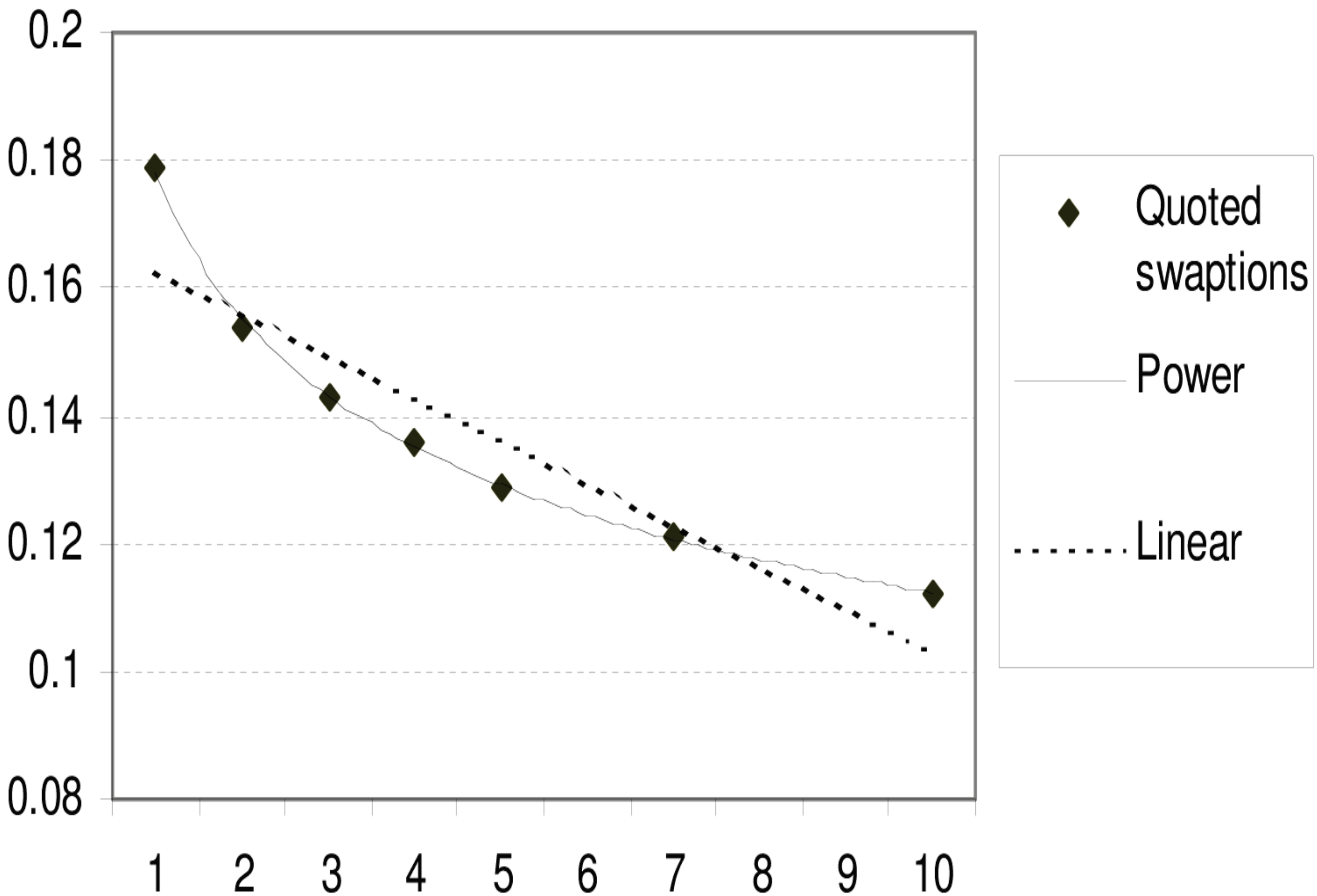
Cascade calibration: Further numerical studies I

In all previous cascade tests negative or complex σ's occur only for input swaptions **artificial** volatilities obtained by local linear interpolation along the columns of the swaption matrix.
On the contrary, volatilities obtained before such artificial interpolated values are all real and positive.
Let us check whether the linear interpolation is really the most suited for patterns in the swaption market. Following Morini (2002), fit a log-linear (or "power") functional form in the maturity to the matrix columns. For example, with our values, the fitted first column is

$$Y = 0.1785\,(X)^{-0.201}\,, \quad \text{or}\ \ln(Y) = \ln(0.1785) - 0.201\,\ln(X),$$

where Y denotes the swaption volatility and X the maturity.

1st columns of the swaptions data with fitted linear and log-linear parametric forms



Cascade calibration: Further numerical studies I

The power fitting form appears clearly closer to the real market pattern than the linear one, as is further confirmed by standard diagnostics concerning the optimization output.

Also a graphical comparison regarding the other columns confirms the superiority of the power form. In order to make sure this was not a one-off coincidence, we tried the same with quotations referring to some months later, finding analogous results.

However, we must recall what is reported in Rebonato and Joshi (2001) about typical swaption configurations. According to this work, two are the common shape patterns that can be found in the Euro swaption market: a humped one, called *normal* and typical of periods of stability, and a monotonically decreasing one, called *excited* since associated with periods immediately following large movement in the yield curve and in the swaption matrix.

Cascade calibration: Further numerical studies II

Our data appear easily to belong to the second pattern. Of course, in periods characterized by humped patterns, a similar form would be likely to prove inadequate.

It is natural to wonder whether, using such a more realistic interpolation for missing maturities, it is possible to change the output of the cascade calibration.

Keep now the original swaption matrix entries of february 1 except for the 7y row. Replace the 7y row by the fitted log-linear values and add the 6y, 8y and 9y maturity rows computed by this fitted form. Errors for the replaced 7th row are (upper part of the matrix, 4 entries)

Errors (differences)	-0.00028	-0.00119	-0.00079	0.00049
% Errors	-0.23272%	-1.03388%	-0.70780%	0.45776%

Cascade calibration: Further numerical studies III

Keeping Rebonato three-parameters pivot as exogenous ρ, and calibrating to the upper part of the swaption matrix, **the previously found negative $\sigma_{10,7}$ disappears at any rank for the exogenous ρ. Even reaching full 19 rank, all volatilities are real and positive.** This is not necessarily the solution, but shows that the choice of the interpolation technique is all but irrelevant!!

Endogenous Interpolation Cascade Calibration I

A much more interesting step would the possibility to develop an analytical calibration **relying only on directly available market data** with no exogenous data interpolation.
We construct a new algorithm assuming σ parameters to be related in a pre-specified way when, due to the lack of market data needed to make a specific discernment, they surface as multiple unknowns. This way one can invert the industry swaption formula via cascade methods even in presence of “holes” in the market swaption matrix.
This method allows to have an exact consistent calibration based on *all* available market swaption quotes, and *only* on them. The new algorithm amounts to carrying out an endogenous interpolation, therefore it is called *Cascade Calibration with Endogenous Interpolation* Algorithm (*EICCA*).

Endogenous Interpolation Cascade Calibration II

Below we consider the simplest and most natural hypothesis on σ parameters, assuming the volatility of forward rates to be constant when no data are available to infer possible changes. We present the algorithm already extended for a complete calibration to the entire swaption matrix.

Endogenous Interpolation Cascade Calibration I

1. Fix s, final dimension of the swaption matrix, and set

$$K := \{k \in \{0..s-1\} : v_{k,y} \text{ missing for } \quad y = k+1, \ldots, k+s\}$$

2. Set $\alpha = 0$;
3. a. If $\alpha \in K$, set $\sigma_{j,m+1} = \sigma_{j,m} = \ldots = \sigma_{j,\alpha+1} =: \sigma_j$ (*), $\alpha + 1 \leq j \leq s$,, $m = \min\{i = \alpha+1, \ldots, s-1, \; i \notin K\}$. Set $\gamma = \alpha$ and $\alpha = m$.
   b. If $\alpha \notin K$, set $\gamma = \alpha$.
   Set $\beta = \alpha + 1$.
4. a. If $\gamma \in K$, solve the cascade 2nd order equation in σ_β with constraints (*).
   b. If $\gamma \notin K$, solve the cascade 2nd order equation in $\sigma_{\beta,\alpha+1}$.
5. Set $\beta = \beta + 1$. If $\beta < s + \gamma$ go to point 4. If $\beta = s + \gamma$, set $\sigma_{\beta,\alpha+1} = \sigma_{\beta,\alpha} = \ldots = \sigma_{\beta,1}$ and solve the cascade 2nd order equation in $\sigma_{\beta,\alpha+1}$. If $\beta < s + \alpha$, repeat point 5, else set $\alpha = \alpha + 1$.
6. If $\alpha < s$, go to point 3, else stop.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA) I

Lest one should get confused by notation, notice that K is the set of indices for missing maturities, which obviously cannot include the last maturity considered, and m in point 3a) represents the index of the first market quoted maturity after missing maturity α.

When Algorithm 5 is applied to a typical Euro swaption matrix we have

$$K = \{5, 7, 8\},$$

namely the maturities at 6, 8 and 9 years after today. The algorithm determines all volatility parameters related to available swaptions, while correctly skipping the others.

For instance, with these missing maturities the volatility buckets $\sigma_{6,6}$, $\sigma_{8,8}$, $\sigma_{9,8}$ and $\sigma_{9,9}$ are not determined by the algorithm. In fact, notice that no market quoted swaption volatilities depend on them, and they

Endogenous Interpolation Cascade Calibration Algorithm (EICCA) II

do not affect the algorithm, which determines independently the other volatility buckets.
When needed, for example for presenting diagnostic structures, we use for these four buckets the homogeneity assumption $\sigma_{k,\beta(t)} =: \eta_{k-(\beta(t)-1)}$ getting $\sigma_{6,6} := \sigma_{5,5}$, $\sigma_{8,8} := \sigma_{7,7}$, $\sigma_{9,8} := \sigma_{8,7}$ and $\sigma_{9,9} := \sigma_{8,8}$.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA) I

Having discarded the influence of exogenous artificial data, we can now check how cascade calibration really works on market data. We see below how algorithm EICC performs in practice.
As a first example, wee apply EICCA to previously used market data of February 1, 2002, with historically estimated correlation at full rank. This corresponds to one of the worst possible situations using basic CCA with exogenous artificial data, giving imaginary and negative entries in the upper triangular calibration considered, and many more if extending to the entire swaption matrix. With the new algorithm EICC results are:

Endogenous Interpolation Cascade Calibration I

0.179									
0.167	0.140								
0.153	0.138	0.138							
0.142	0.148	0.130	0.122						
0.135	0.131	0.134	0.135	0.109					
0.142	0.135	0.106	0.118	0.112	0.109				
0.155	0.126	0.145	0.098	0.130	0.087	0.087			
0.150	0.141	0.118	0.099	0.103	0.142	0.142	0.087		
0.130	0.092	0.136	0.153	0.095	0.122	0.122	0.142	0.087	
0.109	0.127	0.116	0.116	0.130	0.088	0.088	0.112	0.112	0.112
0.123	0.123	0.115	0.112	0.166	0.115	0.115	0.118	0.118	0.118
0.111	0.111	0.111	0.165	0.056	0.147	0.147	0.081	0.081	0.081
0.118	0.118	0.118	0.118	0.107	0.102	0.102	0.083	0.083	0.083
0.117	0.117	0.117	0.117	0.117	0.145	0.145	0.097	0.097	0.097
0.127	0.127	0.127	0.127	0.127	0.127	0.127	0.106	0.106	0.106
0.104	0.104	0.104	0.104	0.104	0.104	0.104	0.135	0.135	0.135
0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114	0.114
0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120	0.120
0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166	0.166

Endogenous Interpolation Cascade Calibration II

namely we have only real and positive σ's still allowing a perfect recovery of all market swaptions quotes.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA) I

Considering earlier CCA tests, now with EICCA based only on market quotations all previously found numerical problems disappear, even for the previously highly problematic set of May 16, 2000 with its typical correlation matrix.
In addition data sets of February 1, 2002, December 10, 2002, and October 10, 2003, have been considered for general complete calibration testing, using as exogenous correlations the corresponding historically estimated matrices and their reduced rank versions. The historical estimations have been performed using one year of data prior to the trading day used for swaption data.
We considered in our tests reduced rank versions of all possible ranks from 2 to full rank 19. Results are summarized as follows.

Endogenous Interpolation Cascade Calibration Algorithm (EICCA) II

Upper Triangular Calibration. This calibration was the typical reference case in the earlier CCA tests. Results included various anomalous results. Now with EICC no anomalous results or numerical problems have been found in any test outputs, at any correlation rank considered with any rank reduction method.

Complete Rectangular Calibration. This calibration was almost always highly problematic with previous cascade calibration. Now, with EICCA, no anomalous results have been found in any test outputs, at any correlation rank with the eigenvalue zeroing by iteration rank reduction method (Morini and Webber, 2004).
Considering the angles parameterization rank reduction methodology seen above, results were analogously satisfactory with one single exception. For 2002 data, in the test with rank 4 correlation, we found two almost-zero negative volatilities, highly influenced by both

Endogenous Interpolation Cascade Calibration Algorithm (EICCA) III

homogeneity assumptions used, so that more realistic and flexible hypotheses could avoid them. But in practice it suffices to use the eigenvalue zeroing by iteration rank reduction technique, or S&C2 parametric form, to obtain positive σ's.

This exception is useful to notice that the fine details of volatility parameters have a precise dependence on the fine details of the correlation structure. Since usually instantaneous correlations are deemed not have a strong influence on swaption prices, this sensitivity can appear a flaw. On the other hand, it gives us a precise indication on the influence of instantaneous correlations on calibration, that with other methods can be hard to detect.

Cascade calibration: Further numerical studies I

Possible integration of the Cascade Calibration with the cap market

The first point to address is the annualization of semi-annual caps data, so as to make them consistent with usually annual swaptions data. We have used the method in the earlier examples of joint calibration with ρ as calibration outputs.

Consider three instants $0 < S < T < U$, all six-months spaced, and assume we are dealing with an $S \times 1$ swaption and with S and T-expiry six-month caplets. Let us denote by v^2_{Black} the Black's swaption volatility and by $\sigma_1(t)$ and $\sigma_2(t)$, respectively, the instantaneous volatilities of the two semi-annual forward rates $F_1(t)$ and $F_2(t)$ associated with the two caplets, whereas $F(t)$ is the annual S-expiry

Cascade calibration: Further numerical studies II

forward rate. It is easy to derive the following approximate relationship to connect the above quantities:

$$v_{\text{Black}}^2 \approx \frac{1}{S}\Big[u_1^2(0)\int_0^S \sigma_1(t)^2\,dt + u_2^2(0)\int_0^S \sigma_2(t)^2\,dt$$
$$+ 2\rho u_1(0)u_2(0)\int_0^S \sigma_1(t)\sigma_2(t)\,dt\Big],$$
$$u_{1,2}(t) = \frac{1}{F(t)}\left(\frac{F_{1,2}(t)}{2} + \frac{F_1(t)F_2(t)}{4}\right),$$

where ρ is the *infra-correlation* between the two semi-annual forward rates. When assuming constant inst vols, we have

$$v_{\text{Black}}^2 \approx u_1^2(0)v_{S-\text{caplet}}^2 + u_2^2(0)v_{T-\text{caplet}}^2 + 2\rho u_1(0)u_2(0)v_{S-\text{caplet}}v_{T-\text{caplet}},$$

Given the last formulas, and setting infra-ρ's to 1, we can simply replace the first column of the input swaption matrix, containing volatilities for unitary length swaptions, with the corresponding array of

Cascade calibration: Further numerical studies III

annualized caplet volatilities. This is the method we used earlier for joint calibration. With the data of February 1 below

Cascade calibration: Further numerical studies IV

	Swaption volatilities	Semi-annual	rates	Caplet	volatilities
1	0,1790	0,0436	0,0480	0,1805	0,1720
2	0,1540	0,0483	0,0508	0,1911	0,1745
3	0,1430	0,0508	0,0523	0,1641	0,1575
4	0,1360	0,0532	0,0545	0,1546	0,1517
5	0,1290	0,0550	0,0560	0,1516	0,1480
6	0,1250	0,0559	0,0566	0,1445	0,1409
7	0,1210	0,0566	0,0572	0,1374	0,1352
8	0,1180	0,0560	0,0562	0,1329	0,1307
9	0,1150	0,0568	0,0571	0,1285	0,1262
10	0,1120	0,0575	0,0577	0,1240	0,1231

Table: Volatilities and forward rates on February 1, 2002

Cascade calibration: Further numerical studies V

we obtain the annualized caplet vols

.178	.185	0.163	0.155	0.152	0.145	0.138	0.134	0.129	.125

Except for the first one, these values are all higher than the swaption volatilities they are to replace.
Replacing the corresponding Sx1 swaptions vols with these annualized cap vols and using exogenous Rebonato 3 parameters pivot correlation at rank four (most standard situation, used earlier), with a cascade calibration we obtain all real and positive σ's, with diagnostics similar to the first cascade calibration tests we performed with year 2000 data. In particular, we have a rapidly increasing TSV.

Cascade calibration: Further numerical studies VI

Assuming again constant inst vols and implying instead ρ's from both caplets and swaptions data, by inverting

$$v_{\text{Black}}^2 \approx u_1^2(0)v_{S-\text{caplet}}^2 + u_2^2(0)v_{T-\text{caplet}}^2 + 2\rho u_1(0)u_2(0)v_{S-\text{caplet}}v_{T-\text{caplet}},$$

we get

1.022	0.388	.543	.536	.444	.493	.533	.56	.586	.598

1.022	0.388	.543	.536	.444	.493	.533	.56	.586	.598

Besides the fact that the first value is outside the viable range for correlations, the other values appear too low to represent real correlations between adjacent rates. A possible reason for this is the aforementioned bias due to the chosen volatility parameterization. Again, more realistic hypothesis can lead to different results.
But the really relevant reasons calling for a cautious interpretation of such results are of a different nature. Indeed, relations and

Cascade calibration: Further numerical studies VII

discrepancies between caps and swaptions tend to be influenced by causes concerning the market fundamentals. Does there exist a basic congruence between the cap and swaption markets, that a model can successfully detect and incorporate? Rebonato (2001) seems to warn against excessive enthusiasm in considering such a possibility.
Rebonato recalls that problems such as illiquidities, agency problems and value-at-risk based limit structures strongly reduce the effectiveness of the *quasi-arbitrageurs* who are supposed to maintain the internal consistency between the two markets.
Accordingly, simple artificial values such as the infra-correlations above are likely to be actually influenced by many different external factors that are hard to detect and measure.

Cascade calibration: Conclusions I

We remarked that some fundamental features make the Cascade methodology particularly appealing:
it is automatic and analytical, and hence instantaneous;
if a common industry approximation is used for pricing, it is free from any calibration error;
it allows for a direct correspondence between market swaption volatilities and LIBOR volatility parameters.
We pointed out that a further opportunity is given by the exogenous nature of the forward rates correlation matrix. Accordingly, we both calibrated with an exogenous historically estimated correlation matrix and considered regular and parsimonious parameterizations, being led to a simple and intuitive methodology to fix parameters consistently with general market tendencies. In this way instantaneous correlation matrices that are rather realistic, regular and simple to control and modify can be easily obtained. Moreover, as we showed, regular

Cascade calibration: Conclusions II

terminal correlations and a satisfactory evolution of the term structure of volatilities are possible, even though our tests revealed a possible trade-off between regularity of the evolution of the TSV and realism of TC's, depending on the level of the rank in the exogenous correlation matrix.

Further, we have given suggestions on the choice of the exogenous correlation matrix and on the interpolation technique for the swaption matrix that avoid negative or complex σ's.

LIBOR model pricing: General approximation I

Freeze part of the drift of the LIBOR dynamics so as to obtain a "multi-dimensional" geometric Brownian motion. This was done earlier to derive approximated formulas for swap volatilities and terminal correlations. Recall: under the T_i-forward-adjusted measure Q^i we have the exact dynamics:

$$dF_k(t) = \mu_{i,k}(t) F_k(t)\, dt + \sigma_k(t) F_k(t)\, dZ_k^i(t)\,,$$

where $\mu_{i,k}(t) := \sigma_k(t)\mu_i^k(t)$ for $i < k$, $\mu_{i,i}(t) := 0$ and $\mu_{i,k}(t) := -\sigma_k(t)\mu_k^i(t)$ for $i > k$. To sum up:

$$\begin{aligned} \mu_{i,k}(t) &:= -\sigma_k(t) \sum_{j=k+1}^{i} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}, \quad k < i \\ \mu_{i,k}(t) &:= 0, \quad k = i \\ \mu_{i,k}(t) &:= \sigma_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}, \quad k > i. \end{aligned}$$

LIBOR model pricing: General approximation II

The distributions or statistical laws of the F_k under Q^i are unknown for $i \neq k$. This is a problem, because prices are expectations under pricing measures, and if we do not know the laws of the random variables we cannot compute the expectations analytically. We are forced to resort to numerical methods. Can we escape this situation in some cases?

LIBOR model pricing: General approximation III

Consider the approximated **lognormal** dynamics:

$$dF_k(t) = \bar{\mu}_{i,k}(t)F_k(t)\,dt + \sigma_k(t)F_k(t)\,dZ_k^i(t)\,,$$

$$\begin{aligned}
\bar{\mu}_{i,k}(t) &:= -\sigma_k(t)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(\mathbf{0})}{1+\tau_jF_j(\mathbf{0})}, \quad k<i\\
\bar{\mu}_{i,k}(t) &:= 0, \quad k=i\\
\bar{\mu}_{i,k}(t) &:= \sigma_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(\mathbf{0})}{1+\tau_jF_j(\mathbf{0})}, \quad k>i.
\end{aligned}$$

This dynamics gives access, in some cases, to a number of techniques which have been developed for the basic Black and Scholes setup, for example, in equity and FX markets. Moreover, this

LIBOR model pricing: General approximation IV

"freezing-part-of-the-drift" technique can be combined with drift interpolation so as to allow for rates that are not in the fundamental (spanning) family $T_0, T_1, \ldots, T_M$ corresponding to the particular model being implemented. Finally, even resorting to MC allows now for a "one-shot" propagation of the dynamics with no infra-discretization, thus reducing memory requirements and simulation time. A similar idea may work also in some smile extensions.

Libor in Arrears (In Advance Swaps) I

An in-advance swap (or LIBOR in arrears) is an IRS that resets at dates $T_{\alpha+1}, \ldots, T_\beta$ and pays at the same dates, with unit notional amount and with fixed-leg rate K.
With respect to standard swaps, the LIBOR payments are “in arrears”, since the libor pays immediately when it resets, and not one period later.
More precisely, the discounted payoff of an in-advance swap (of “payer” type) can be expressed via

$$\sum_{i=\alpha+1}^{\beta} \frac{B(0)}{B(T_i)} \tau_{i+1}(L(T_i, T_{i+1}) - K) =$$

$$= \sum_{i=\alpha+1}^{\beta} \frac{B(0)}{B(T_i)} \tau_{i+1}(F_{i+1}(T_i) - K).$$

Libor in Arrears (In Advance Swaps) II

The value of such a contract is, therefore,

$$\textbf{IAS} = E^B \left[\sum_{i=\alpha+1}^{\beta} \frac{B(0)}{B(T_i)} \tau_{i+1} (F_{i+1}(T_i) - K) \right].$$

Before calculating the expectations, it is convenient to make some adjustments. We shall use the following identity (obtained easily via

Libor in Arrears (In Advance Swaps) III

iterated conditioning):

$$E^B\left[X_T\frac{B(0)}{B(T)}\right] = E^B\left[\frac{P(T,S)}{P(T,S)}\frac{B(0)}{B(T)}X_T\right] =$$

$$= E^B\left\{\frac{1}{P(T,S)}X_T\frac{B(0)}{B(T)}E^B\left[\frac{B(T)}{B(S)}1\middle|\text{Info}_T\right]\right\} =$$

$$= E^B\left\{E^B\left[\frac{1}{P(T,S)}X_T\frac{B(0)}{B(T)}\frac{B(T)}{B(S)}1\middle|\text{Info}_T\right]\right\} =$$

$$= E^B\left\{E^B\left[\frac{1}{P(T,S)}X_T\frac{B(0)}{B(S)}\middle|\text{Info}_T\right]\right\} =$$

$$= E^B\left[\frac{1}{P(T,S)}X_T\frac{B(0)}{B(S)}\right] \quad \text{so that}$$

$$\boxed{E^B\left[X_T\frac{B(0)}{B(T)}\right] = E^B\left[\frac{X_T\frac{B(0)}{B(S)}}{P(T,S)}\right]} \quad \text{for all } 0 < T < S,$$

Libor in Arrears (In Advance Swaps) IV

where X is a T-measurable random variable, known from Info$_T$
To value the above contract, notice that

$$E\left\{\sum_{i=\alpha+1}^{\beta}\frac{B(0)}{B(T_i)}\tau_{i+1}(F_{i+1}(T_i)-K)\right\}$$
$$=E\left\{\sum_{i=\alpha+1}^{\beta}D(0,T_i)\left[\frac{1}{P(T_i,T_{i+1})}-(1+\tau_{i+1}K)\right]\right\}=\ldots$$

Now use our previous result with $T=T_i$, $S=T_{i+1}$, $X_T=1/P(T_i,T_{i+1})$ to get

$$E\left[\frac{B(0)}{B(T_i)}\frac{1}{P(T_i,T_{i+1})}\right]=E\left[\frac{\frac{1}{P(T_i,T_{i+1})}\frac{B(0)}{B(T_{i+1})}}{P(T_i,T_{i+1})}\right]=$$

Libor in Arrears (In Advance Swaps) V

$$= E\left[\frac{1}{P(T_i, T_{i+1})^2}\frac{B(0)}{B(T_{i+1})}\right]$$

and substitute:

$$= E\left\{\sum_{i=\alpha+1}^{\beta}\left[\frac{B(0)}{B(T_{i+1})}\frac{1}{P(T_i, T_{i+1})^2} - \frac{B(0)}{B(T_i)}(1+\tau_{i+1}K)\right]\right\}$$

Libor in Arrears (In Advance Swaps)

$$= E \sum_{i=\alpha+1}^{\beta} \left[\frac{B(0)}{B(T_{i+1})} \frac{1}{P(T_i, T_{i+1})^2} - \frac{B(0)}{B(T_i)}(1 + \tau_{i+1} K) \right]$$

$$= \sum_{i=\alpha+1}^{\beta} E^{\boxed{B}} \left[\frac{\boxed{B(0)}}{\boxed{B(T_{i+1})}} \frac{1}{P(T_i, T_{i+1})^2} \right]$$

$$- \sum_{i=\alpha+1}^{\beta} E^{B} \left[\frac{B(0)}{B(T_i)}(1 + \tau_{i+1} K) \right]$$

$$= \sum_{i=\alpha+1}^{\beta} \boxed{P(0, T_{i+1})} E^{\boxed{i+1}} \left[\frac{1}{\boxed{P(T_{i+1}, T_{i+1})}} \frac{1}{P(T_i, T_{i+1})^2} \right]$$

$$- \sum_{i=\alpha+1}^{\beta} P(0, T_i)(1 + \tau_{i+1} K) =$$

Libor in Arrears (In Advance Swaps)

$$= \sum_{i=\alpha+1}^{\beta} P(0,T_{i+1}) E^{i+1}\left[(1+\tau_{i+1}F_{i+1}(T_i))^2\right]$$

$$- \sum_{i=\alpha+1}^{\beta} P(0,T_i)(1+\tau_{i+1}K).$$

Libor in Arrears (In Advance Swaps)

Computing the expected value is an easy task, since we know that, under Q^{i+1}, F_{i+1} has the driftless (martingale) lognormal dynamics

$$dF_{i+1}(t) = \sigma_{i+1}(t)F_{i+1}(t)dZ_{i+1}(t) \ ,$$

so that (Ito formula $\phi(F) = F^2, \ \phi'(F) = 2F, \ \phi''(F) = 2$),

$$\begin{aligned} dF^2_{i+1}(t) &= 2F_{i+1}(t)dF_{i+1}(t) + \frac{1}{2}2dF_{i+1}(t)dF_{i+1}(t) \\ &= \sigma_{i+1}(t)^2 F^2_{i+1}(t)dt + 2\sigma_{i+1}(t)F^2_{i+1}(t)dZ_{i+1}(t) \ , \end{aligned}$$

so that we still have a geometric brownian motion for F^2:

Libor in Arrears (In Advance Swaps)

$$dF_{i+1}^2(t) = \sigma_{i+1}(t)^2 F_{i+1}^2(t)dt + 2\sigma_{i+1}(t)F_{i+1}^2(t)dZ_{i+1}(t) \; ,$$

and the mean of this process is known to be

$$E^{i+1}\left(F_{i+1}^2(T_i)\right) = F_{i+1}^2(0)\exp\left[\int_0^{T_i}\sigma_{i+1}^2(t)dt\right] = F_{i+1}^2(0)\exp(T_i v_i^2)$$

where the v's have been defined earlier and are caplet volatilities for $T_i - T_{i+1}$.

Libor in Arrears (In Advance Swaps)

By expanding the square and substituting we obtain

$$\begin{aligned}\textbf{IAS} &= \sum_{i=\alpha+1}^{\beta} \{P(0,T_{i+1})\,[1+2\tau_{i+1}F_{i+1}(0)+ \\ &+ \tau_{i+1}^2 F_{i+1}^2(0)\exp(v_i^2 T_i)\Big] - (1+\tau_{i+1}K)P(0,T_i)\}.\end{aligned}$$

Contrary to the plain-vanilla case, this price depends on the volatility of forward rates through the caplet volatilities v. Notice however that correlations between different rates are not involved in this product, as one expects from the additive and “one-rate-per-time” nature of the payoff.

Ratchet Caps and Floors I

A ratchet cap is a cap where the strike is updated at each caplet reset, based on the previous realization of the relevant interest rate.
A simple ratchet cap first resetting at T_α and paying at $T_{\alpha+1}, \dots, T_\beta$ pays the following discounted payoff:

$$\sum_{i=\alpha+1}^{\beta} D(0, T_i)\tau_i \left[L(T_{i-1}, T_i) - (L(T_{i-2}, T_{i-1}) + X)\right]^+ ,$$

Notice that if we set $K_i := L(T_{i-2}, T_{i-1}) + X$ for all i's this is a set of caplets with (random) strikes K_i.
X is a margin, which can be either positive or negative.

Ratchet Caps and Floors II

A **sticky** ratchet cap is instead given by:

$$\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i \left[L(T_{i-1},T_i) - X_i\right]^+ ,$$

$$X_i = \max\left(L(T_{i-2},T_{i-1}) \pm \bar{X}, X_{i-1} \pm \bar{X}\right),$$

$$X_\alpha := L(T_{\alpha-1},T_\alpha).$$

There are versions with "min" replacing "max" in the X_i's definition. The quantity $\bar{X}$ is a spread that can be positive or negative.

Ratchet caps and floors I

In general a sticky ratchet cap has to be valued through Monte Carlo simulation. We have

$$E\{\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i\,[L(T_{i-1},T_i)-X_i]^+\}$$
$$= P(0,T_\beta)\sum_{i=\alpha+1}^{\beta}\tau_i E^\beta\left\{\frac{[L(T_{i-1},T_i)-X_i]^+}{P(T_i,T_\beta)}\right\}.$$

Since the Q^β forward-rate dynamics of $F_{\beta(t)}(t),\dots,F_\beta(t)$ can be discretized via the usual scheme, Monte Carlo pricing can be carried out in the usual manner. We can use also the lognormal frozen-drift approximation to implement a faster MC simulation.
However, for the non-sticky ratchet cap payoff we may investigate possible analytical approximations based on the usual “freezing the drift” technique for the LIBOR market model.

Non-Sticky Ratchets: Analytical approximation I

All we need to compute is the expectation

$$E\{D(0,T_i)\,[L(T_{i-1},T_i) - (L(T_{i-2},T_{i-1}) + X)]^+\}$$

$$= P(0,T_i)E^i\{[F_i(T_{i-1}) - F_{i-1}(T_{i-2}) - X]^+\} =: P(0,T_i)m_i$$

and then add terms. In the above expectation, the rates evolve as follows under the measure Q^i: $dF_i(t) = \sigma_i(t)F_i(t)dZ_i(t)$,

$$dF_{i-1}(t) = -\frac{\rho_{i-1,i}\tau_i\sigma_i(t)F_i(t)}{1+\tau_iF_i(t)}\sigma_{i-1}(t)F_{i-1}(t)dt + \sigma_{i-1}(t)F_{i-1}(t)dZ_{i-1}(t)$$

As usual, in such dynamics we do not know the distribution of $F_{i-1}(t)$. But, since F_{i-1} and F_i's reset times are adjacent, we may freeze the

Non-Sticky Ratchets: Analytical approximation II

drift in F_{i-1} and be rather confident on the resulting approximations. We thus replace the second SDE by

$$\begin{aligned} dF_{i-1}(t) &= \bar{\mu}(t)F_{i-1}(t)dt + \sigma_{i-1}(t)F_{i-1}(t)dZ_{i-1}(t), \\ \bar{\mu}(t) &:= -\frac{\rho_{i-1,i}\tau_i\sigma_i(t)F_i(0)}{1+\tau_i F_i(0)}\sigma_{i-1}(t). \end{aligned}$$

Now both F_{i-1} and F_i follow (correlated) geometric Brownian motions as in the Black and Scholes model.

Non-Sticky Ratchets: Analytical approximation I

Spread option

Now consider the case $X > 0$.
If we set $S_1 := F_i$, $S_2 := F_{i-1}$, $r - q_1 := 0$, $r - q_2 := \bar{\mu}(t)1\{t < T_{i-2}\}$, $\sigma_1 := \sigma_i(t)$, $\sigma_2 := \sigma_{i-1}(t)1\{t < T_{i-2}\}$, $a = 1$, $b = -1$, and $\omega = 1$, we may view our dynamics as the two-dimensional Black Scholes dynamics $d[S_1, S_2]$ and our payoff as a spread option payoff, by slightly adjusting to the fact that no discounting should occur in our case.
Consider two assets whose prices S_1 and S_2 evolve, under the risk neutral measure, according to

$$dS_1(t) = S_1(t)[(r - q_1)dt + \sigma_1 dW_1^Q(t)], \quad S_1(0) = s_1,$$
$$dS_2(t) = S_2(t)[(r - q_2)dt + \sigma_2 dW_2^Q(t)], \quad S_2(0) = s_2,$$

where W_1^Q and W_2^Q are Brownian motions under Q with instantaneous correlation ρ.

Non-Sticky Ratchets: Analytical approximation II

Spread option

Fix a maturity T, a positive real number a, a negative real number b, a strike price K. The spread-option payoff at time T is then defined by

$$H = (awS_1(T) + bwS_2(T) - wK)^+ , \tag{43}$$

where $w = 1$ for a call and $w = -1$ for a put.

Non-Sticky Ratchets: Analytical approximation I

Spread option

Price of the spread option:

$$\pi_t = e^{-r(T-t)} E_t^Q \left\{ (awS_1(T) + bwS_2(T) - wK)^+ \right\},$$

A pseudo-analytical formula can be derived. The unique arbitrage-free price is

$$\pi_t = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} f(v) dv,$$

Non-Sticky Ratchets: Analytical approximation II

Spread option

where

$$f(v) = awS_1(t)\exp\left[-q_1\tau - \frac{1}{2}\rho^2\sigma_1^2\tau + \rho\sigma_1\sqrt{\tau}v\right]\cdot$$
$$\cdot\Phi\left(w\frac{\ln\frac{aS_1(t)}{h(v)} + [\mu_1 + (\frac{1}{2} - \rho^2)\sigma_1^2]\tau + \rho\sigma_1\sqrt{\tau}v}{\sigma_1\sqrt{\tau}\sqrt{1-\rho^2}}\right)$$
$$-wh(v)e^{-r\tau}\Phi\left(w\frac{\ln\frac{aS_1(t)}{h(v)} + (\mu_1 - \frac{1}{2}\sigma_1^2)\tau + \rho\sigma_1\sqrt{\tau}v}{\sigma_1\sqrt{\tau}\sqrt{1-\rho^2}}\right)$$

and

$$h(v) = K - bS_2(t)e^{(\mu_2 - \frac{1}{2}\sigma_2^2)\tau + \sigma_2\sqrt{\tau}v}, \quad \mu_{1,2} = r - q_{1,2}, \quad \tau = T - t.$$

proof based on standard bivariate Gaussian variables comp.

Non-Sticky Ratchets: Analytical approximation. I

If $X < 0$ we just switch the definitions of S_1 and S_2 above, $S_1 := F_{i-1}$, $S_2 = F_i$ etc., and then take $\omega = -1$. In the calculations below we assume $X > 0$.
As a matter of fact, our coefficients here are time-dependent, but this does not change substantially the derivation. It follows that our expected value

$$m_i := E^i\{[F_i(T_{i-1}) - F_{i-1}(T_{i-2}) - X]^+\}$$

is given by our formula above for the spread option when taking into account the above substitutions, i.e. one needs to apply said formula

Non-Sticky Ratchets: Analytical approximation. II

with $a = 1,\ \omega = 1,\ t = 0,$

$$S_1(t) = F_i(0),\ q_1\tau = 0,\ q_2\tau = -\int_0^{T_{i-2}} \bar{\mu}(u)du,$$
$$r = 0,\quad \sigma_1^2\tau = \int_0^{T_{i-1}} \sigma_i^2(u)du,\ \sigma_2^2\tau = \int_0^{T_{i-2}} \sigma_{i-1}^2(u)du,$$
$$\rho = \rho_{i-1,i},\quad K = X.$$

Once we have the m_i's, our ratchet price is given by

$$\sum_{i=\alpha+1}^{\beta} P(0, T_i)\tau_i m_i.$$

Non-Sticky Ratchets: Analytical approximation I

Case $X = 0$.

The above price depends on a one-dimensional numerical integration. There is a case, though, where this is not necessary. Indeed, if $X = 0$, we obtain a special ratchet cap that, under the lognormal assumption, we may value analytically through the **Margrabe formula for the option exchanging one asset for another**.

We map the ratchet payoff terms

$$E^i\left[(F_i(T_{i-1}) - F_{i-1}(T_{i-2}))^+\right]$$

into equity payoffs

$$H = (S_1(T) - S_2(T))^+$$

This payoff is the so called "option to exchange one asset (S_1) for another (S_2)". Indeed, if we hold S_2, when we are at T the option pays

$$(S_1(T) - S_2(T))^+ = \max(S_1(T) - S_2(T), 0) = S_1(T) - S_2(T)$$

Non-Sticky Ratchets: Analytical approximation II

Case $X = 0$.

if $S_1(T) > S_2(T)$, and 0 otherwise. Recall we are holding S_2. By getting the option payoff in this case where $S_1(T) > S_2(T)$, we get a total of

$$S_2(T) + (S_1(T) - S_2(T)) = S_1(T).$$

Non-Sticky Ratchets: Analytical approximation I

Case $X = 0$.

So, when holding S_2, if $S_1(T) > S_2(T)$ by means of the option we end up with S_1, so we have exchanged S_2 with the more valuable S_1. On the contrary, if $S_1(T) < S_2(T)$, the option expires worthless and there is no exchange, so we keep the more valuable S_2.
This means that the exchange is a right but no obligation, since it occurs only when it favors us.
This is then indeed an option to exchange one asset for another. In the market this kind of option is priced with Margrabe's formula, which we derive below by using the change of numeraire technique.

Non-Sticky Ratchets: Analytical approximation I

Case $X = 0$.

We derive a formula now for

$$E^B\left[\frac{B(0)}{B(T)}(S_1(T) - S_2(T))^+\right] = E^{S_1}\left[\frac{S_1(0)}{S_1(T)}(S_1(T) - S_2(T))^+\right] =$$

$$= S_1(0)E^{S_1}\left[\left(1 - \frac{S_2(T)}{S_1(T)}\right)^+\right] = S_1(0)E^{S_1}\left[(1 - Y(T))^+\right]$$

where $Y_t = S_2(t)e^{-\int_t^T q_2(s)ds}/S_1(t)$. Note that we took S_1 as numeraire, assuming $q_1 = 0$, since the numeraire has to be a positive non-dividend paying asset ($q_1 = 0$). Notice also that in the numerator, to have the price of a tradable asset, we got rid of the dividend by inserting the forward price

$$E_t^B\left[\frac{B(t)}{B(T)}S_2(T)\right] = S_2(t)e^{-\int_t^T q_2(s)ds}$$

Non-Sticky Ratchets: Analytical approximation II

Case $X = 0$.

(without dividends the forward price would be $S_2(t)$ itself).
Now we need to derive the dynamics of Y_t under the S_1 measure. We know this is a martingale, since Y_t is a ratio between a tradable asset and our numeraire S_1, so that by FACT ONE on the change of numeraire (earlier lecture) we have that Y_t is a martingale (=zero drift).
Compute then, first under Q^B:

$$dY_t = d\left(\frac{S_2(t)e^{-\int_t^T q_2(s)ds}}{S_1(t)}\right) = d\left(e^{-\int_t^T q_2(s)ds}\frac{S_2(t)}{S_1(t)}\right) =$$

First notice that the first term would only give a "dt" contribution when differentiated. Then we compute directly

$$= e^{-\int_t^T q_2(s)ds} d\left(\frac{S_2(t)}{S_1(t)}\right) + (\ldots)dt =$$

Non-Sticky Ratchets: Analytical approximation III

Case $X = 0$.

$$= e^{-\int_t^T q_2(s)ds} \left[\frac{1}{S_1(t)} d\left(S_2(t)\right) + S_2(t) d\left(\frac{1}{S_1(t)}\right) + \right.$$

$$\left. + \, dS_2(t) \, d\left(\frac{1}{S_1(t)}\right) \right] + (\ldots)dt =$$

$$= e^{-\int_t^T q_2(s)ds} \left[\frac{1}{S_1(t)} d\left(S_2(t)\right) + S_2(t) d\left(\frac{1}{S_1(t)}\right) \right] + (\ldots)dt =$$

$$= e^{-\int_t^T q_2(s)ds} \left\{ \frac{1}{S_1(t)} S_2(t)[(r - q_2)dt + \sigma_2 dW_2^B] + \right.$$

$$\left. + \, S_2(t) d\left(\frac{1}{S_1(t)}\right) \right\} + (\ldots)dt = \ldots \rightarrow$$

Non-Sticky Ratchets: Analytical approximation IV

Case $X = 0$.

Since (Ito $\phi(S) = \frac{1}{S}, \phi'(S) = -\frac{1}{S^2}, \phi''(S) = 2/(S^3)$)

$$d\left(\frac{1}{S_1(t)}\right) = \frac{-1}{S_1^2} dS_1 + \frac{1}{2}\frac{2}{S_1^3} dS_1 dS_1 = -\frac{1}{S_1}[r dt + \sigma_1 dW_1^B] + (\ldots)dt,$$

by substituting we obtain

$$\rightarrow \ldots = e^{-\int_t^T q_2(s)ds}\left[\frac{1}{S_1(t)} S_2(t)\sigma_2 dW_2^B + S_2(t)\left(-\frac{1}{S_1}\sigma_1 dW_1^B\right)\right]$$

$$+(\ldots)dt =$$

Recalling that $Y_t = S_2 e^{-\int q_2}/S_1$, we may then write

$$dY_t = -Y_t\sigma_1 dW_1^B + Y_t\sigma_2 dW_2^B + (...)dt$$

Non-Sticky Ratchets: Analytical approximation V

Case $X = 0$.

Now, if we change numeraire, the diffusion part does not change. Since we already know that under the S_1 measure Y is a martingale, this means that the equation of Y under S_1 will have the same diffusion parts and zero drift. We get

$$dY_t = -Y_t \sigma_1 dW_1^{S_1} + Y_t \sigma_2 dW_2^{S_1}$$

or also

$$dY_t = Y_t(-\sigma_1 dW_1^{S_1} + \sigma_2 dW_2^{S_1})$$

From the point of view of the law, this process is the same as a process with a single brownian motion

$$dY_t = Y_t(\sigma_0 dW_0^{S_1})$$

Non-Sticky Ratchets: Analytical approximation VI

Case $X = 0$.

provided that

$$\mathrm{Var}(\sigma_2 dW_2^{S_1} - \sigma_1 dW_1^{S_1}) = \mathrm{Var}(\sigma_0 dW_0^{S_1})$$

This equation reads

$$\mathrm{Var}(\sigma_2 dW_2^{S_1} - \sigma_1 dW_1^{S_1}) = \sigma_1^2 dt + \sigma_2^2 dt - 2\rho\sigma_1\sigma_2 dt$$

and since

$$\mathrm{Var}(\sigma_0 dW_0^{S_1}) = \sigma_0^2 dt$$

we have

$$\sigma_0^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$$

Let us now go back to

$$S_1(0)E^{S_1}\left[(1 - Y(T))^+\right]$$

Non-Sticky Ratchets: Analytical approximation VII

Case $X = 0$.

with the dynamics

$$dY_t = Y_t(\sigma_0 dW_0^{S_1})$$

This is a put option with strike 1 for which we get the formula

$$S_1(0)[\Phi(-d_2) - Y(0)\Phi(-d_1)]$$

with

$$d_{1,2} = \frac{\ln(Y(0)/1) \pm \frac{1}{2}\int_0^T \sigma_0^2(t)dt}{\left(\int_0^T \sigma_0^2(t)dt\right)^{\frac{1}{2}}}$$

Recalling the expressions for Y and σ_0 we get

$$[S_1(0)\Phi(-d_2) - S_2(0)e^{-\int_0^T q_2(t)dt}\Phi(-d_1)]$$

Non-Sticky Ratchets: Analytical approximation VIII

Case $X = 0$.

$$d_{1,2} = \frac{\ln(S_2(0)/S_1(0)) - \int_0^T q_2(t)dt \pm \frac{1}{2}\int_0^T [...]dt}{\left(\int_0^T [\sigma_1^2(t) + \sigma_2^2(t) - 2\rho\sigma_1(t)\sigma_2(t)]dt\right)^{\frac{1}{2}}}$$

As before, set

$$S_1(t) = F_i(0),\ q_1 = 0,\ \int_0^T q_2(t)dt = -\int_0^{T_{i-2}} \bar{\mu}(u)du,$$

$$r = 0,\ \ \int_0^T \sigma_1^2(t)dt = \int_0^{T_{i-1}} \sigma_i^2(t)dt,$$

$$\int_0^T \sigma_2^2(t)dt = \int_0^{T_{i-2}} \sigma_{i-1}^2(t)dt,\ \rho = \rho_{i-1,i}$$

to get:

Non-Sticky Ratchets: Analytical approximation I

Case $X = 0$.

$$E\left\{\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i\left[L(T_{i-1},T_i) - L(T_{i-2},T_{i-1})\right]^+\right\}$$

$$= E\left\{\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i\left[F_i(T_{i-1}) - F_{i-1}(T_{i-2})\right]^+\right\}$$

$$\approx \sum_{i=\alpha+1}^{\beta} \tau_i\, P(0,T_i)\left[F_i(0)\Phi(d_1^i) - F_{i-1}(0)\exp\left(\int_0^{T_{i-2}}\bar{\mu}(u)du\right)\Phi(d_2^i)\right],$$

$$d_{1,2}^i = \frac{\ln(F_i(0)/F_{i-1}(0)) - \int_0^{T_{i-2}}\bar{\mu}(u)du}{R_i} \pm \frac{1}{2}R_i,$$

$$R_i = \left(\int_0^{T_{i-1}}\sigma_i^2(u)du + \int_0^{T_{i-2}}(\sigma_{i-1}^2(u) - 2\rho_{i-1,i}\sigma_{i-1}(u)\sigma_i(u))du\right)^{\frac{1}{2}}$$

Non-Sticky Ratchets: Analytical approximation II

Case $X = 0$.

In this section we dealt with ratchet caps. The treatment of ratchet floors is analogous.

Zero Coupon Swaption I

A payer (receiver) zero-coupon swaption is a contract giving the right to enter a payer (receiver) zero-coupon IRS at a future time. A zero-coupon IRS is an IRS where a single fixed payment is due at the unique (final) payment date T_β for the fixed leg in exchange for a stream of usual floating payments $\tau_i L(T_{i-1}, T_i)$ at times T_i in $T_{\alpha+1}, T_{\alpha+2}, \ldots, T_\beta$ (usual floating leg). In formulas, the discounted payoff of a payer zero-coupon IRS is, at time $t \leq T_\alpha$:

$$\frac{B(t)}{B(T_\alpha)} \left[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta} K \right],$$

where $\tau_{\alpha,\beta}$ is the year fraction between T_α and T_β. The analogous payoff for a receiver zero-coupon IRS is obviously given by the opposite quantity.

Zero Coupon Swaption II

Taking risk-neutral expectation, we obtain easily the contract value as

$$P(t, T_\alpha) - P(t, T_\beta) - \tau_{\alpha,\beta} K P(t, T_\beta),$$

which is the typical value of a floating leg minus the value of a fixed leg with a single final payment.

Zero Coupon Swaption I

The value of K that renders the contract fair is obtained by equating to zero the above value. $K = F(t; T_\alpha, T_\beta)$. Indeed, the value of the swap is independent of the number of payments on the floating leg, since the floating leg always values at par, no matter the number of payments. Therefore, we might as well have taken a floating leg paying only in T_β the amount $\tau_{\alpha,\beta} L(T_\alpha, T_\beta)$. This would have given us again a standard swaption, standard in the sense that the two legs of the underlying IRS have the same payment dates (collapsing to T_β) and the unique reset date T_α. In such a one-payment case, the swap rate collapses to a forward rate, so that we should not be surprised to find out that the forward swap rate in this particular case is simply a forward rate.

Zero Coupon Swaption I

An option to enter a payer zero-coupon IRS is a payer zero-coupon swaption, and the related payoff is

$$\frac{B(t)}{B(T_\alpha)}[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta}K]^+,$$

or, equivalently, by expressing the F's in terms of discount factors,

$$\frac{B(t)}{B(T_\alpha)}\left[1 - P(T_\alpha, T_\beta) - P(T_\alpha, T_\beta)\tau_{\alpha,\beta}K\right]^+ ,$$

which in turn can be written as

$$\frac{B(t)}{B(T_\alpha)}\tau_{\alpha,\beta}P(T_\alpha, T_\beta)\left[F(T_\alpha; T_\alpha, T_\beta) - K\right]^+ .$$

Zero Coupon Swaption I

$$\frac{B(t)}{B(T_\alpha)}\tau_{\alpha,\beta}P(T_\alpha, T_\beta)\left[F(T_\alpha; T_\alpha, T_\beta) - K\right]^+ .$$

Notice that, from the point of view of the payoff structure, this is merely a caplet. As such, it can be priced easily through Black's formula for caplets. The problem, however, is that such a formula requires the integrated percentage variance (volatility) of the forward rate $F(\cdot; T_\alpha, T_\beta)$, which is a forward rate over a non-standard period. Indeed, $F(\cdot; T_\alpha, T_\beta)$ is not in our usual family of spanning forward rates, unless we are in the trivial case $\beta = \alpha + 1$.

Zero Coupon Swaption I

Therefore, since the market provides us (through standard caps and swaptions) with volatility data for standard forward rates, we need a formula for deriving the integrated percentage volatility of the forward rate $F(\cdot; T_\alpha, T_\beta)$ from volatility data of the standard forward rates $F_{\alpha+1}, \ldots, F_\beta$. The reasoning is once again based on the “freezing the drift” procedure, leading to an approximately lognormal dynamics for our standard forward rates.

Zero Coupon Swaption I

Denote for simplicity $F(t) := F(t; T_\alpha, T_\beta)$ and $\tau := \tau_{\alpha,\beta}$.
We begin by noticing that, through straightforward algebra, we have (write everything in terms of discount factors to check)

$$1 + \tau F(t) = \prod_{j=\alpha+1}^{\beta} (1 + \tau_j F_j(t)).$$

It follows that

$$\ln(1 + \tau F(t)) = \sum_{j=\alpha+1}^{\beta} \ln(1 + \tau_j F_j(t)),$$

so that $d \ln(1 + \tau F(t)) =$

$$= \sum_{j=\alpha+1}^{\beta} d \ln(1 + \tau_j F_j(t)) = \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1 + \tau_j F_j(t)} + (\ldots)dt.$$

Zero Coupon Swaption II

Since $dF(t) = \frac{1+\tau F(t)}{\tau} d\ln(1+\tau F(t)) + (\ldots)dt$,

we obtain from the above expression

$$dF(t) = \frac{1+\tau F(t)}{\tau} \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1+\tau_j F_j(t)} + (\ldots)dt.$$

Zero Coupon Swaption I

$$dF(t) = \frac{1+\tau F(t)}{\tau} \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1+\tau_j F_j(t)} + (\dots)dt.$$

Take variance (conditional on t) on both sides:

$$\mathrm{Var}\left(\frac{dF(t)}{F(t)}\right) = \left[\frac{1+\tau F(t)}{\tau F(t)}\right]^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{i,j} \sigma_i(t) \sigma_j(t) F_i(t) F_j(t)}{(1+\tau_i F_i(t))(1+\tau_j F_j(t))} dt.$$

Freeze all $t's$ to 0 except for the σ's, and integrate over $[0, T_\alpha]$:
$(v^{zc}_{\alpha,\beta})^2 := (1/T_\alpha)\times$

$$\left[\frac{1+\tau F(0)}{\tau F(0)}\right]^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{i,j} F_i(0) F_j(0)}{(1+\tau_i F_i(0))(1+\tau_j F_j(0))} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t) dt.$$

Zero Coupon Swaption II

To price the zero-coupon swaption it is then enough to put this quantity into the related Black's Caplet formula:

$$\mathbf{ZCPS} = \tau P(0, T_\beta)[F(0)\Phi(d_1(F(0), K, v^{zc}_{\alpha,\beta}))$$

$$-K\Phi(d_2(F(0), K, v^{zc}_{\alpha,\beta}))].$$

Zero Coupon Swaption I

We have checked the accuracy of this formula against the usual Monte Carlo pricing based on the exact dynamics of the forward rates. In the tests all swaptions are at-the-money. We have done this under a number of situations , corresponding to possible modifications of the data coming from a standard calibrations of the LIBOR model to at-the-money swaptions data.
All cases show the formula to be sufficiently accurate for practical purposes.
When using the formula we notice that the at-the-money standard swaption has always a lower volatility (and hence price) than the corresponding at-the-money zero-coupon swaption. We may wonder whether this is a general feature. Indeed, we have the following.

Zero Coupon Swaption I

Comparison between zero-coupon swaptions and corresponding standard swaptions: A first remark is due for a comparison between the zero-coupon swaption volatility $v^{zc}_{\alpha,\beta}$ and the corresponding European-swaption approximation $v^{\text{LMM}}_{\alpha,\beta}$. If we rewrite the latter as

$$T_\alpha (v^{\text{LMM}}_{\alpha,\beta})^2 = \sum_{i,j=\alpha+1}^{\beta} \rho_{i,j}\lambda_i\lambda_j \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt, \quad \lambda_i = \frac{w_i(0)F_i(0)}{S_{\alpha,\beta}(0)},$$

it is easy to check that

$$T_\alpha (v^{zc}_{\alpha,\beta})^2 = \sum_{i,j=\alpha+1}^{\beta} \rho_{i,j}\mu_i\mu_j \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt,$$

where

$$\mu_i = \frac{P(0,T_\alpha)}{P(0,T_i)}\lambda_i \geq \lambda_i,$$

Zero Coupon Swaption II

the discrepancy increasing with the payment index i. It follows that, for positive correlations, the zero-coupon swaption volatility is always larger than the corresponding plain vanilla swaption volatility, the difference increasing with the tenor $T_\beta - T_\alpha$, for each given T_α.

Constant Maturity Swaps (CMS's) I

A constant-maturity swap is a financial product structured as follows. We assume a unit nominal amount. Let us denote by $\{T_0, \dots, T_n\}$ a set of payment dates at which coupons are to be paid. At time T_{i-1} (in some variants at time T_i), $i \geq 1$, institution A pays to B the c-year swap rate resetting at time T_{i-1} in exchange for a fixed rate K. Formally, at time T_{i-1} institution A pays to B

$$S_{i-1,i-1+c}(T_{i-1})\,\tau_i\,,$$

instead of

$$L(T_{i-1}, T_i)\tau_i = F_i(T_{i-1})\,\tau_i\,,$$

as would be natural (standard Interest Rate Swap with model independent valuation, see earlier Lecture).

Constant Maturity Swaps (CMS's) I

The net value of the contract to B at time 0 is

$$E^B\left(\sum_{i=1}^n \frac{B(0)}{B(T_{i-1})}(S_{i-1,i-1+c}(T_{i-1})-K)\tau_i\right)$$

$$=\sum_{i=1}^n \tau_i E^{\boxed{B}}\left[\frac{\boxed{B(0)}}{\boxed{B(T_{i-1})}} S_{i-1,i-1+c}(T_{i-1})\right] - K\sum_{i=1}^n \tau_i P(0,T_{i-1})$$

We can change numeraire in two ways: choose a rolling numeraire in each different term, $P(\cdot, T_{i-1})$, or choose the single "final" numeraire $P(\cdot, T_n)$

$$1: \quad \rightarrow = \sum_{i=1}^n \tau_i P(0,T_{i-1})\left[E^{i-1}\left(S_{i-1,i-1+c}(T_{i-1})\right) - K\right]$$

$$2: \quad \rightarrow = \sum_{i=1}^n \tau_i \left(P(0,T_n)\,E^n\left(\frac{S_{i-1,i-1+c}(T_{i-1})}{P(T_{i-1},T_n)}\right) - KP(0,T_{i-1})\right).$$

CMS's I

We need only compute either

$$E^{i-1}\left[S_{i-1,i-1+c}(T_{i-1})\right] \quad \text{or} \quad E^{n}[S_{i-1,i-1+c}(T_{i-1})/P(T_{i-1},T_n)]$$

At first sight, one might think to discretize the dynamics of the forward swap rate in the swap model under the relevant forward measure, and compute the required expectation through a Monte Carlo simulation. However, notice that forward rates appear in the drift of such equation, so that we are forced to evolve forward rates anyway. As a consequence, we can build forward swap rates as functions of the forward LIBOR rates obtained by the Monte Carlo simulated dynamics of the LIBOR model. Find the swap rate $S_{i-1,i-1+c}(T_{i-1})$ from the T_{i-1} values of the (Monte Carlo generated) spanning forward rates

$$F_i(T_{i-1}), F_{i+1}(T_{i-1}), \ldots, F_{i-1+c}(T_{i-1}).$$

CMS's II

Analogously to earlier cases, such forward rates can be generated according to the usual discretized (Milstein) dynamics based on Gaussian shocks and under the unique measure Q^n for example.

CMS's I

Alternatively, resort to $S_{\alpha,\beta}(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) F_i(T_\alpha)$ and compute

$$E^\alpha S_{\alpha,\beta}(T_\alpha) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) E^\alpha F_i(T_\alpha)$$

$$\approx \sum_{i=\alpha+1}^{\beta} w_i(0) e^{\int_0^{T_\alpha} \bar{\mu}_{\alpha,i}(t)dt} F_i(0)$$

We have frozen again the drift in the F_i's dynamics of the F's under Q^α.
This can be compared with classical market convexity adjustments.
The two methods give similar results when volatilities are not too high.

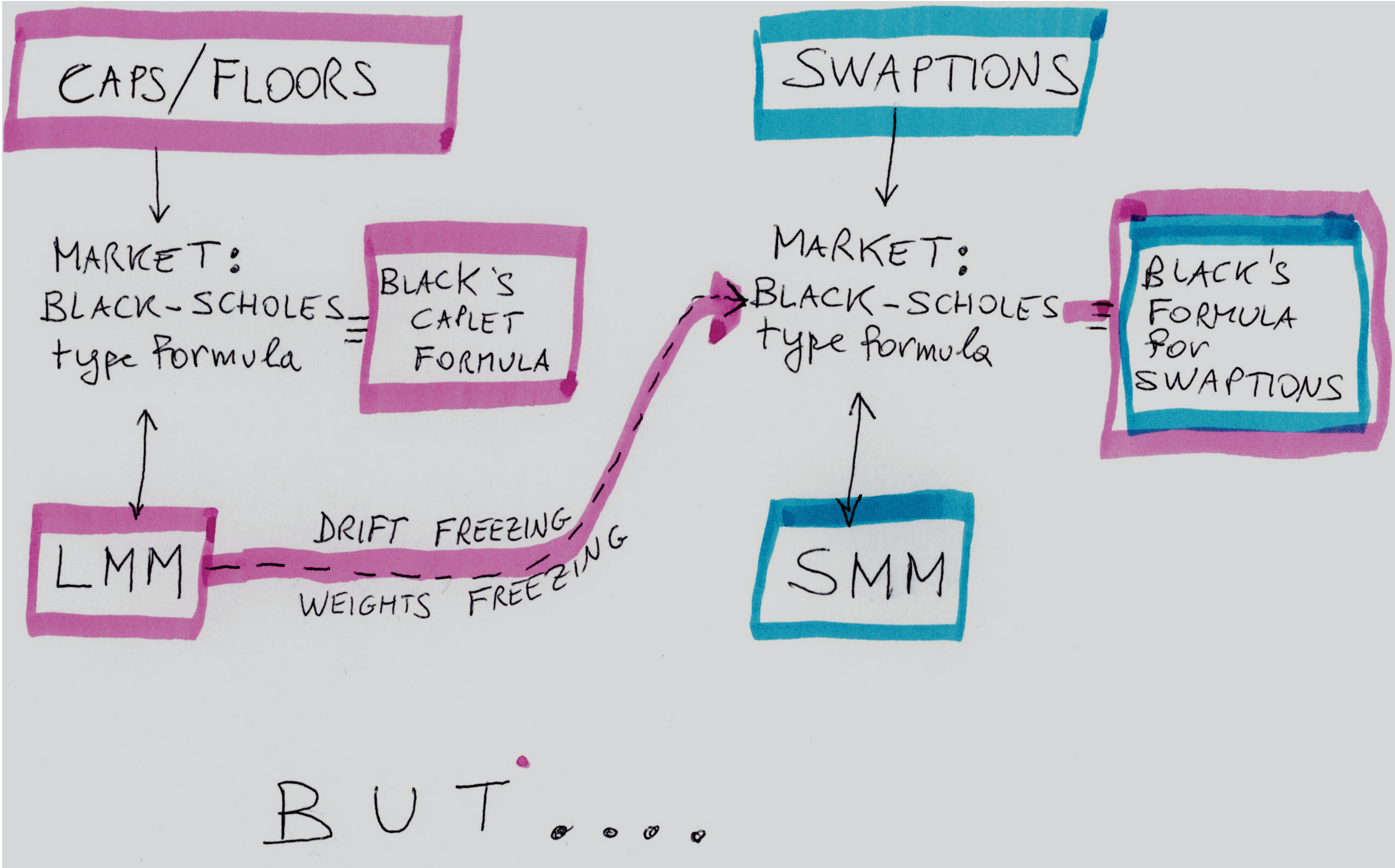
Notation for $\bar{\mu}$ was given at the beginning of this unit.

CMS's II

The method is general and can be used whenever swap rates or forward rates are paid at times that are not "natural" in swaps and similar contracts. A dynamics can be obtained by the freezing procedures outlined above.

ADDING SMILE TO LIBOR MODELS I

- Guided tour to the caplet and swaption smile problems.
- Theoretical results on smile modeling in general (Breeden and Litzenberger, Dupire, local volatility models and stochastic volatility models)
- Displaced diffusion LIBOR model
- CEV LIBOR model
- The local volatility lognormal mixture dynamics (LVLMD) LIBOR model
- The Stochastic Volatility SABR (stochastic alpha beta rho) model.


CAPS/FLOORS
MARKET:
BLACK-SCHOLES type Formula
=
BLACK'S CAPLET FORMULA
LMM
DRIFT FREEZING
WEIGHTS FREEZING
SWAPTIONS
MARKET:
BLACK-SCHOLES type formula
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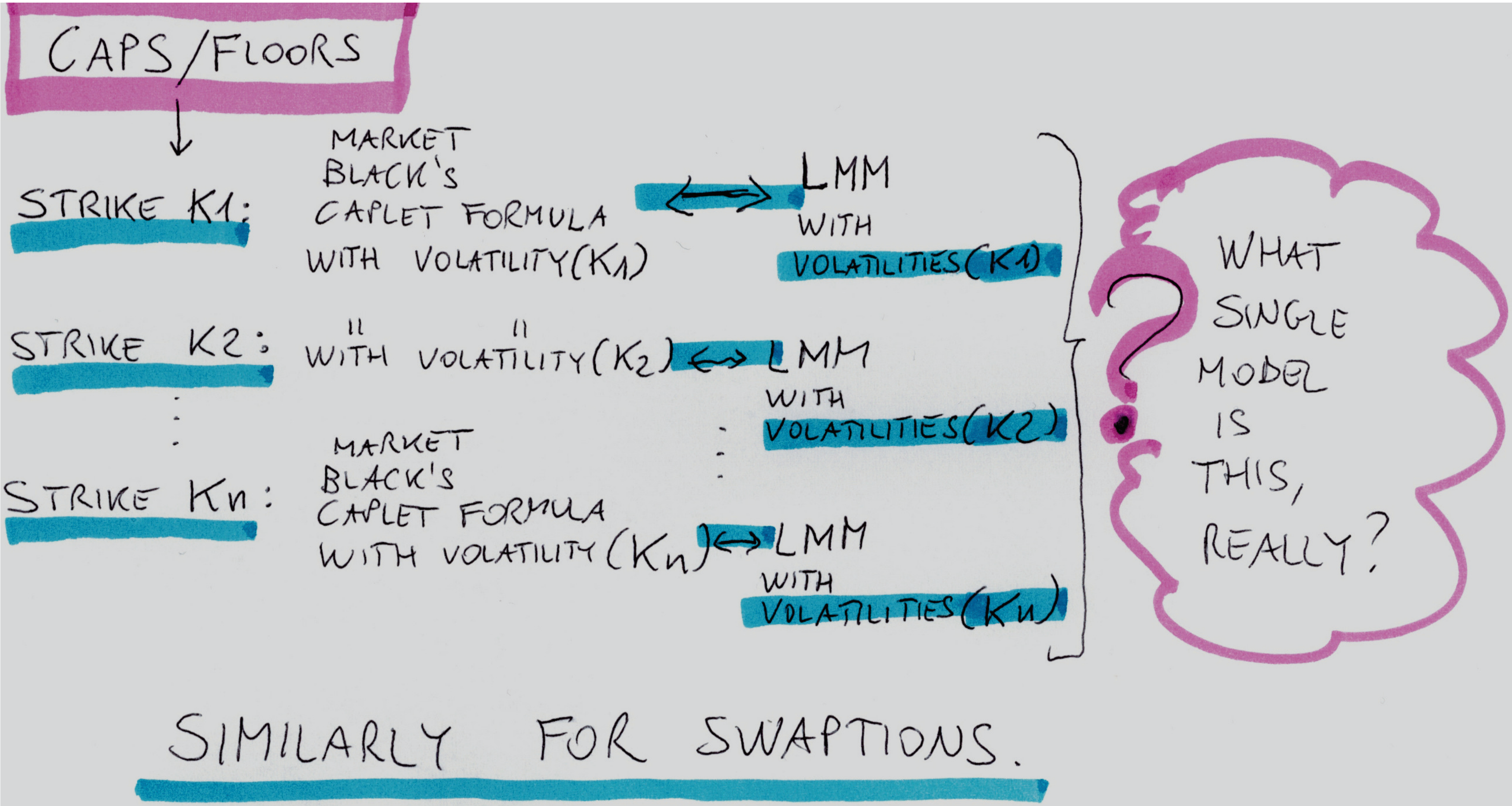

CAPS/FLOORS
STRIKE K1:
MARKET BLACK'S CAPLET FORMULA WITH VOLATILITY(K1)
LMM WITH VOLATILITIES(K1)
STRIKE K2:
" WITH " VOLATILITY(K2)
LMM WITH VOLATILITIES(K2)
STRIKE Kn:
MARKET BLACK'S CAPLET FORMULA WITH VOLATILITY(Kn)
LMM WITH VOLATILITIES(Kn)
WHAT SINGLE MODEL IS THIS, REALLY?
SIMILARLY FOR SWAPTIONS.

Caplet Smile Modeling: Guided Tour I

We have seen earlier that Black's formula for caplets. To fix ideas, let us consider again the time-0 price of a T_2-maturity caplet resetting at time T_1 $(0 < T_1 < T_2)$ with strike K

$$P(0, T_2)\tau E_0^2[(F(T_1; T_1, T_2) - K)^+].$$

The dynamics for F in the above expectation under the T_2-forward measure is the lognormal LMM dynamics

$$dF(t; T_1, T_2) = \sigma_2(t)F(t; T_1, T_2)\, dW_t\,. \tag{44}$$

Lognormality of the T_1-marginal distribution of this dynamics implies that the above expectation results in Black's formula

$$\begin{aligned} \mathrm{Cpl}^{\mathrm{Black}}(0, T_1, T_2, K) &= P(0, T_2)\tau \mathrm{Bl}(K, F_2(0), v_2(T_1))\,, \\ v_2(T_1)^2 &= \int_0^{T_1} \sigma_2^2(t)dt\,. \end{aligned}$$

Caplet Smile Modeling: Guided Tour II

The average volatility of the forward rate in $[0, T_1]$, i.e. $v_2(T_1)/\sqrt{T_1}$, does not depend on the strike K of the option. In this formulation, volatility is a characteristic of the forward rate underlying the contract, and has nothing to do with the nature of the contract itself. In particular, it has nothing to do with the strike K of the contract.

Caplet Smile Modeling: Guided Tour I

Now take two different strikes K_1 and K_2. Suppose that the market provides us with the prices of the two related caplets with the same underlying forward rates and the same maturity.
Does there exist a *single* volatility $v_2(T_1)$ such that both

$$\mathrm{Cpl}^{\mathrm{MKT}}(0, T_1, T_2, K_1) = P(0, T_2)\tau \mathrm{Bl}(K_1, F_2(0), v_2(T_1))$$
$$\mathrm{Cpl}^{\mathrm{MKT}}(0, T_1, T_2, K_2) = P(0, T_2)\tau \mathrm{Bl}(K_2, F_2(0), v_2(T_1))$$

hold? The answer is a resounding "no". In general, market caplet prices do not behave like this. What one sees when looking at the market is that two *different* volatilities $v_2(T_1, K_1)$ and $v_2(T_1, K_2)$ are required to match the observed market prices if one is to use Black's formula:

$$\mathrm{Cpl}^{\mathrm{MKT}}(0, T_1, T_2, K_1) = P(0, T_2)\tau \mathrm{Bl}(K_1, F_2(0), v_2^{\mathrm{MKT}}(T_1, K_1)),$$
$$\mathrm{Cpl}^{\mathrm{MKT}}(0, T_1, T_2, K_2) = P(0, T_2)\tau \mathrm{Bl}(K_2, F_2(0), v_2^{\mathrm{MKT}}(T_1, K_2)).$$

Caplet Smile Modeling: Guided Tour II

In other terms, each caplet market price requires its own Black volatility $v_2^{\mathrm{MKT}}(T_1, K)$ depending on the caplet strike K.
The market therefore uses Black's formula simply as a metric to express caplet prices as volatilities.

Caplet Smile Modeling: Guided Tour I

The curve $K \mapsto v_2^{\text{MKT}}(T_1, K)/\sqrt{T_1}$ is the so called volatility smile of the T_1-expiry caplet. If Black's formula were consistent along different strikes, this curve would be flat, since volatility should not depend on the strike K. Instead, this curve is commonly seen to exhibit "smiley" or "skewed" shapes.
Clearly, only some strikes $K = K_i$ are quoted by the market, so that usually the remaining points have to be determined through interpolation or through an alternative model. Interpolation in K, for a fixed expiry T_1, can be easy but it does not give any insight as to the underlying forward-rate dynamics compatible with such prices.

Caplet Smile Modeling: Guided Tour I

Let p_2 be the density of $F_2(T_1)$ under the T_2-forward measure (if Black's formula were ok, this density would be lognormal). It is easy to see that (Breeden and Litzenberger (1978)),

$$\frac{\partial^2 \mathrm{Cpl}^{\mathrm{MKT}}(0, T_1, T_2, K)}{\partial K^2} = P(0, T_2)\tau p_2(K),$$

so that by differentiating the interpolated-prices curve we can find p_2.

Caplet Smile Modeling: Guided Tour I

This is based on the following: we know that (differentiation is in the sense of distributions)

$$(d/dK)[(F-K)^+] = -1_{\{K<F\}}, \quad (d^2/dK^2)[(F-K)^+] = \delta(K-F)$$

where δ is the Dirac delta function centered in 0. Now,

$$\frac{\partial^2 \mathrm{Cpl}^{\mathrm{MKT}}(0, T_1, T_2, K)}{\partial K^2} = P(0, T_2)\tau \frac{\partial^2 E_0^2[(F_2(T_1)-K)^+]}{\partial K^2},$$

$$\frac{\partial^2 E_0^2[(F_2(T_1)-K)^+]}{\partial K^2} = E_0^2\left[\frac{\partial^2 (F_2(T_1)-K)^+}{\partial K^2}\right] =$$

$$= E_0^2[\delta(K - F_2(T_1))] = \int \delta(K-x) p_2(x) dx = p_2(K)$$

Thus Breeden and Litzenberger's result ensures that by by differentiating the interpolated-prices curve we can find the density p_2.

Caplet Smile Modeling: Guided Tour II

However, the method of interpolation may interfere with the recovery of the density, since a second derivative of the interpolated curve is involved. Moreover, what kind of F dynamics does the density p_2 come from?

Caplet Smile Modeling: Guided Tour I

$$\frac{\partial^2 \mathrm{Cpl}^{\mathrm{MKT}}(0, T_1, T_2, K)}{\partial K^2} = P(0, T_2)\tau p_2(K).$$

Starting from this result, Dupire looks for a diffusion coefficient for an assumed diffusion dynamics of the underlying such that also the derivatives of prices with respect to the time-to-maturity are retrieved. Substantially, by assuming also a continuum of traded maturities, a further differentiation with respect to the time to maturity may lead to the possibility to invert the Kolmogorov forward (or Fokker-Planck) equation for the assumed diffusion, thus retrieving the diffusion coefficient from knowledge of the density evolution consistent with the market quotations.

Caplet Smile Modeling: Guided Tour I

There is a problem in case of the caplet market, though. Indeed, it makes no sense to assume a continuum of traded maturities for options on the forward rate F_2. The only instant of interest in a forward rate is typically its reset date T_1, since at that instant it becomes a LIBOR rate. And payoffs contain LIBOR rates, not Forward-LIBOR rates. This means that we might have caplets on $L(T_1, T_2) = F_2(T_1)$ (maturity T_2), $L(T_2, T_3) = F_3(T_2)$ (maturity T_3), $L(T_3, T_4) = F_4(T_3)$ (maturity T_4) and so on. But the forward rates involved are different, so we cannot assume to have options on more maturities T_2, T_3, T_4... for the same F, as Dupire's method would require. This can work in the equity or FX market, where the asset is always the same.

Caplet Smile Modeling: Guided Tour I

Dupire's general method would require to have options on more maturities $T_2, T_3, T_4 \ldots$ for the same F, which is not the case in the interest-rate option market.

Dupire's method is in fact nonparametric, since it aims at deriving a diffusion coefficient as a function of a whole market surface in maturity and strike.

We need to work only in the strike dimension, since maturity is fixed for a caplet.

We may then proceed the other way around (parametric-dynamics approach)

We assume a dynamics a priori, depending on given parameters.

We price options with the right maturity with said dynamics.

Prices will depend on the parameters

We set the parameters so as to match the relevant options prices for the given maturities. In detail:

Caplet Smile Modeling: Guided Tour I

A partial answer to these issues can be given the other way around, by starting from a **parametric alternative dynamics**

$$dF(t;T_1,T_2) = \nu(t, F(t;T_1,T_2))\, dW_t \tag{45}$$

This alternative dynamics generates a smile, which is obtained as follows.

Caplet Smile Modeling: Guided Tour I

1. Set K to a starting value;
2. Compute the model caplet price

$$\Pi(K) = P(0,T_2)\tau E_0^2(F(T_1;T_1,T_2)-K)^+$$

   with F obtained through the alternative dynamics (45).
3. Invert Black's formula for this strike, i.e. solve

$$\Pi(K) = P(0,T_2)\tau \mathrm{Bl}(K_1, F_2(0), v(K))$$

   in $v(K)$.
4. Change K and restart from point 2.

The fact that the alternative dynamics is not lognormal implies that we obtain a smile curve $K \mapsto v(K)/\sqrt{T_1}$ that is not flat.
Calibration: Choose $\nu(\cdot,\cdot)$ so that $v(K)$ is as close as possible to $v_2^{\mathrm{MKT}}(T_1, K)$ for all quoted K.

Alternative dynamics for the smile I

$$dF(t;T_1,T_2) = \nu(t, F(t;T_1,T_2))\, dW_t$$

ν can be either a deterministic or a stochastic function of F. In the latter case we would be using a so called “stochastic-volatility model”, where for example

$$\nu(t,F) = \sqrt{\xi(t)}F,\ \ d\xi(t) = k(\theta - \xi(t))dt + \eta\sqrt{\xi(t)}dZ(t),$$

with $dZdW = \rho_{W,Z}dt$. Volatility acquires a “stochastic life”.
Here we will concentrate on a deterministic $\nu(t,\cdot)$, leading to “local-volatility models” such as for example $\nu(t,F) = \sigma_2(t)F^\gamma$ (CEV model), where $0 \leq \gamma \leq 1$ and where σ_2 is deterministic.
The only exception will be the SABR model, that is a stochastic volatility model where $\sqrt{\xi(t)} = V_t$ where V is a new stochastic process given by a driftless geometric browian motion, $dV_t = \epsilon V_t dZ_t$.

Alternative dynamics for the smile I

- Local volatility models have the problem that the smile in the future, conditional on future information, tends to flatten.
- For example, conditional on a future time $u > 0$, consider the smile for the maturity $u + T$ conditional on the information at u.
- As u moves forward, the smile for maturity $T + u$ tends to flatten with local volatility models.
- Instead, stochastic volatility models are capable of not flattening the smile as u moves forward, and this is considered to be important.

Summing up the smile problem I

Summing up: The “true” forward-adjusted density p_2 of F_2 is linked to Caplet (Call on F) or Floorlet (Put on F) market prices through second-order differentiation wrt strikes.
Need dF dynamics as compatible as possible with density p_2.
Dupire works on p's extracted from prices through interpolation rather than on prices directly, and based on this obtains dF. However, interpolation interferes strongly with the result and the method is unstable;
One can instead parameterize dF and fit the prices implied by the parameterized dF to the market prices $\text{Cpl}^{\text{MKT}}(0, T_1, T_2, K_i)$ for the quoted strikes K_i.
The problem is that the parameterization has to be flexible and has to lead to a tractable model.
We finally point out that one has to deal, in general, with an implied-volatility surface, since we have a caplet-volatility curve for

Summing up the smile problem II

each considered expiry. The calibration issues, however, are essentially unchanged, apart from the obviously larger computational effort required when trying to fit a bigger set of data.

Shifted lognormal (displaced diffusion) model for the smile I

A very simple way of constructing forward-rate dynamics that implies non-flat volatility structures is by shifting the generic lognormal dynamics. Indeed, let us assume that the forward rate F_j evolves, under its associated T_j-forward measure, according to

$$F_j(t) = X_j(t) + \alpha, \quad dX_j(t) = \beta(t) X_j(t)\, dW_t,$$

where α is a real constant, β is a deterministic function of time and W is a standard Brownian motion. We have

$$dF_j(t) = \beta(t)(F_j(t) - \alpha)\, dW_t.$$

The distribution of $F_j(T)$, conditional on $F_j(t)$, $t < T \leq T_{j-1}$, is a shifted lognormal distribution. The resulting model for F_j preserves the

Shifted lognormal (displaced diffusion) model for the smile II

analytically tractability of the geometric Brownian motion X. Notice indeed that

$$E_t^j\{[F_j(T_{j-1}) - K]^+\} = E_t^j\{[X_j(T_{j-1}) - (K - \alpha)]^+\},$$

so that, for $\alpha < K$, the caplet price $\mathrm{Cpl}(t, T_{j-1}, T_j, K)$ is simply given by

$$\tau P(t, T_j)\mathrm{Bl}\left(K - \alpha, F_j(t) - \alpha, \left(\int_t^{T_{j-1}} \beta^2(u)du\right)^{1/2}\right).$$

Shifted lognormal model for the caplet smile I

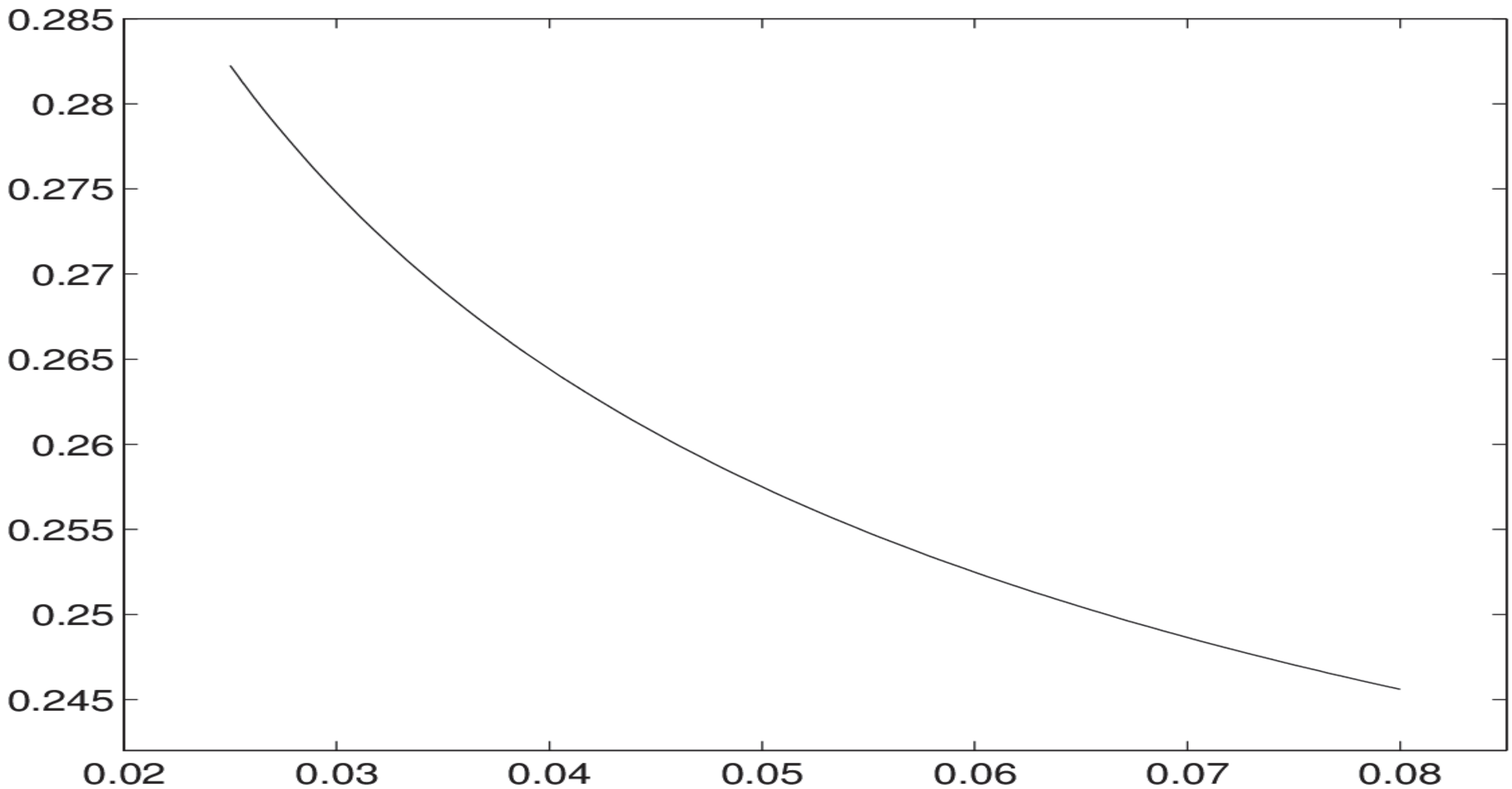
$$dF_j(t) = \beta(t)(F_j(t) - \alpha)\, dW_t.$$

The implied Black vol $\hat{v}/\sqrt{T_{j-1}} = \hat{v}(K,\alpha)/\sqrt{T_{j-1}}$ (say at $t = 0$) is obtained by backing out the volatility parameter $\hat{v}$ in Black's formula that matches the model price:

$$\text{Bl}(K, F, \hat{v}(K,\alpha)\,) = \text{Bl}\left(K - \alpha, F - \alpha, \left(\int_0^{T_{j-1}} \beta^2(u)du\right)^{1/2}\right).$$

with $F = F_j(0)$. An example of the skewed volatility structure $K \mapsto \hat{v}(K,\alpha)/\sqrt{T_1}$ is shown below.

Caplet volatility structure $K \mapsto \hat{v}(K,\alpha)/\sqrt{T_{j-1}}$ implied, at time $t=0$, by the forward-rate dynamics above where we set $T_{j-1}=1$, $T_j=1.5$, $\alpha=-0.015$, $\beta(t)=0.2$ for all t and $F_j(0)=0.055$



Shifted lognormal model for the caplet smile I

$$dF_j(t) = \beta(t)(F_j(t) - \alpha)\, dW_t.$$

Introducing a non-zero α has two effects on the implied caplet volatility structure, which for $\alpha = 0$ is flat at the constant level.
First, it leads to a strictly decreasing ($\alpha < 0$) or increasing ($\alpha > 0$) curve.
Second, it moves the curve upwards ($\alpha < 0$) or downwards ($\alpha > 0$).
More generally, ceteris paribus, increasing α shifts the volatility curve $K \mapsto \hat{v}(K, \alpha)$ down, whereas decreasing α shifts the curve up.
Shifting a lognormal diffusion can then help in recovering skewed volatility structures. However, such structures are often too rigid, and highly negative slopes are impossible to recover.
Moreover, the best fitting of market data is often achieved for decreasing implied volatility curves, which correspond to negative

Shifted lognormal model for the caplet smile II

values of the α parameter, and hence to a support of the forward-rate density containing negative values. Even though the probability of negative rates may be negligible in practice, many people regard this drawback as an undesirable feature.

CEV model for the caplet smile I

CEV model of Cox (1975) and Cox and Ross (1976).

$$dF_j(t) = \sigma_j(t)[F_j(t)]^\gamma \, dW_t,$$
$$F_j = 0 \;\text{ absorbing boundary when }\; 0 < \gamma < 1/2.$$

For $0 < \gamma < 1/2$ this equation does not have a unique solution unless we specify a boundary condition at $F_j = 0$. This is why we take $F_j = 0$ as an absorbing boundary.
Time dependence of σ_j can be dealt with through a deterministic time change. Indeed, by setting

$$v(\tau, T) = \int_\tau^T \sigma_j(s)^2 ds, \quad \widetilde{W}(v(0,t)) := \int_0^t \sigma_j(s) dW(s),$$

CEV model for the caplet smile II

we obtain a Brownian motion $\widetilde{W}$ with time parameter v. We substitute this time change in the above equation by setting $f_j(v(t)) := F_j(t)$ and obtain

$$df_j(v) = f_j(v)^\gamma d\widetilde{W}(v), \quad f_j = 0 \text{ abs boun when } 0 < \gamma < 1/2.$$

This can be transformed into a Bessel process via a change of variable. Straightforward manipulations lead then to the transition density of $F_j(T)$ conditional on $F_j(t)$, $t < T \leq T_{j-1}$ (noncentral chi squared).

CEV model for the caplet smile I

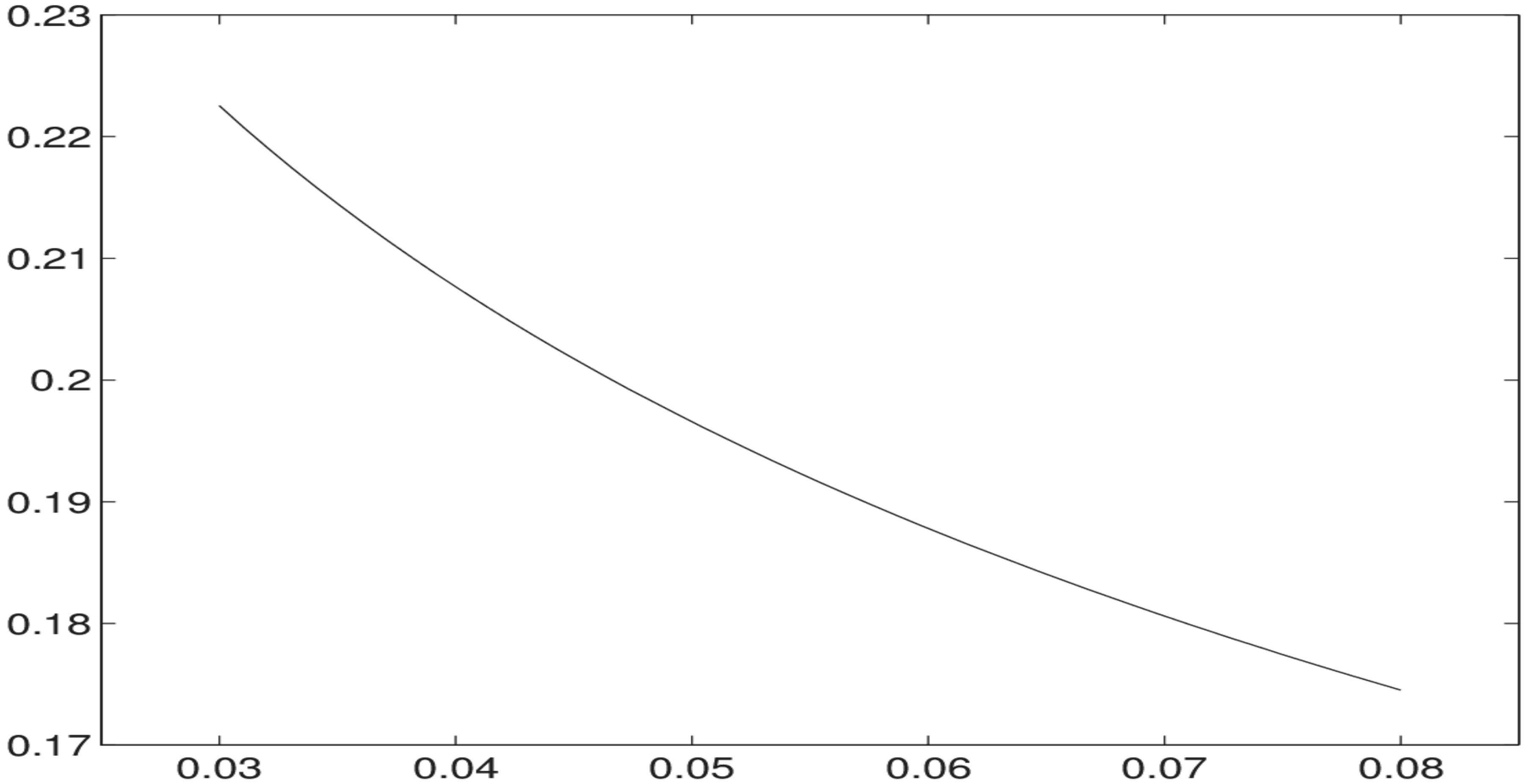
$$dF_j(t) = \sigma_j(t)[F_j(t)]^\gamma \, dW_t, \;\; F_j = 0 \;\text{ abs b when }\; 0 < \gamma < 1/2.$$

This model features analytical tractability, allowing for the known n.c.-χ^2 transition density. The following explicit formula can be derived:
$\mathrm{Cpl}(t, T_{j-1}, T_j, \tau, N, K) =$

$$\tau N P(t,T_j)\left[F_j(t)\left(1-\chi^2\left(2K^{1-\gamma};\frac{1}{1-\gamma}+2,2u\right)\right)\right.$$
$$\left.-K\chi^2\left(2u;\frac{1}{1-\gamma},2kK^{1-\gamma}\right)\right]$$

$$k = \frac{1}{2v(t,T)(1-\gamma)^2}, \quad u = k[F_j(t)]^{2(1-\gamma)}.$$

Caplet volatility structure implied by CEV at time $t = 0$, where we set $T_{j-1} = 1$, $T_j = 1.5$, $\sigma_j(t) = 1.5$ for all t, $\gamma = 0.5$ and $F_j(0) = 0.055$.



CEV model for the caplet smile I

As previously done in the case of a geometric Brownian motion, an extension of the above model can be proposed based on displacing the CEV process. The introduction of the extra parameter α determining the density shifting may improve the calibration to market data. Finally, there is the possibly annoying feature of absorption in $F = 0$. While this does not necessarily constitute a problem for caplet pricing, it can be an undesirable feature from an empirical point of view. Also, it is not clear whether there could be some problems when pricing more exotic structures. As a remedy to this absorption problem, Andersen and Andreasen (2000) propose a "Limited" CEV (LCEV) process, where instead of $\phi(F) = F^\gamma$ they set

$$\phi(F) = F \ \min(\epsilon^{\gamma-1}, F^{\gamma-1}) \ ,$$

where ϵ is a small positive real number.

CEV model for the caplet smile II

As far as the calibration of the CEV model to swaptions is concerned, approximated swaption prices based on “freezing the drift” and “collapsing all measures” are also derived (analogous to the lognormal case in the LMM). See Andersen and Andreasen (2000) for the details.

Brigo & Mercurio's Mixture Dynamics for the caplet smile I

For each time t let us consider N lognormal densities

$$p_t^i(y) = \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{F_j(0)} + \frac{1}{2} V_i^2(t) \right]^2 \right\},$$

$$V_i(t) := \sqrt{\int_0^t \sigma_i^2(u) du}, \quad p_0^i(x) = \delta(x - F_j(0)),$$

where all σ_i's are positive and deterministic time functions.
Brigo and Mercurio (2000a) showed that it is possible to determine the *local volatility* σ in the Q^j-forward-rate dynamics:

$$dF_j(t) = \sigma^{\text{mix}}(t, F_j(t))F_j(t)\, dW_t,$$

Brigo & Mercurio's Mixture Dynamics for the caplet smile II

In such a way that the SDE admits a unique strong solution whose marginal density, at each time $t \leq T_{j-1}$, is given by the mixture of lognormals

$$p_{F_j(t)}(y) := \frac{d}{dy} Q^j\{F_j(t) \leq y\} = \sum_{i=1}^{N} \lambda_i p_t^i(y),$$

with $\lambda_i > 0$ such that $\sum_{i=1}^{N} \lambda_i = 1$. Notice:

$$\int_0^{+\infty} y p_t(y) dy = \sum_{i=1}^{N} \lambda_i \int_0^{+\infty} y p_t^i(y) dy = \sum_{i=1}^{N} \lambda_i F_j(0) = F_j(0).$$

B & M's Mixture dynamics for the caplet smile I

$$dF_j(t) = \sigma^{\text{mix}}(t, F_j(t))F_j(t)\, dW_t, \quad p_{F_j(t)}(y) := \sum_{i=1}^{N} \lambda_i p_t^i(y).$$

The local volatility $\sigma^{\text{mix}}(t,\cdot)$ is backed out from the Fokker-Planck equation associated with the above dynamics.

Assume that each σ_i is continuous and bounded from above and below by (strictly) positive constants, and that there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$. Then, if we set

$$\sigma^{\text{mix}}(t,y)^2 := \frac{\sum_{i=1}^{N} \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln \frac{y}{F_j(0)} + \frac{1}{2}V_i^2(t)\right]^2\right\}}{\sum_{i=1}^{N} \lambda_i \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln \frac{y}{F_j(0)} + \frac{1}{2}V_i^2(t)\right]^2\right\}},$$

B & M's Mixture dynamics for the caplet smile II

for $(t, y) > (0, 0)$ and $\nu(t, y) = \sigma_0$ for $(t, y) = (0, F_j(0))$, the above SDE has a unique strong solution whose marginal density is given by the above mixture of lognormals

B & M's Mixture dynamics for the caplet smile I

$$dF_j(t) = \sigma^{\text{mix}}(t, F_j(t))F_j(t)\, dW_t, \quad p_{F_j(t)}(y) := \sum_{i=1}^{N} \lambda_i p_t^i(y).$$

$\sigma^{\text{mix}}(t, y)^2$ can be viewed as a weighted average of the squared basic volatilities $\sigma_1^2(t), \ldots, \sigma_N^2(t)$, where the weights are all functions of the chosen lognormal basic densities:

$$\sigma^{\text{mix}}(t, y)^2 = \sum_{i=1}^{N} \Lambda_i(t, y)\sigma_i^2(t), \quad \Lambda_i(t, y) := \frac{\lambda_i p_t^i(y)}{\sum_{i=1}^{N} \lambda_i p_t^i(y)}.$$

As a consequence, for each $t > 0$ and $y > 0$, the function σ^{mix} is bounded from below and above by (strictly) positive constants.

$$\sigma_* \leq \sigma^{\text{mix}}(t, y) \leq \sigma^* \quad \text{for each } t, y > 0,$$

B & M's Mixture dynamics for the caplet smile II

$$\sigma_* = \inf_{t \geq 0} \min_{i=1,\ldots,N} \sigma_i(t) > 0, \quad \sigma^* = \sup_{t \geq 0} \max_{i=1,\ldots,N} \sigma_i(t) < +\infty.$$

The function $\sigma^{\text{mix}}(t, y)$ can be extended by continuity to $\{(0, y) : y > 0\}$ and $\{(t, 0) : t \geq 0\}$ by setting $\sigma^{\text{mix}}(0, y) = \sigma_0$ and $\sigma^{\text{mix}}(t, 0) = \nu^*(t)$, where $\nu^*(t) := \sigma_{i^*}(t)$ and $i^* = i^*(t)$ is such that $V_{i^*}(t) = \max_{i=1,\ldots,N} V_i(t)$. In particular, $\sigma^{\text{mix}}(0, 0) = \sigma_0$.
Indeed, for every $\bar{y} > 0$ and every $\bar{t} \geq 0$,

$$\lim_{t \to 0} \sigma^{\text{mix}}(t, \bar{y}) = \sigma_0, \quad \lim_{y \to 0} \sigma^{\text{mix}}(\bar{t}, y) = \nu^*(t).$$

B & M's Mixture dynamics for the caplet smile I

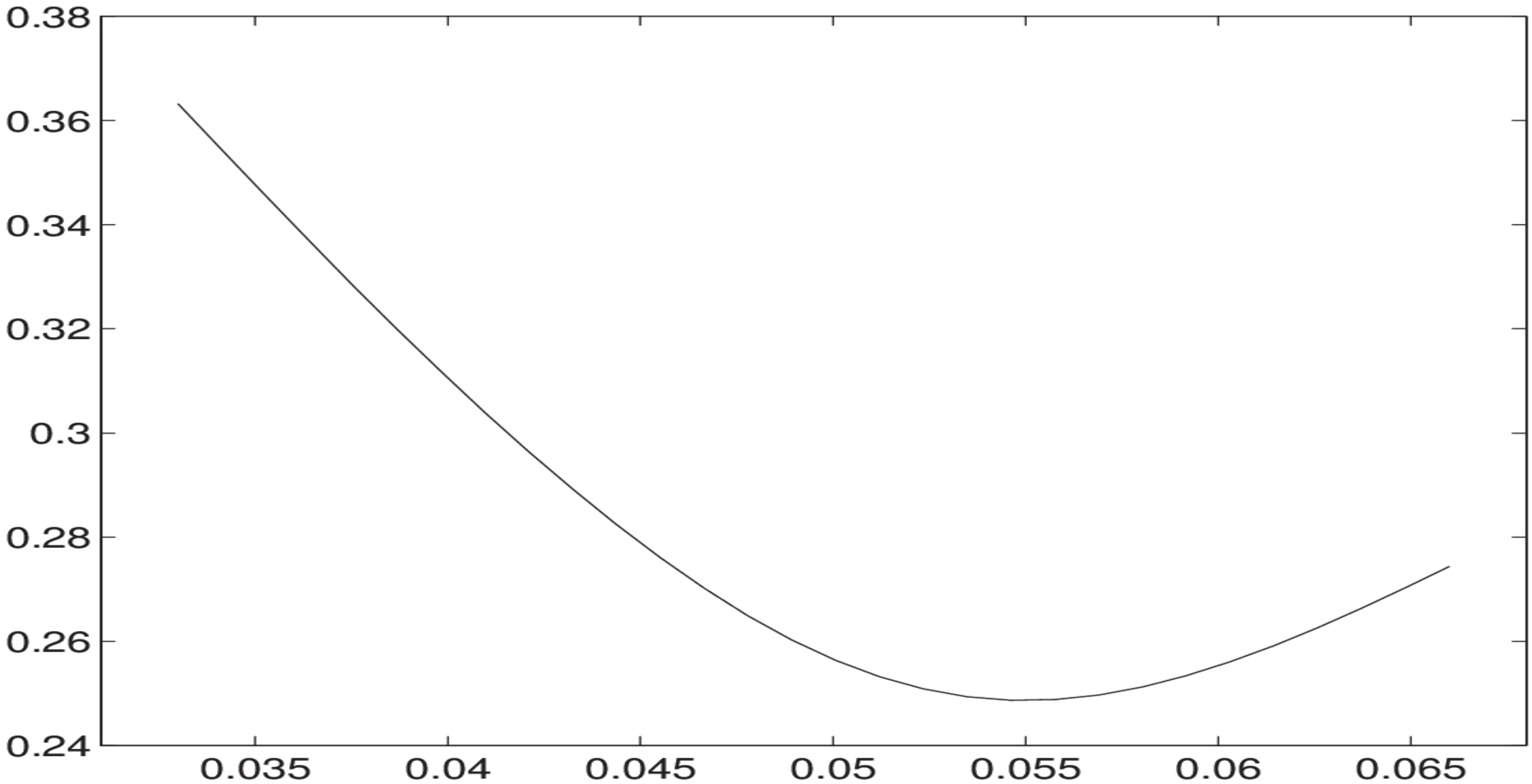
$$dF_j(t) = \sigma^{\text{mix}}(t, F_j(t))F_j(t)\, dW_t, \quad p_{F_j(t)}(y) := \sum_{i=1}^{N} \lambda_i p_t^i(y).$$

At time $t = 0$, $E^j\{[F_j(T_{j-1}) - K]^+\} =$

$$\int_0^{+\infty} (y - K)^+ p_{F(T_{j-1})}(y) dy = \sum_{i=1}^{N} \lambda_i \int_0^{+\infty} (y - K)^+ p^i_{T_{j-1}}(y) dy,$$

$$\text{Cpl}(0, T_{j-1}, T_j, K) = \tau P(0, T_j) \sum_{i=1}^{N} \lambda_i \text{Bl}(K, F_j(0), V_i(T_{j-1})).$$

Caplet smile implied by the mixture dynamics with $T_{j-1} = 1$, $N = 3$, $(V_1(1), V_2(1), V_3(1)) = (0.6, 0.1, 0.2)$, $(\lambda_1, \lambda_2, \lambda_3) = (0.2, 0.3, 0.5)$ and $F_j(0) = 0.055$



B & M's Mixture dynamics for the caplet smile I

When proposing alternative dynamics, it can be quite problematic to come up with analytical formulas for caplet prices. Here, instead, such problem can be avoided from the beginning, just because the use of analytically-tractable densities $p^i_{T_{j-1}}$ immediately leads to explicit caplet prices for the process F_j. **This is fundamental for calibration purposes**.

Moreover, the absence of bounds on the parameter N implies that **a virtually unlimited number of parameters can be introduced in the forward-rate dynamics** and used for a better calibration to market data.

A last remark concerns the classic economic interpretation of a mixture of densities. We can indeed view F_j as a process whose density at time t coincides with the basic density $p^i_{T_{j-1}}$ with probability λ_i. This is related to an **uncertain volatility model** of which the diffusion model we presented is a projection on 1-dimensional diffusions.

B & M's shifted Mixture dynamics for the smile I

The earlier mixture dynamics implies that the vol smile has a minimum at the at-the-money forward level, i.e. for $K = F_j(0)$. Brigo and Mercurio (2000b) proposed a simple way to generalize the mixture dynamics in order to introduce more asymmetry and shift the minimum. The basic lognormal-mixture model is combined with the displaced-diffusion technique. Set

$$F_j(t) = \alpha + \bar{F}_j(t), \quad dF_j(t) = \sigma^{\text{mix}}\left(t, F_j(t) - \alpha\right)(F_j(t) - \alpha)dW_t$$

where α is a real constant and $\bar{F}_j$ evolves according to the basic "lognormal mixture" dynamics.

The analytical expression for the marginal density of such process is given by the shifted mixture of lognormals $p_{F_j(t)}(y) =$

$$= \sum_{i=1}^{N} \lambda_i \frac{1}{(y-\alpha)V_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y-\alpha}{F_j(0)-\alpha} + \tfrac{1}{2}V_i^2(t)\right]^2\right\}, \quad y > \alpha.$$

B & M's shifted Mixture dynamics for the smile I

$$dF_j(t) = \sigma^{\text{mix}}\left(t, F_j(t) - \alpha\right)(F_j(t) - \alpha)dW_t$$

This model for the forward-rate process preserves the analytical tractability of the original process $\bar{F}_j$. Indeed,

$$E^j\left\{[F_j(T_{j-1}) - K]^+\right\} = E^j\left\{\left[\bar{F}(T_{j-1}) - (K - \alpha)\right]^+\right\},$$

so that, for $\alpha < K$, we have $\text{Cpl}(0, T_{j-1}, T_j, K) =$

$$= \tau P(t, T_j)\sum_{i=1}^{N}\lambda_i \text{Bl}(K - \alpha, F_j(0) - \alpha, V_i(T_{j-1})).$$

B & M's shifted Mixture dynamics for the smile II

The introduction of α has the effect that, decreasing α, the variance of the asset price at each time increases while maintaining the correct expectation. Indeed, $E(F_j(t)) = F_j(0)$ and

$$\mathrm{Var}(F_j(t)) = (F_j(0) - \alpha)^2 \left(\sum_{i=1}^{N} \lambda_i e^{V_i^2(t)} - 1 \right).$$

α affects the implied vol curve. First, the level: changing α leads to an almost parallel shift. Second, it moves the strike with minimum volatility: if $\alpha > 0$ (< 0) the minimum is attained for strikes lower (higher) than the ATM's $F_j(0)$. In general α can be used to add asymmetry without shifting the curve. Finally, once again approximated swaption prices based on the "freezing the drift" approach can be attempted.

THE SABR MODEL I

Hagan, Kumar, Lesniewski and Woodward (2002) propose a stochastic-volatility model for the evolution of the forward price of an asset under the asset's canonical measure.

This model is widely used in practice because of its simplicity and tractability (but brace for Horror stories!!).

Here, we apply the model to forward rates. Precisely, the forward rate F_k is assumed to evolve under the associated measure Q^k as

$$\begin{aligned} dF_k(t) &= V(t)F_k(t)^\beta \, dZ_k(t), \\ dV(t) &= \epsilon V(t)\, dW_k(t), \\ V(0) &= \alpha, \end{aligned}$$

where Z_k and W_k are Q^k-standard Brownian motions with $dZ_k(t)\, dW_k(t) = \rho\, dt$ and where $\beta \in (0, 1]$, ϵ and α are positive constants and $\rho \in [-1, 1]$.

THE SABR MODEL I

Using singular perturbation techniques, a closed-form approx for the price at time $t = 0$ of a T_k-maturity caplet is

$$\mathrm{Cpl}(0, T_{k-1}, T_k, \tau_k, K) = \tau_k P(0, T_k)[F_k(0)\Phi(d_+) - K\Phi(d_-)]$$

$$d_\pm = \frac{\ln(F_k(0)/K) \pm \frac{1}{2}\sigma^{\mathrm{imp}}(K, F_k(0))^2 T_{k-1}}{\sigma^{\mathrm{imp}}(K, F_k(0))\sqrt{T_{k-1}}}$$

$$\sigma^{\mathrm{imp}}(K,F) = \frac{\alpha}{(FK)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2}{24}\ln^2\left(\frac{F}{K}\right) + \frac{(1-\beta)^4}{1920}\ln^4\left(\frac{F}{K}\right) + \cdots\right]}\frac{z}{x(z)}$$
$$\cdot\left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24(FK)^{1-\beta}} + \frac{\rho\beta\epsilon\alpha}{4(FK)^{\frac{1-\beta}{2}}} + \epsilon^2\frac{2-3\rho^2}{24}\right]T_{k-1} + \cdots\right\},$$

THE SABR MODEL I

with

$$z := \frac{\epsilon}{\alpha}(FK)^{\frac{1-\beta}{2}} \ln\left(\frac{F}{K}\right),$$

$$x(z) := \ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.$$

THE SABR MODEL I

The ATM (caplet) implied volatility is immediately obtained by setting $K = F = F_k(0)$:

$$\sigma^{ATM} = \sigma^{\text{imp}}(F_k(0), F_k(0)) = \frac{\alpha}{F_k(0)^{1-\beta}} \cdot$$

$$\cdot \left\{ 1 + \left[\frac{(1-\beta)^2\alpha^2}{24 F_k(0)^{2-2\beta}} + \frac{\rho\beta\epsilon\alpha}{4F_k(0)^{1-\beta}} + \epsilon^2 \frac{2-3\rho^2}{24} \right] T_{k-1} + .. \right\}.$$

The ATM volatility, as a function of the forward rate $F_k(0)$, traces a curve that is called *backbone*.

The leading term in σ^{ATM} is $\alpha/F_k(0)^{1-\beta}$, meaning that α and β concur in determining both the level and slope of ATM implied volatilities (the other parameters have less relevant impacts).

THE SABR MODEL II

The SABR dynamics lead to skews in the implied volatilities both through a β smaller than one ("non-lognormal" case) and through a non-zero correlation.
In practice, it can be difficult to disentangle the contributions of the two parameters, since market implied volatilities can be fitted equally well by different choices of β ranging from zero (zero excluded) to one.

THE SABR MODEL I

Hagan et al. suggest to determine β either by a-priori choice (personal taste) or by historical calibration. In the latter case:

$$\ln \sigma^{ATM} = \ln \alpha - (1-\beta) \ln F_k(0) + \ln\{1 + \cdots\},$$

so that β can be found with a linear regression applied to a historical plot of $(\ln F_k(0), \ln \sigma^{ATM})$.

Remark 1. Hagan et al. postulate the evolution of a single forward asset. Their model, therefore, is not a proper extension of the LMM. In a LIBOR market model, in fact, not only has one to specify the joint evolution of forward rates under a common measure, but also to clarify the relations among the volatility dynamics of each forward rate.

Remark 2. The SABR model can be equivalently used for modeling a swap-rate evolution and, consequently, for the (analytical) pricing of swaptions. In fact, one can assume that under the swap measure $Q^{a,b}$

THE SABR MODEL II

$$
\begin{aligned}
dS_{a,b}(t) &= V(t)S_{a,b}(t)^{\beta}\, dZ^{a,b}(t),\\
dV(t) &= \epsilon V(t)\, dW^{a,b}(t), \quad V(0) = \alpha.
\end{aligned}
$$

In practice, this model is widely used by financial institutions to quote implied volatility smiles and skews for swaptions.

THE SABR MODEL I

We now consider an example of calibration of the SABR model to swaption volatilities and CMS swap spreads.
The reason why we resort to such a joint calibration is because implied volatilities by themselves do not allow to uniquely identify the four parameters of the SABR model.
In fact, several are the combinations of parameters β and ρ that produce (almost) equivalent fittings to the finite set of market volatilities available for given maturity and tenor.
Our examples of calibration, based on Euro data as of 28 September 2005, are performed by minimizing the sum of square percentage differences between model quantities (volatilities and CMS spreads) and the corresponding market ones.

THE SABR MODEL I

Swaption volatilities are quoted by the market for different strikes K as a difference $\Delta\sigma^{\text{M}}_{a,b}$ with respect to the ATM level

$$\Delta\sigma^{\text{M}}_{a,b}(\Delta K) := \sigma^{\text{M}}_{a,b}(K^{\text{ATM}} + \Delta K) - \sigma^{\text{ATM}}_{a,b}$$

usually for $\Delta K = \pm 200, \pm 100, \pm 50, \pm 25$ basis points.

The market also quotes the spread $X_{n,c}$ over LIBOR that sets to zero the value of a CMS swap paying the c-year swap rate on dates T'_i, $i = 1, \ldots, n$.

Denoting by $S'_{i,c}$ the c-year (forward) swap rate setting at $T'_i - \delta$, the spread is explicitly given in terms of CMS convexity adjustments as:

$$X_{n,c} = \frac{\sum_{i=1}^{n}\left(S'_{i,c}(0) + \mathbf{CA}(S'_{i,c};\delta)\right) P(0,T'_i)}{\sum_{i=1}^{n} P(0,T'_i)} - \frac{1 - P(0,T'_n)}{\delta\sum_{i=1}^{n} P(0,T'_i)}$$

where all the accrual periods are equal to $\delta = 3m$.

THE SABR MODEL I

We use the following formula for implied volatilities: $\sigma^{\text{imp}}(K, S_{a,b}(0)) \approx$

$$\approx \frac{\alpha}{(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\left[1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{S_{a,b}(0)}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{S_{a,b}(0)}{K}\right)\right]}$$

$$\cdot\frac{z}{x(z)}\left\{1+\left[\frac{(1-\beta)^2\alpha^2}{24(S_{a,b}(0)K)^{1-\beta}}+\frac{\rho\beta\epsilon\alpha}{4(S_{a,b}(0)K)^{\frac{1-\beta}{2}}}+\epsilon^2\frac{2-3\rho^2}{24}\right]T_a\right\},$$

where $z := \frac{\epsilon}{\alpha}(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\ln\left(\frac{S_{a,b}(0)}{K}\right)$ and

$x(z) := \ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.$

Even though this is only an approximation, it is market practice to consider it as exact and to use it as a functional form mapping strikes into implied volatilities.

THE SABR MODEL I

We then use the following formula for convexity adjustments:

$$\mathbf{CA}(S_{a,b};\delta) := E^{T_a+\delta}[S_{a,b}(T_a)] - S_{a,b}(0)$$

$$\approx S_{a,b}(0)\,\theta(\delta)\left(\frac{E^{a,b}(S^2_{a,b}(T_a))}{S^2_{a,b}(0)} - 1\right)$$

$$= S_{a,b}(0)\,\theta(\delta)\left(\frac{2}{S^2_{a,b}(0)}\int_0^{\infty}\mathrm{Bl}(K, S_{a,b}(0), v^{\mathrm{imp}}(K, S_{a,b}(0)))\,dK - 1\right),$$

$$v^{\mathrm{imp}}(K, S_{a,b}(0)) := \sigma^{\mathrm{imp}}(K, S_{a,b}(0))\sqrt{T_a}$$

$$\theta(\delta) := 1 - \frac{\tau S_{a,b}(0)}{1+\tau S_{a,b}(0)}\left(\delta + \frac{b-a}{(1+\tau S_{a,b}(0))^{b-a} - 1}\right)$$

and δ is the accrual period of the swap rate.

THE SABR MODEL I

				Strike			
Exp-Ten	-200	-100	-50	-25	25	50	100	200
1y - 10y	11.51%	3.24%	1.03%	0.37%	-0.22%	-0.22%	0.21%	2.13%
5y - 10y	7.80%	2.63%	1.02%	0.44%	-0.33%	-0.53%	-0.63%	-0.17%
10y - 10y	6.39%	2.25%	0.91%	0.40%	-0.31%	-0.52%	-0.71%	-0.47%
20y - 10y	5.86%	2.07%	0.85%	0.37%	-0.30%	-0.51%	-0.73%	-0.62%
30y - 10y	5.44%	1.92%	0.79%	0.35%	-0.29%	-0.52%	-0.79%	-0.85%
1y - 20y	9.45%	2.74%	1.17%	0.46%	-0.24%	-0.25%	0.15%	1.62%
5y - 20y	7.43%	2.56%	1.00%	0.43%	-0.32%	-0.51%	-0.60%	-0.10%
10y - 20y	6.59%	2.34%	0.94%	0.41%	-0.32%	-0.54%	-0.72%	-0.43%
20y - 20y	6.11%	2.19%	0.90%	0.40%	-0.32%	-0.55%	-0.77%	-0.61%
30y - 20y	5.46%	1.92%	0.79%	0.35%	-0.29%	-0.50%	-0.72%	-0.69%
1y - 30y	9.17%	2.67%	1.19%	0.47%	-0.25%	-0.27%	0.13%	1.58%
5y - 30y	7.45%	2.58%	1.01%	0.44%	-0.33%	-0.52%	-0.61%	-0.13%
10y - 30y	6.73%	2.38%	0.96%	0.42%	-0.33%	-0.53%	-0.68%	-0.35%
20y - 30y	6.20%	2.22%	0.91%	0.40%	-0.32%	-0.54%	-0.74%	-0.55%
30y - 30y	5.39%	1.90%	0.78%	0.35%	-0.28%	-0.50%	-0.72%	-0.68%

THE SABR MODEL II

Eur market volatility smiles across expiry, tenor and strike. Strikes are expressed as absolute differences in basis points *w.r.t* the at-the-money values.

THE SABR MODEL I

Expiry	Tenor 10y	20y	30y
1y	17.60%	15.30%	14.60%
5y	16.00%	14.80%	14.30%
10y	14.40%	13.60%	13.10%
20y	13.10%	12.10%	11.90%
30y	12.90%	12.30%	12.30%

Table: Market at-the-money volatilities.

THE SABR MODEL II

	Tenor		
Maturity	10y	20y	30y
5y	94.1	124.1	130.3
10y	82.0	104.8	110.6
15y	72.5	91.3	98.3
20y	66.7	84.2	92.9
30y	64.6	85.2	97.9

Table: Market CMS swap spreads in basis points.

THE SABR MODEL I

Remark. In practice, β needs to be bounded for a successful calibration. In fact, values of β approaching one lead to divergent values for convexity adjustments (for $\beta = 1$ the correction is infinite). As a numerical confirmation, we show below the CMS swap spreads $X_{n,10}(\beta)$ for a ten-year underlying swap rate and for different maturities n, after calibration, with fixed β, to whole swaption smile.

	β						
Maturity	0.2	0.3	0.4	0.5	0.6	0.7	0.8
5	93.4	94.0	94.0	94.0	94.1	94.1	94.9
10	80.6	81.3	81.5	81.8	82.2	83.0	85.3
15	70.4	71.6	72.1	72.9	74.3	78.5	129.8
20	63.0	65.8	66.6	68.1	71.2	82.1	306.1
30	56.2	62.0	63.7	66.6	73.5	104.2	1206.4

THE SABR MODEL: CALIBRATION RESULTS I

		Strike								
Expiry	**Tenor**	-200	-100	-50	-25	0	25	50	100	200
5y	10y	2.1	1.2	0.9	1.0	1.0	1.2	1.5	1.4	1.7
10y	10y	1.5	0.7	1.1	0.7	0.5	0.7	1.1	1.2	1.2
20y	10y	1.9	1.1	1.7	0.3	0.5	0.8	1.1	1.4	1.4
5y	20y	2.6	1.8	1.1	0.7	0.7	0.8	1.4	1.9	2.0
10y	20y	1.9	1.2	0.8	0.4	0.8	0.4	1.0	1.7	1.5
20y	20y	2.4	1.2	1.4	0.9	0.6	0.4	1.6	1.8	1.7
5y	30y	2.3	1.5	0.8	1.1	0.8	1.5	1.1	1.1	1.5
10y	30y	1.5	0.5	1.1	0.8	0.7	1.0	1.1	0.8	1.1
20y	30y	2.7	1.7	1.7	0.6	0.5	0.8	1.4	1.6	1.7

Absolute differences in bps between market and SABR implied volatilities.

THE SABR MODEL: CALIBRATION RESULTS II

	Tenor		
Maturity	10y	20y	30y
5y	0.1	0.2	0.9
10y	0.2	0.9	2.6
15y	0.4	1.0	3.3
20y	1.4	0.4	2.7
30y	2.1	0.2	1.5

Absolute differences in bps between market and SABR CMS swap spreads.

THE SABR MODEL: CALIBRATION RESULTS III

HORROR STORIES:

After the beginning of the financial crisis, with periods of Low rates and High Volatilities, the SABR expansion formula for implied volatility breaks.

Prices computed with that formula imply negative probability densities for forward and swap rates

Market is struggling to find a standard model to go beyond SABR

Herd mentality is part of the problem

Conclusions on LIBOR models with smile effects I

We have seen some possibilities to include (caplet) smile effects in the LIBOR market model by means of alternative dynamics:

- **Displaced Diffusion**. One parameter for each maturity, implies monotonic smile, can fit only few data, parametrically poor but analytically tractable.
- (Shifted) **CEV model**. One (two) parameter(s) for each maturity, monotonic smile, can fit only few data, parametrically poor but analytically tractable.
- (Shifted) **Mixture dynamics**. As many parameters as needed, non-monotonic smile, can fit several data, analytically tractable, interesting uncertain volatility version.

Conclusions on LIBOR models with smile effects II

- **SABR (stochastic Alpha Beta Rho) Model**. Stochastic volatility model. Very popular. Market oriented, used by brokers and practitioners. Does not flatten future smiles. Based on perturbation theory, not fully rigorous. Problems in extending it properly to a full LIBOR model for all tenors and maturities under a single pricing measure.

Conclusions on LIBOR models with smile effects I

Open problems:
Swaptions smile associated with the caplet-smile calibrated LIBOR model? Can one connect the two smiles, perhaps playing with instantaneous correlations?
Analytical approximation for swaption prices in the LIBOR models with smile? Partial answers for CEV and displaced diffusion...
More numerical tests, implied future smiles conditional on future realizations of underlying rates, diagnostics....

The crisis (2008-current). Multiple curves

Following the 7[8] credit events happening to Financials in one month of 2008,

Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir and Kaupthing [and Merrill Lynch]

the market broke up and interest rates that used to be very close to each other and were used to model risk free rates for different maturities started to diverge.

Multiple curves: LIBOR?

Credit/Default-free interest rates r_t, $L(t, T)$, $F(t, T_{i-1}, T_i)$ etc?

So it is not clear what is the risk free rate r_t anymore, but especially credit/default-free interest rates with finite (rather than infinitesimal) tenor $T - t$ are hard to define: What is the credit/default-free $L(t, T)$? In the above course we identified it with LIBOR interbank rates, ie interest rates banks charge each other for lending and borrowing. However, after the credit events above, banks can no longer be considered as default free, so that Interbank rates, and LIBOR rates in particular, are contaminated by counterparty credit risk and liquidity risk.

LIBOR has been also subject to illegal manipulation (see the LIBOR rigging scandal involving a number of major banks), but this is fraud risk and is another story.

Multiple curves: OIS?

Besides LIBOR, other rates have been considered as default/credit risk free rates in the past. One of the most popular is the overnight rate. This is an interest rate $O(t_{i-1}, t_i)$ applied at time t_{i-1} to a loan that is closed one or two days later at t_i. Hence the credit risk embedded in the overnight rate is only on one day and is limited. Furthermore, overnight rates are harder to manipulate illegally (some are quoted by central banks).

There are *swaps* built on overnight rates, and they are called Ovenight Indexed Swaps (OIS).

Multiple curves: OIS?

OIS have been introduced back in the mid nineties. The maturities T of OISs range from 1 week to 2 years or longer.

Overnight swaps

At maturity T, the swap parties calculate the final payment as a difference between the accrued interest of the fixed rate K and the geometric average $L^O(0, T)$ of the floating index rates $O(t_{i-1}, t_i)$ on the swap notional for t_i ranging from the initial time $t_{\text{first}} = 0$ to the swap maturity $t_{\text{last}} = T$. Since the net difference is exchanged, rather than swapping the actual rates, OISs have little counterparty credit risk.

Overnight swaps vs LIBOR indexed swaps: Counterparty risk

In a LIBOR based swap where we pay L and receive K, if our counterparty defaults (say with zero recovery) we still pay L and we lose the whole K. If the net rate were exhanged as in OIS, at default we would only lose $K - L$ if positive.

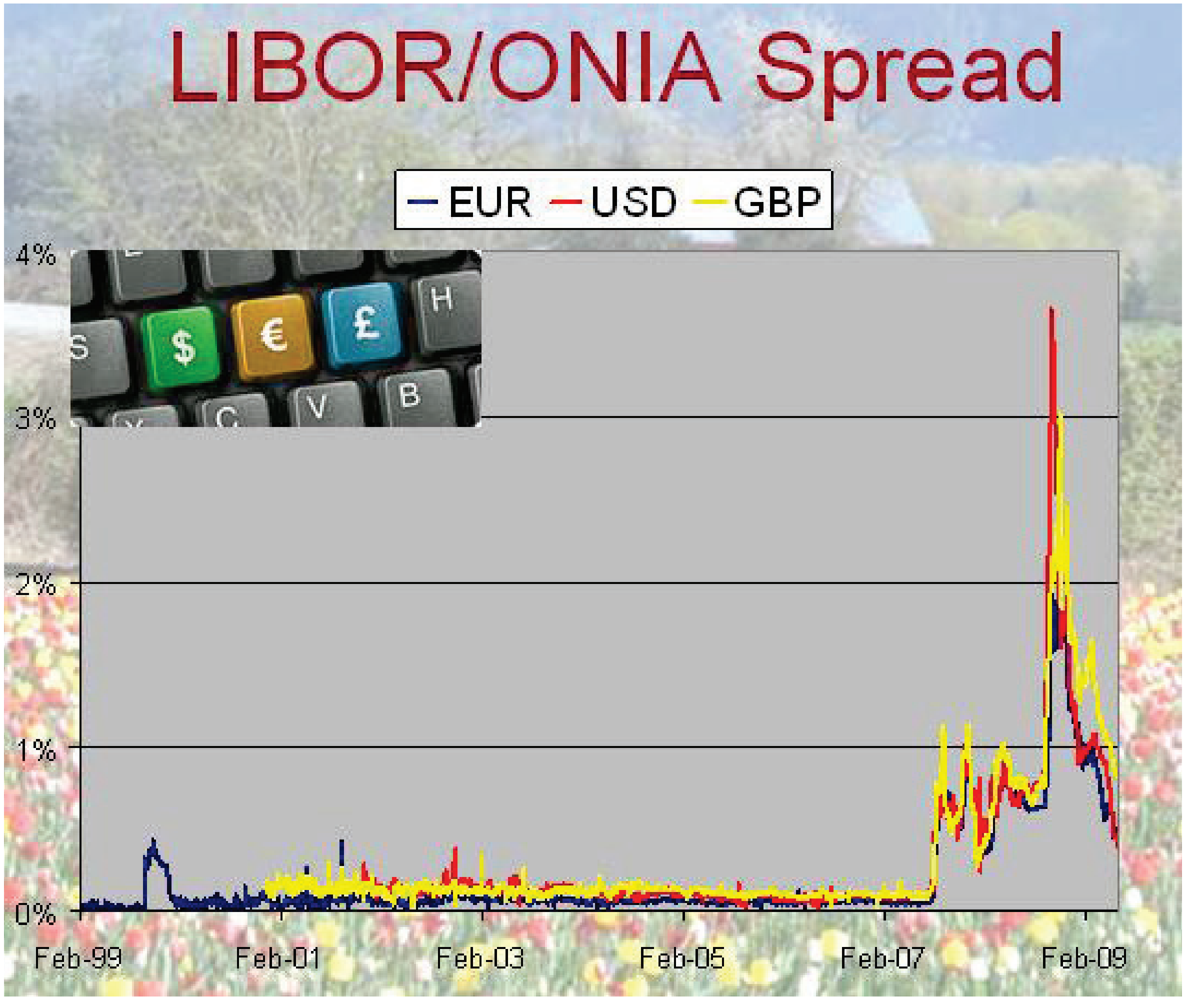


Figure: Spread between 3 months Libor and 3 months ONIA (OIS) swaps. Plotting $t \mapsto L(t, t + 3m) - L^O(t, t + 3m)$ (proxy of credit and liquidty risk). Taken from a talk of Aaron Brown (2011)

The crisis (2008-current). Multiple curves

At the moment it is no longer realistic to neglect credit risk and liquidity effects in interest rate modeling, pretending there is a risk free rate that is governing the LIBOR and interbank markets.

The OIS rate partly solves the problem as it is a best proxy for a default- and liquidity-free interest rate. Residual credit risk is still present and liquidity effects may still be visible, especially under strong stress scenarios.

These days one tends to use overnight swap rates as proxies for the risk free rates, whereas LIBOR and LIBOR-based swap rates have to be managed more carefully. **There are multiple curves that can be built for discounting, some LIBOR based, other OIS based, and yet other different ones.**

The following table is taken by a presentation of Marco Bianchetti (2011)

The crisis (2008-current). Multiple curves

	Libor	Euribor	Eonia	Eurepo
Definition	London InterBank Offered Rate	Euro InterBank Offered Rate	Euro OverNight Index Average	Euro Repurchase Ageement rate
Market	London Interbank	Euro Interbank	Euro Interbank	Euro Interbank
Side	Offer	Offer	Offer	Offer
Rate quotation specs	EURLibor = Euribor, Other currencies: minor differences (e.g. act/365, T+0, London calendar for GBPLibor).	TARGET calendar, settlement *T+2*, act/360, three decimal places, modified following, end of month, tenor variable.	TARGET calendar, settlement *T+1*, act/360, three decimal places, tenor 1d.	As Euribor
Maturities	1d-12m	1w, 2w, 3w,1m,...,12m	1d	T/N-12m
Publication time	12.30 CET	11:00 am CET	6:45-7:00 pm CET	As Euribor
Panel banks	8-16 banks (London based) per currency	42 banks from 15 EU countries + 4 international banks	Same as Euribor	34 EU banks plus some large international bank from non-EU countries
Calculation agent	Reuters	Reuters	European Central Bank	Reuters
Transactions based	No	No	Yes	No
Collateral	No (unsecured)	No (unsecured)	No (unsecured)	Yes (secured)
Counterparty risk	Yes	Yes	Low	Negligible
Liquidity risk	Yes	Yes	Low	Negligible
Tenor basis	Yes	Yes	No	No

The crisis (2008-current). Multiple curves

The uncertainty on which rate could be considered as a natural discounting rate is pushing banks to use multiple curves, trying to patch them together, at times in inconsistent ways.

Much work needs to be done to include consistently credit and liquidity effects in interest rate theory from the start, thus avoiding the confusion of unexplained multiple curves. The industry is looking at this now.

Multiple curves explained as synthesis of more fundamental Credit, Liquidity and Funding effects

Multiple curves explained as synthesis of more fundamental Credit, Liquidity and Funding effects.

Rather than taking the curves as fundamental objects, we need to interpret them as incorporating fundamental effects that need to be modeled first.

These effects are Credit Risk and Liquidity Funding Risk.

We face this challenge now.

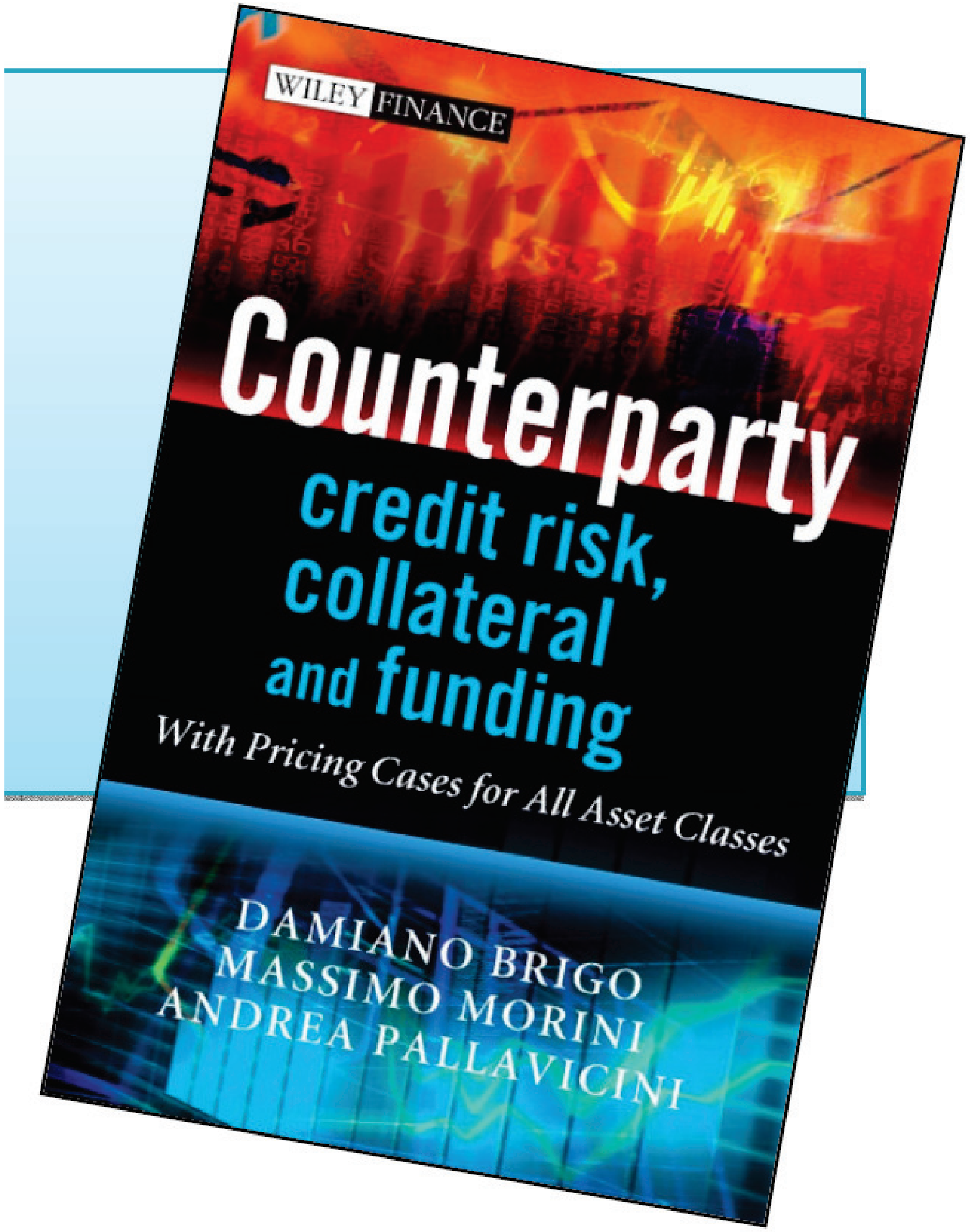
PART II: PRICING CREDIT RISK, COLLATERAL AND FUNDING

In this Part we look at how we may include Counterparty Credit Risk into the Valuation from the start rather than through unexplained ad-hoc discount (multiple) curves.

This leads to the notions of Credit and Debit Valuation Adjustments (CVA DVA).

We also hint at Funding Valuation Adjustments (FVA).

Presentation based on the Forthcoming Book



Intro to Basic Credit Risk Products and Models

Before dealing with the current topical issues of Counterparty Credit Risk, CVA, DVA and Funding, we need to introduce some basic elements of Credit Risk Products and Credit Risk Modelling.

We now briefly look at:

- Products: Credit Default Swaps (CDS) and Defaultable Bonds
- Payoffs and prices of such products
- Market implied $\mathbb{Q}$ probabilities of default defined by such models
- Intensity models and probabilities of defaults as credit spreads
- Credit spreads as possibly constant, curved or even stochastic
- Credit spread volatility (stochastic credit spreads)

Defaultable (corporate) zero coupon bonds

We started this course by defining the zero coupon bond price $P(t, T)$. Similarly to $P(t, T)$ being one of the possible fundamental quantities for describing the interest-rate curve, we now consider a defaultable bond $\bar{P}(t, T)$ as a possible fundamental variable for describing the defaultable market.

DEFAULT FREE		
time t		time T
:	$\longleftarrow$	:
$P(t, T)$		1

with DEFAULT		
time t		time T :
:	$\longleftarrow$	NO DEFAULT: 1
$\bar{P}(t, T)$		DEFAULT: 0

When considering default, we have a random time τ representing the time at which the bond issuer defaults. $\boxed{\tau \text{ : Default time of the issuer}}$

Defaultable (corporate) zero coupon bonds I

The value of a bond issued by the company and promising the payment of 1 at time T, as seen from time t, is the risk neutral expectation of the discounted payoff
BondPrice = Expectation[Discount x Payoff]

$$P(t,T) = \mathbb{E}\{D(t,T)\;1\;|\mathcal{F}_t\}, \qquad \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T) := \mathbb{E}\{D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\}$$

where $\mathcal{G}_t$ represents the flow of information on whether default occurred before t and if so at what time exactly, and on the default free market variables (like for example the risk free rate r_t) up to t. The filtration of default-free market variables is denoted by $\mathcal{F}_t$. Formally, we assume

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \le u\},\; 0 \le u \le t).$$

D is the stochastic discount factor between two dates, depending on interest rates, and represents discounting.

Defaultable (corporate) zero coupon bonds II

The "indicator" function $\mathbf{1}_{\text{condition}}$ is 1 if "condition" is satisfied and 0 otherwise. In particular, $\mathbf{1}_{\{\tau>T\}}$ reads 1 if default τ did not occur before T, and 0 in the other case.

We understand then that (ignoring recovery) $\mathbf{1}_{\{\tau>T\}}$ is the correct payoff for a corporate bond at time T: the contract pays 1 if the company has not defaulted, and 0 if it defaulted before T, according to our earlier stylized description.

Defaultable (corporate) zero coupon bonds

If we include a recovery amount REC to be paid at default τ in case of early default, we have as discounted payoff at time t

$$D(t,T)\mathbf{1}_{\{\tau>T\}} + \text{R}_{\text{EC}} D(t,\tau)\mathbf{1}_{\{\tau\leq T\}}$$

If we include a recovery amount REC paid at maturity T, we have as discounted payoff

$$D(t,T)\mathbf{1}_{\{\tau>T\}} + \text{R}_{\text{EC}} D(t,T)\mathbf{1}_{\{\tau\leq T\}}$$

Taking $\mathbb{E}[\cdot|\mathcal{G}_t]$ on the above expressions gives the price of the bond.

Fundamental Credit Derivatives: Credit Default Swaps

Credit Default Swaps are basic protection contracts that became quite liquid on a large number of entities after their introduction.

CDS's are now actively traded and have become a sort of basic product of the credit derivatives area, analogously to interest-rate swaps and FRA's being basic products in the interest-rate derivatives world.

As a consequence, the need is not to have a model to be used to value CDS's, but rather to consider a model that can be *calibrated* to CDS's, i.e. to take CDS's as model inputs (rather than outputs), in order to price more complex derivatives.

As for options, single name CDS options have never been liquid, as there is more liquidity in the CDS index options. We may expect models will have to incorporate CDS index options quotes rather than price them, similarly to what happened to CDS themselves.

Fundamental Credit Derivatives: CDS's

A CDS contract ensures protection against default. Two companies "A" (Protection buyer) and "B" (Protection seller) agree on the following.
If a third company "C" (Reference Credit) defaults at time τ, with $T_a < \tau < T_b$, "B" pays to "A" a certain (deterministic) cash amount L_{GD}.
In turn, "A" pays to "B" a rate R at times $T_{a+1}, \ldots, T_b$ or until default.
Set $\alpha_i = T_i - T_{i-1}$ and $T_0 = 0$.

Protection Seller B	$\rightarrow$	protection L_{GD} at default τ_C if $T_a < \tau_C \leq T_b$	$\rightarrow$	Protection Buyer A
	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\leftarrow$	

(protection leg and premium leg respectively). The cash amount L_{GD} is a *protection* for "A" in case "C" defaults. Typically L_{GD} = notional, or "notional - recovery" $= 1 - \text{R}_{\text{EC}}$.

Fundamental Credit Derivatives: CDS's

A typical stylized case occurs when "A" has bought a corporate bond issued by "C" and is waiting for the coupons and final notional payment from "C": If "C" defaults before the corporate bond maturity, "A" does not receive such payments. "A" then goes to "B" and buys some protection against this risk, asking "B" a payment that roughly amounts to the loss on the bond (e.g. notional minus deterministic recovery) that A would face in case "C" defaults.

Or again "A" has a portfolio of several instruments with a large exposure to counterparty "C". To partly hedge such exposure, "A" enters into a CDS where it buys protection from a bank "B" against the default of "C".