# Interest Rate Modelling and Derivative Pricing

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# Part II

# Yield Curves and Linear Products

# Outline

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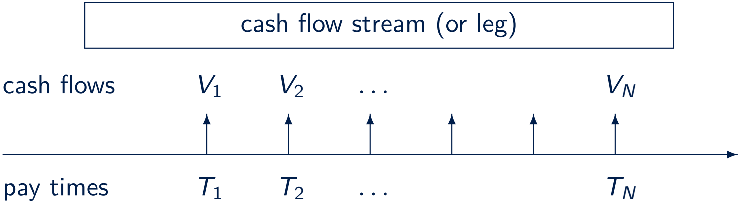
# DCF method requires knowledge of today's ZCB prices




- Assume $t = 0$ and deterministic cash flows, then

$$V(0) = \sum_{i=1}^{N} P(0, T_i) \cdot V_i .$$

How do we get today's ZCB prices $P(0, T_i)$?

# Yield curve is fundamental object for interest rate modelling

- A yield curve (YC) at an observation time $t$ is the function of zero coupon bonds $P(t,\cdot):[t,\infty)\to\mathbb{R}^+$ for maturities $T\geq t$.
- YCs are typically represented in terms of interest rates (instead of zero coupon bond prices).
- Discretely compounded zero rate curve $z_p(t,T)$ with frequency $p$, such that
$$P(t,T)=\left(1+\frac{z_p(t,T)}{p}\right)^{-p\cdot(T-t)}.$$
- Simple compounded zero rate curve $z_0(t,T)$ (i.e. $p=1/(T-t)$), such that
$$P(t,T)=\frac{1}{1+z_0(t,T)\cdot(T-t)}.$$
- Continuous compounded zero rate curve $z(t,T)$ (i.e. $p=\infty$), such that
$$P(t,T)=\exp\left\{-z(t,T)\cdot(T-t)\right\}.$$

# For interest rate modelling we also need continuous compounded forward rates

## Definition (Continuous Forward Rate)

Suppose a given observation time $t$ and zero bond curve $P(t, \cdot) : [t, \infty) \to \mathbb{R}^+$ for maturities $T \geq t$. The continuous compounded forward rate curve is given by

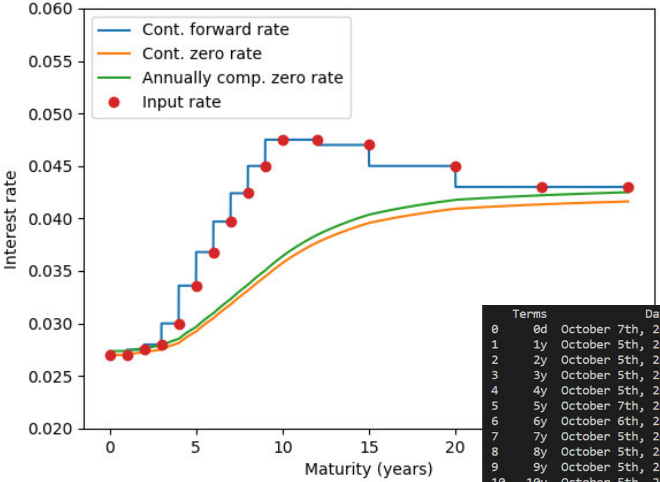
$$f(t, T) = -\frac{\partial \ln (P(t, T))}{\partial T}.$$

From the definition follows

$$P(t, T) = \exp \left\{ -\int_t^T f(t, s) ds \right\}.$$

- For static yield curve modelling and (simple) linear instrument pricing we are interested particularly in curves at $t = 0$.
- For (more complex) option pricing we are interested in modelling curves at $t > 0$.

# We show a typical yield curve example




| | Terms | Dates | Times | Rates |
|---|---|---|---|---|
| 0 | 0d | October 7th, 2019 | 0 | 0.027 |
| 1 | 1y | October 5th, 2020 | 0.99726 | 0.027 |
| 2 | 2y | October 5th, 2021 | 1.99726 | 0.0275 |
| 3 | 3y | October 5th, 2022 | 2.99726 | 0.028 |
| 4 | 4y | October 5th, 2023 | 3.99726 | 0.03 |
| 5 | 5y | October 7th, 2024 | 5.00548 | 0.0336 |
| 6 | 6y | October 6th, 2025 | 6.00274 | 0.0368 |
| 7 | 7y | October 5th, 2026 | 7 | 0.0397 |
| 8 | 8y | October 5th, 2027 | 8 | 0.0424 |
| 9 | 9y | October 5th, 2028 | 9.00274 | 0.045 |
| 10 | 10y | October 5th, 2029 | 10.0027 | 0.0475 |
| 11 | 12y | October 6th, 2031 | 12.0055 | 0.0475 |
| 12 | 15y | October 5th, 2034 | 15.0055 | 0.047 |
| 13 | 20y | October 5th, 2039 | 20.0082 | 0.045 |
| 14 | 25y | October 5th, 2044 | 25.0137 | 0.043 |
| 15 | 30y | October 5th, 2049 | 30.0164 | 0.043 |

# The market data for curve calibration is quoted by market data providers

```
                   Euribor vs 6 mth           3/6 basis        Swap Spreads
                                         Spot Starting Date       (Gadget)
1 Yr   -0.226/-0.266  16Yrs   1.295/1.255   1 Yr    4.30
2 Yrs  -0.128/-0.168  17Yrs   1.334/1.294   2 Yrs   4.80       5y  59.3
3 Yrs   0.010/-0.030  18Yrs   1.367/1.327   3 Yrs   5.35      10y  66.0
4 Yrs   0.154/0.114   19Yrs   1.393/1.353   4 Yrs   5.90
5 Yrs   0.293/0.253   20Yrs   1.415/1.375   5 Yrs   6.40
6 Yrs   0.429/0.389                         6 Yrs   6.70   Page live in
7 Yrs   0.558/0.518   21Yrs   1.432/1.392   7 Yrs   6.85   London hours ONLY
8 Yrs   0.678/0.638   22Yrs   1.446/1.406   8 Yrs   6.90   (between 0700 - 1800)
9 Yrs   0.790/0.750   23Yrs   1.457/1.417   9 Yrs   6.90
10Yrs   0.892/0.852   24Yrs   1.465/1.425   10Yrs   6.85
                      25Yrs   1.471/1.431  This page will close 30th April
11Yrs   0.983/0.943                        6.00pm and re open 7.00am 2nd May
12Yrs   1.064/1.024   26Yrs   1.476/1.436              10X12   0.192/0.152
13Yrs   1.135/1.095   27Yrs   1.480/1.440              10X15   0.378/0.338
14Yrs   1.197/1.157   28Yrs   1.484/1.444              10X20   0.543/0.503
15Yrs   1.250/1.210   29Yrs   1.486/1.446              10X25   0.599/0.559
                      30Yrs   1.488/1.448              10X30   0.616/0.576
                      35Yrs   1.491/1.451              10X35   0.619/0.579
                      40Yrs   1.486/1.446              10X40   0.614/0.574
                      45Yrs   1.476/1.436              10X45   0.604/0.564
                      50Yrs   1.466/1.426              10X50   0.594/0.554
Disclaimer <IDIS>     60Yrs   1.456/1.416              10X60   0.584/0.544
```

# Outline

# Recall the introductory swap example

## Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

Dates

Market conventions

(annually, 30/360 day count, modified following, Target calendar)


Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

(semi-annually, act/360 day count, modified following, Target calendar)

How do we get from description to cash flow stream?

# There are a couple of market conventions that need to be taken into account in practice

- Holiday calendars define at which dates payments can be made.
- Business day conventions specify how dates are adjusted if they fall on a non-business day.
- Schedule generation rules specify how regular dates are calculated.
- Day count conventions define how time is meassured between dates.

# Outline

# Dates are represented as triples day/month/year or as serial numbers

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | **Date** | **Serial** | **EUR Payment System (TARGET)** | **London Bank Holiday** |
| 4 | | Friday, July 27, 2018 | 43308 | FALSE | FALSE |
| 5 | | Monday, August 27, 2018 | 43339 | FALSE | TRUE |
| 6 | | Thursday, September 27, 2018 | 43370 | FALSE | FALSE |
| 7 | | Saturday, October 27, 2018 | 43400 | TRUE | TRUE |
| 8 | | Tuesday, November 27, 2018 | 43431 | FALSE | FALSE |
| 9 | | Thursday, December 27, 2018 | 43461 | FALSE | FALSE |
| 10 | | Sunday, January 27, 2019 | 43492 | TRUE | TRUE |
| 11 | | Wednesday, February 27, 2019 | 43523 | FALSE | FALSE |
| 12 | | Wednesday, March 27, 2019 | 43551 | FALSE | FALSE |
| 13 | | Saturday, April 27, 2019 | 43582 | TRUE | TRUE |
| 14 | | Monday, May 27, 2019 | 43612 | FALSE | TRUE |
| 15 | | | | | |
| 16 | | Sunday, January 1, 1900 | 1 | | |
| 17 | | | | | |

# A calender specifies business days and non-business days

## Holiday Calendar

A holiday calendar $\mathcal{C}$ is a set of dates which are defined as holidays or non-business days.

- A particular date $d$ is a non-business day if $d \in \mathcal{C}$.
- Holiday calendars are specific to a region, country or market segment.
- Need to be specified in the context of financial product.
- Typically contain weekends and special days of the year.
- May be joined (e.g. for multi-currency products), $\bar{\mathcal{C}} = \mathcal{C}_1 \cup \mathcal{C}_2$.
- Typical examples are TARGET calendar and LONDON calendar.

# Outline

# A business day convention maps non-business days to adjacent business days

## Business Day Convention (BDC)

- A business day convention is a function $\omega_{\mathcal{C}} : \mathcal{D} \to \mathcal{D}$ which maps a date $d \in \mathcal{D}$ to another date $\bar{d}$.
- It is applied in conjunction with a calendar $\mathcal{C}$.
- Good business days are unchanged, i.e. $\omega_{\mathcal{C}}(d) = d$ if $d \notin \mathcal{C}$.

## Following

$$\omega_{\mathcal{C}}(d) = \min\left\{\bar{d} \in \mathcal{D}\backslash\mathcal{C} \mid \bar{d} \geq d\right\}$$

## Preceding

$$\omega_{\mathcal{C}}(d) = \max\left\{\bar{d} \in \mathcal{D}\backslash\mathcal{C} \mid \bar{d} \leq d\right\}$$

## Modified Following

$$\omega_{\mathcal{C}}(d) = \begin{cases} \omega_{\mathcal{C}}^{\text{Following}}(d), & \text{if Month}\,[d] = \text{Month}\left[\omega_{\mathcal{C}}^{\text{Following}}(d)\right] \\ \omega_{\mathcal{C}}^{\text{Preceeding}}(d), & \text{else} \end{cases}$$

# Outline

# Schedules represent sets of regular reference dates

| | Annual Frequency | TARGET Calendar | Modified Following |
|---|---|---|---|
| **Start** | Thu, 31 Oct 2019 | FALSE | Thu, 31 Oct 2019 |
| | Sat, 31 Oct 2020 | TRUE | Fri, 30 Oct 2020 |
| | Sun, 31 Oct 2021 | TRUE | Fri, 29 Oct 2021 |
| | Mon, 31 Oct 2022 | FALSE | Mon, 31 Oct 2022 |
| | Tue, 31 Oct 2023 | FALSE | Tue, 31 Oct 2023 |
| | Thu, 31 Oct 2024 | FALSE | Thu, 31 Oct 2024 |
| | Fri, 31 Oct 2025 | FALSE | Fri, 31 Oct 2025 |
| | Sat, 31 Oct 2026 | TRUE | Fri, 30 Oct 2026 |
| | Sun, 31 Oct 2027 | TRUE | Fri, 29 Oct 2027 |
| | Tue, 31 Oct 2028 | FALSE | Tue, 31 Oct 2028 |
| | Wed, 31 Oct 2029 | FALSE | Wed, 31 Oct 2029 |
| | Thu, 31 Oct 2030 | FALSE | Thu, 31 Oct 2030 |
| | Fri, 31 Oct 2031 | FALSE | Fri, 31 Oct 2031 |
| | Sun, 31 Oct 2032 | TRUE | Fri, 29 Oct 2032 |
| | Mon, 31 Oct 2033 | FALSE | Mon, 31 Oct 2033 |
| | Tue, 31 Oct 2034 | FALSE | Tue, 31 Oct 2034 |
| | Wed, 31 Oct 2035 | FALSE | Wed, 31 Oct 2035 |
| | Fri, 31 Oct 2036 | FALSE | Fri, 31 Oct 2036 |
| | Sat, 31 Oct 2037 | TRUE | Fri, 30 Oct 2037 |
| | Sun, 31 Oct 2038 | TRUE | Fri, 29 Oct 2038 |
| **End** | Mon, 31 Oct 2039 | FALSE | Mon, 31 Oct 2039 |

# Schedule generation follows some rules/conventions as well

1. Consider direction of roll-out: forward or backward (relevant for front/back stubs).
   1.1 Forward, roll-out from start (or effective) date to end (or maturity) date
   1.2 Backward, roll-out from end (or maturity) date to start (or effective) date
2. Roll out unadjusted dates according to frequency or tenor, e.g. annual frequency or 3 month tenor
3. If first/last period is broken consider short stub or long stub.
   3.1 Short stub is an unregular last period smaller then tenor.
   3.2 Long stub is an unregular last period larger then tenor
4. Adjust unadjusted dates according to calendar and BDC.

# Outline

# Day count conventions map dates to times or year fractions

## Day Count Convention

A day count convention is a function $\tau : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ which measures a time period between dates in terms of years.

We give some examples:

## Act/365 Fixed Convention

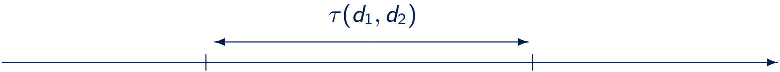
$\tau(d_1, d_2) = (d_2 - d_1)/365$

- Typically used to describe time in financial models.

## Act/360 Convention

$\tau(d_1, d_2) = (d_2 - d_1)/360$

- Often used for Libor floating rate payments.



# 30/360 methods are slightly more involved

## General 30/360 Method

- Consider two dates $d_1$ and $d_2$ represented as triples of day/month/year, i.e. $d_1 = [D_1, M_1, Y_1]$ and $d_2 = [D_2, M_2, Y_2]$ with $D_{1/2} \in \{1, \ldots, 31\}$, $M_{1/2} \in \{1, \ldots, 12\}$ and $Y_{1/2} \in \{1, 2, \ldots\}$.

- Obviously, only valid dates are allowed (no Feb. 30 or similar).

- Adjust $D_1 \mapsto \bar{D}_1$ and $D_2 \mapsto \bar{D}_2$ according to specific rules.

- Calculate

$$\tau(d_1, d_2) = \frac{360 \cdot (Y_2 - Y_1) + 12 \cdot (M_2 - M_1) + \left(\bar{D}_2 - \bar{D}_1\right)}{360}.$$

# Some specific 30/360 rules are given below

## 30/360 Convention (or 30U/360, Bond Basis)

1. $\bar{D}_1 = \min\{D_1, 30\}$.
2. If $\bar{D}_1 = 30$ then $\bar{D}_2 = \min\{D_2, 30\}$ else if $\bar{D}_2 = D_2$.

## 30E/360 Convention (or Eurobond)

1. $\bar{D}_1 = \min\{D_1, 30\}$.
2. $\bar{D}_2 = \min\{D_2, 30\}$.

# Outline

# Now we have all pieces to price a deterministic coupon leg

Coupon is calculated as

$$\text{Coupon} = \text{Notional} \times \text{Rate} \times \text{YearFraction}$$
$$= 100,000,000\text{EUR} \times 3\% \times \tau$$

| | | | |
|---|---|---|---|
| **ValDate** | Tue, 01 Oct 2019 | **Sum** | **41.787.358** |

| | Annual Frequency | TARGET Calendar | Modified Following | _D1 | _D2 | tau | Rate | Coupon | P(0,T) | P(0,T)*Cpn |
|---|---|---|---|---|---|---|---|---|---|---|
| **Start** | Thu, 31 Oct 2019 | FALSE | Thu, 31 Oct 2019 | | | | | | | |
| | Sat, 31 Oct 2020 | TRUE | Fri, 30 Oct 2020 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,9712 | 2.913.606 |
| | Sun, 31 Oct 2021 | TRUE | Fri, 29 Oct 2021 | 30 | 29 | 0,997 | 3,00% | 2.991.667 | 0,9451 | 2.827.508 |
| | Mon, 31 Oct 2022 | FALSE | Mon, 31 Oct 2022 | 29 | 31 | 1,006 | 3,00% | 3.016.667 | 0,9191 | 2.772.558 |
| | Tue, 31 Oct 2023 | FALSE | Tue, 31 Oct 2023 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,8927 | 2.677.958 |
| | Thu, 31 Oct 2024 | FALSE | Thu, 31 Oct 2024 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,8644 | 2.593.199 |
| | Fri, 31 Oct 2025 | FALSE | Fri, 31 Oct 2025 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,8343 | 2.502.952 |
| | Sat, 31 Oct 2026 | TRUE | Fri, 30 Oct 2026 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,8029 | 2.408.751 |
| | Sun, 31 Oct 2027 | TRUE | Fri, 29 Oct 2027 | 30 | 29 | 0,997 | 3,00% | 2.991.667 | 0,7705 | 2.305.172 |
| | Tue, 31 Oct 2028 | FALSE | Tue, 31 Oct 2028 | 29 | 31 | 1,006 | 3,00% | 3.016.667 | 0,7371 | 2.223.720 |
| | Wed, 31 Oct 2029 | FALSE | Wed, 31 Oct 2029 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,7037 | 2.111.106 |
| | Thu, 31 Oct 2030 | FALSE | Thu, 31 Oct 2030 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,6711 | 2.013.239 |
| | Fri, 31 Oct 2031 | FALSE | Fri, 31 Oct 2031 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,6400 | 1.919.866 |
| | Sun, 31 Oct 2032 | TRUE | Fri, 29 Oct 2032 | 30 | 29 | 0,997 | 3,00% | 2.991.667 | 0,6104 | 1.826.132 |
| | Mon, 31 Oct 2033 | FALSE | Mon, 31 Oct 2033 | 29 | 31 | 1,006 | 3,00% | 3.016.667 | 0,5821 | 1.755.979 |
| | Tue, 31 Oct 2034 | FALSE | Tue, 31 Oct 2034 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,5553 | 1.665.987 |
| | Wed, 31 Oct 2035 | FALSE | Wed, 31 Oct 2035 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,5300 | 1.589.867 |
| | Fri, 31 Oct 2036 | FALSE | Fri, 31 Oct 2036 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,5059 | 1.517.640 |
| | Sat, 31 Oct 2037 | TRUE | Fri, 30 Oct 2037 | 30 | 30 | 1,000 | 3,00% | 3.000.000 | 0,4832 | 1.449.640 |
| | Sun, 31 Oct 2038 | TRUE | Fri, 29 Oct 2038 | 30 | 29 | 0,997 | 3,00% | 2.991.667 | 0,4617 | 1.381.390 |
| **End** | Mon, 31 Oct 2039 | FALSE | Mon, 31 Oct 2039 | 29 | 31 | 1,006 | 3,00% | 3.016.667 | 0,4412 | 1.331.088 |

# Outline

# Outline

# Recall the introductory swap example

Pays 3% on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

(annually, 30/360 day count, modified following, Target calendar)


Stochastic interest rates

Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

(semi-annually, act/360 day count, modified following, Target calendar)

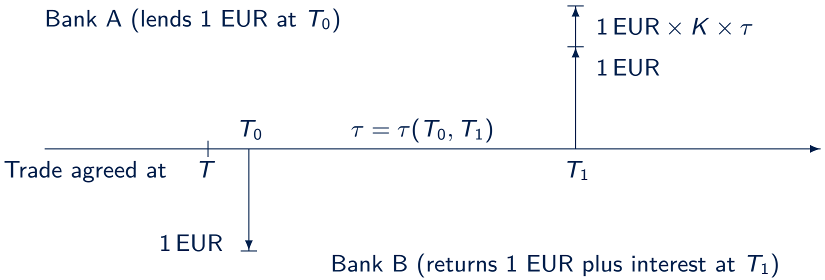
How do we model floating rates?

# We start with some introductory remarks

- London Interbank Offered Rates (Libor) currently are the key building blocks of interest rate derivatives.
- They are fixed for USD, GBP, JPY, CHF (and EUR).
- EUR equivalent rate is Euribor rate (we will use Libor synonymously for Euribor).
- Libor rate modelling has undergone significant changes since financial crisis in 2008.
- This is typically reflected by the term Multi-Curve Interest Rate Modelling.
- Recent developments in the market will lead to a shift from LIbor rates to alternative floating rates in the near future (Ibor Transition).
- We will also touch on potential new alternative rates specifications when discussing OIS swaps.

# Let's start with the classical Libor rate model

What is the fair interest rate $K$ bank A and Bank B can agree on?




We get (via DCF methodology)

$$0 = V(T) = P(T, T_0) \cdot \mathbb{E}^{T_0}\left[-1 \mid \mathcal{F}_T\right] + P(T, T_1) \cdot \mathbb{E}^{T_1}\left[1 + \tau K \mid \mathcal{F}_T\right],$$
$$0 = -P(T, T_0) + P(T, T_1) \cdot (1 + \tau K).$$

# Spot Libor rates are fixed daily and quoted in the market

$$0 = -P(T, T_0) + P(T, T_1) \cdot (1 + \tau K)$$

## Spot Libor rate

The fair rate for an interbank lending deal with trade date $T$, spot starting date $T_0$ (typically 0d or 2d after $T$) and maturity date $T_1$ is

$$L(T; T_0, T_1) = \left[ \frac{P(T, T_0)}{P(T, T_1)} - 1 \right] \frac{1}{\tau}.$$

- Panel banks submit daily estimates for interbank lending rates to calculation agent.
- Relevant periods (i.e. $[T_0, T_1]$) considered are 1m, 3m, 6m and 12m.
- Trimmed average of submissions is calculated and published.

Libor rate fixings currently are the most important reference rates for interest rate derivatives.

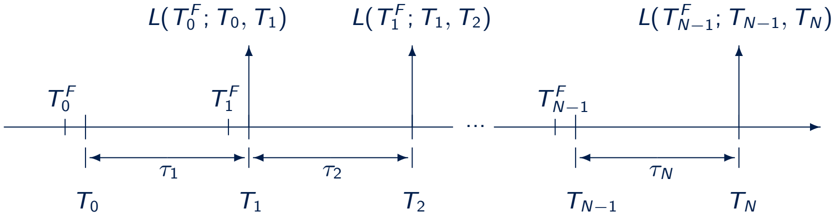
# Example publication at Intercontinental Exchange (ICE)

theice.com/marketdata/reports/170

## ICE LIBOR Historical Rates

| TENOR | PUBLICATION TIME* | USD ICE LIBOR 06-SEP-2018 |
|---|---|---|
| Overnight | 11:55:04 AM | 1.91838 |
| 1 Week | 11:55:04 AM | 1.96100 |
| 1 Month | 11:55:04 AM | 2.13256 |
| 2 Month | 11:55:04 AM | 2.20950 |
| 3 Month | 11:55:04 AM | 2.32706 |
| 6 Month | 11:55:04 AM | 2.54419 |
| 1 Year | 11:55:04 AM | 2.84906 |

# A plain vanilla Libor leg pays periodic Libor rate coupons




We get (via DCF methodology)

$$
\begin{aligned}
V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i}\left[ L(T_{i-1}^F; T_{i-1}, T_i) \cdot \tau_i \mid \mathcal{F}_t \right] \\
&= \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i}\left[ L(T_{i-1}^F; T_{i-1}, T_i) \mid \mathcal{F}_t \right] \cdot \tau_i .
\end{aligned}
$$

Thus all we need is

$$
\mathbb{E}^{T_i}\left[ L(T_{i-1}^F; T_{i-1}, T_i) \mid \mathcal{F}_t \right] = ?
$$

# Libor rate is a martingale in the terminal measure

## Theorem (Martingale property of Libor rate)

*The Libor rate $L(T; T_0, T_1)$ with observation/fixing date $T$, accrual start date $T_0$ and accrual end date $T_1$ is a martingale in the $T_1$-forward measure and*

$$\mathbb{E}^{T_1}\left[L(T; T_0, T_1) \mid \mathcal{F}_t\right] = \left[\frac{P(t, T_0)}{P(t, T_1)} - 1\right]\frac{1}{\tau} = L(t; T_0, T_1).$$

## Proof.

The fair Libor rate at fixing time $T$ is $L(T; T_0, T_1) = \left[P(T, T_0)/P(T, T_1) - 1\right]/\tau$. The zero coupon bond $P(T, T_0)$ is an asset and $P(T, T_1)$ is the numeraire in the $T_1$-forward meassure. Thus FTAP yields that the discounted asset price is a martingale, i.e.

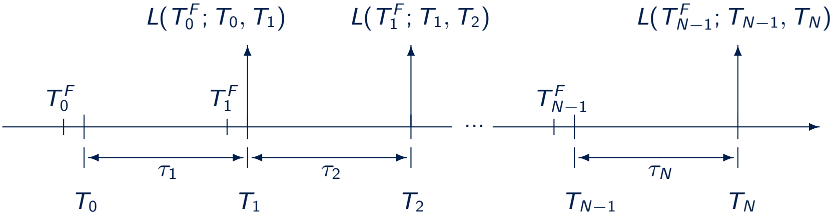
$$\mathbb{E}^{T_1}\left[\frac{P(T, T_0)}{P(T, T_1)} \mid \mathcal{F}_t\right] = \frac{P(t, T_0)}{P(t, T_1)},$$

Linearity of expectation operator yields

$$\mathbb{E}^{T_1}\left[L(T; T_0, T_1) \mid \mathcal{F}_t\right] = \left[\mathbb{E}^{T_1}\left[\frac{P(T, T_0)}{P(T, T_1)}\middle|\mathcal{F}_t\right] - 1\right]\frac{1}{\tau} = \left[\frac{P(t, T_0)}{P(t, T_1)} - 1\right]\frac{1}{\tau} = L(t; T_0, T_1).$$

□

# This allows pricing the Libor leg based on today's knowledge of the yield curve only




Libor leg becomes

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot \mathbb{E}^{T_i}\left[L(T_{i-1}^F; T_{i-1}, T_i) \cdot \tau_i \mid \mathcal{F}_t\right]$$

$$= \sum_{i=1}^{N} P(t, T_i) \cdot L(t; T_{i-1}, T_i) \cdot \tau_i$$

# Libor leg may be simplified in the current single-curve setting

We have

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot L(t; T_{i-1}, T_i) \cdot \tau_i$$

with

$$L(t; T_{i-1}, T_i) = \left[ \frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right] \frac{1}{\tau_i}.$$

This yields

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot \left[ \frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right] \frac{1}{\tau_i} \cdot \tau_i \\ &= \sum_{i=1}^{N} P(t, T_{i-1}) - P(t, T_i) \\ &= P(t, T_0) - P(t, T_N). \end{aligned}$$

We only need discount fators $P(t, T_0)$ and $P(t, T_N)$ at first date $T_0$ and last date $T_N$.

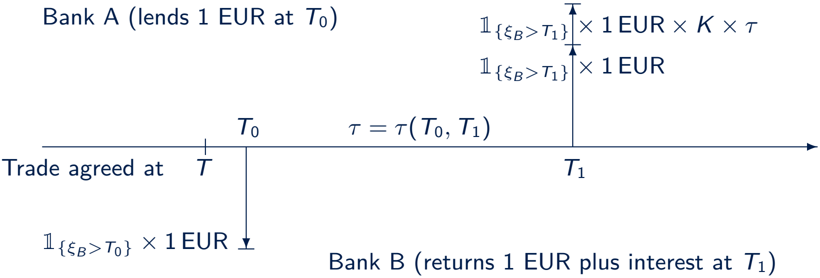
# Outline

# The classical Libor rate model misses an important detail


What if a counterparty defaults?

# What if Bank B defaults prior to $T_0$ or $T_1$?

What is the fair rate $K$ bank A and Bank B can agree on given the risk of default?




- Cash flows are paid only if no default occurs.
- We apply a simple credit model.
- Denote $\mathbb{1}_D$ the indicator function for an event $D$ and random variable $\xi_B$ the first time bank B defaults.

# Credit-risky trade value can be derived using derivative pricing formula

$$\frac{V(T)}{B(T)} = \mathbb{E}^{\mathbb{Q}}\left[ -\mathbb{1}_{\{\xi_B > T_0\}} \cdot \frac{1}{B(T_0)} + \mathbb{1}_{\{\xi_B > T_1\}} \cdot \frac{1 + K \cdot \tau}{B(T_1)} \right].$$

(all expectations conditional on $\mathcal{F}_T$)

Assume independence of credit event $\left\{\xi_B > T_{0/1}\right\}$ and interest rate market, then

$$\frac{V(T)}{B(T)} = -\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_0\}}\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T_0)}\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_1\}}\right] \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{1 + K \cdot \tau}{B(T_1)}\right].$$

Abbreviate survival probability $Q(T, T_{0,1}) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\left\{\xi_B > T_{0,1}\right\}} \,|\, \mathcal{F}_T\right]$ and apply change of measure

$$V(T) = -P(T, T_0)Q(T, T_0)\mathbb{E}^{T_0}[1] + P(T, T_1)Q(T, T_1)\mathbb{E}^{T_1}[1 + K \cdot \tau].$$

# This yields the fair spot rate in the presence of credit risk

$$V(T) = -P(T, T_0)Q(T, T_0)\mathbb{E}^{T_0}[1] + P(T, T_1)Q(T, T_1)\mathbb{E}^{T_1}[1 + K \cdot \tau].$$

If we solve $V(T) = 0$ and set $K = L(T; T_0, T_1)$ we get

$$L(T; T_0, T_1) = \left[\frac{P(T, T_0)}{P(T, T_1)} \cdot \frac{Q(T, T_0)}{Q(T, T_1)} - 1\right]\frac{1}{\tau}.$$

We need a model for the survival probability $Q(T, T_{1,2})$.

Consider, e.g., hazard rate model $Q(T, T_{1,2}) = \exp\left\{-\int_T^{T_{1,2}} \lambda(s)ds\right\}$ with deterministic hazard rate $\lambda(s)$. Then forward survival probability $D(T_0, T_1)$ with

$$D(T_0, T_1) = \frac{Q(T, T_0)}{Q(T, T_1)} = \exp\left\{-\int_{T_0}^{T_1} \lambda(s)ds\right\}$$

is independent of observation time $T$.

# Deterministic hazard rate assumption preserves the martingale property of forward Libor rate

## Theorem (Martingale property of credit-risky Libor rate)

*Consider the credit-risky Libor rate $L(T; T_0, T_1)$ with observation/fixing date $T$, accrual start date $T_0$ and accrual end date $T_1$. If the forward survival probability $D(T_0, T_1)$ is deterministic such that*

$$L(T; T_0, T_1) = \left[ \frac{P(T, T_0)}{P(T, T_1)} \cdot D(T_0, T_1) - 1 \right] \frac{1}{\tau},$$

*then $L(t; T_0, T_1)$ is a martingale in the $T_1$-forward measure and*

$$\mathbb{E}^{T_1} \left[ L(T; T_0, T_1) \mid \mathcal{F}_t \right] = L(t; T_0, T_1) = \left[ \frac{P(t, T_0)}{P(t, T_1)} \cdot D(T_0, T_1) - 1 \right] \frac{1}{\tau}.$$

## Proof.

Follows analogously to classical Libor rate martingale property.

□

# Outline

# Forward Libor rates are typically parametrised via projection curve

- Hazard rate $\lambda(u)$ in $Q(T, T_{1,2}) = \exp\left\{-\int_T^{T_{1,2}} \lambda(u)du\right\}$ is often considered as a tenor basis spread $s(u)$.
- Survival probability $Q(T, T_{1,2})$ can be interpreted as discount factor.
- Suppose we know time-$t$ survival probabilities $Q(t, \cdot)$ for a forward Libor rate $L(t, T_0, T_0 + \delta)$ with tenor $\delta$ (typically 1m, 3m, 6m or 12m). Then we define the projection curve

$$P^\delta(t, T) = P(t, T) \cdot Q(t, T).$$

- With projection curve $P^\delta(t, T)$ the forward Libor rate formula is analogous to the classical Libor rate formula, i.e.

$$L^\delta(t, T_0) = L(t; T_0, T_0 + \delta) = \left[\frac{P^\delta(t, T_0)}{P^\delta(t, T_1)} - 1\right] \frac{1}{\tau}.$$

This yields the multi-curve modelling framework consisting of discount curve $P(t, T)$ and tenor-dependent projection curves $P^\delta(t, T)$.

# There is an alternative approach to multi-curve modelling

Define forward Libor rate $L^\delta(t, T_0)$ for a tenor $\delta$ as

$$L^\delta(t, T_0) = \mathbb{E}^{T_1}\left[L(T; T_0, T_0 + \delta) \mid \mathcal{F}_t\right].$$

(Without any assumptions on default, survival probabilities etc.)

Postulate a projection curve parametrisation

$$L^\delta(t, T_0) = \left[\frac{P^\delta(t, T_0)}{P^\delta(t, T_1)} - 1\right]\frac{1}{\tau}.$$

- We will discuss calibration of projection curve $P^\delta(t, T)$ later.
- This approach alone suffices for linear products (e.g. Libor legs) and simple options.
- It does not specify any relation between projection curve $P^\delta(t, T)$ and discount curve $P(t, T)$.

# Projection curves can also be written in terms of zero rates and continuous forward rates

Consider a projection curve given by (pseudo) discount factors $P^\delta(t, T)$ (observed today).

- Corresponding continuous compounded zero rates are

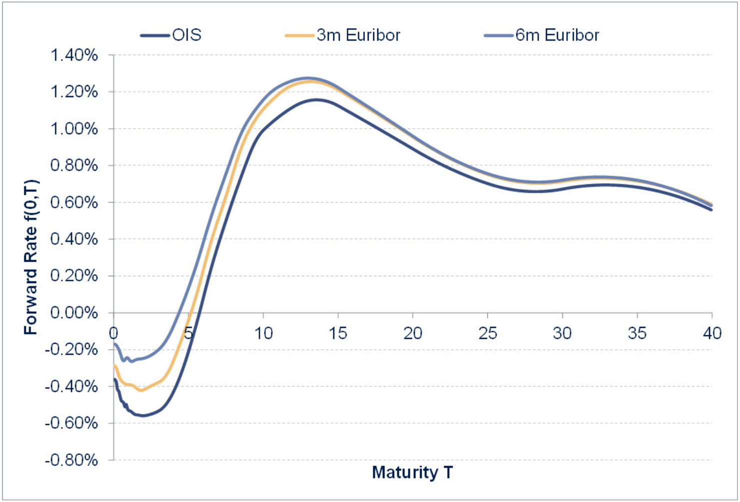
$$z^\delta(t, T) = -\frac{\ln\left[P^\delta(t, T)\right]}{T - t}.$$

- Corresponding continuous compounded forward rates are

$$f^\delta(t, T) = -\frac{\partial \ln\left[P^\delta(t, T)\right]}{\partial T}.$$

# We illustrate an example of a multi-curve set-up for EUR

Market data as of July 2016



# Libor leg pricing needs to be adapted slightly for multi-curve pricing

Classical single-curve Libor leg price is

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} P(t, T_i) \cdot L(t; T_{i-1}, T_i) \cdot \tau_i \\ &= P(t, T_0) - P(t, T_N). \end{aligned}$$

Multi-curve Libor leg pricing becomes

$$V(t) = \sum_{i=1}^{N} P(t, T_i) \cdot L^{\delta}(t, T_{i-1}) \cdot \tau_i$$

with

$$L^{\delta}(t, T_{i-1}) = \left[ \frac{P^{\delta}(t, T_{i-1})}{P^{\delta}(t, T_i)} - 1 \right] \frac{1}{\tau_i}.$$

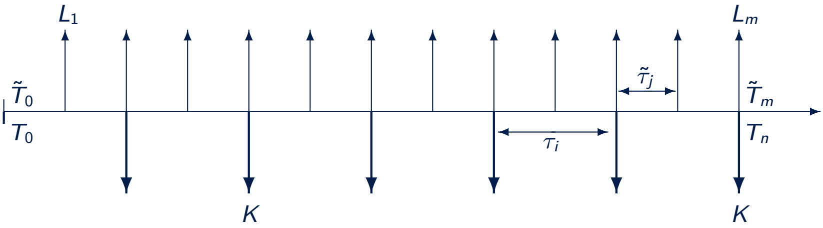
- Note that we need different yield curves for Libor rate projection and cash flow discounting.
- Single-curve pricing formula simplification does not work for multi-curve pricing.

# Outline

# Outline

# With the fixed leg and Libor leg pricing available we can directly price a Vanilla interest rate swap

float leg (EUR conventions: 6m Euribor, Act/360)




fixed leg (EUR conventions: annual, 30/360)

Present value of (fixed rate) payer swap with notional $N$ becomes

$$V(t) = \sum_{j=1}^{m} N \cdot L^{6m}(t, \tilde{T}_{j-1}) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j) - \sum_{i=1}^{n} N \cdot K \cdot \tau_i \cdot P(t, T_i).$$

# Vanilla swap pricing formula allows us to price the underlying swap of our introductory example

Interbank swap deal example

Pays 3% on 100mm EUR

Start date: Oct 31, 2019

End date: Oct 31, 2039

(annually, 30/360 day count, modified following, Target calendar)


Pays 6-months Euribor floating rate on 100mm EUR

Start date: Oct 31, 2019

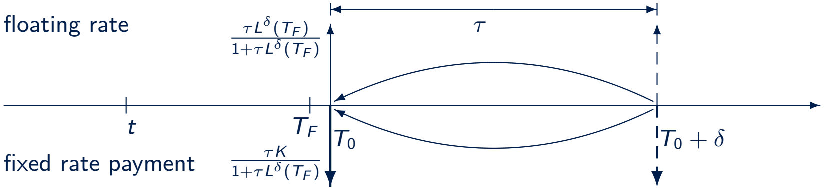
End date: Oct 31, 2039

(semi-annually, act/360 day count, modified following, Target calendar)

# We illustrate swap pricing with QuantLib/Excel...

# Outline

# Forward Rate Agreement yields exposure to single forward Libor rates




- Fixed rate $K$ agreed at trade inception (prior to $t$).
- Libor rate $L^\delta(T_F, T_0)$ fixed at $T_F$, valid for the period $T_0$ to $T_0 + \delta$.
- Payoff paid at $T_0$ is difference $\tau \cdot \left[L^\delta(T_F, T_0) - K\right]$ discounted from $T_1$ to $T_0$ with discount factor $\left[1 + \tau \cdot L^\delta(T_F, T_0)\right]^{-1}$, i.e.

$$V(T_0) = \frac{\tau \cdot \left[L^\delta(T_F, T_0) - K\right]}{1 + \tau \cdot L^\delta(T_F, T_0)}.$$

# Time-$T_F$ FRA price can be obtained via deterministic basis spread model

Note that payoff $V(T_0) = \frac{\tau \cdot \left[L^\delta(T_F, T_0) - K\right]}{1 + \tau \cdot L^\delta(T_F, T_0)}$ is already determined at $T_F$. Thus (via DCF)

$$V(T_F) = P(T_F, T_0) \cdot V(T_0) = P(T_F, T_0) \cdot \frac{\tau \cdot \left[L^\delta(T_F, T_0) - K\right]}{1 + \tau \cdot L^\delta(T_F, T_0)}.$$

Recall that (with $T_1 = T_0 + \delta$)

$$1 + \tau \cdot L^\delta(T_F, T_0) = \frac{P^\delta(T_F, T_0)}{P^\delta(T_F, T_1)} = \frac{P(T_F, T_0)}{P(T_F, T_1)} \cdot D(T_0, T_1).$$

Then

$$\begin{aligned} V(T_F) &= P(T_F, T_0) \cdot \tau \cdot \left[L^\delta(T_F, T_0) - K\right] \cdot \frac{1}{D(T_0, T_1)} \cdot \frac{P(T_F, T_1)}{P(T_F, T_0)} \\ &= P(T_F, T_1) \cdot \tau \cdot \left[L^\delta(T_F, T_0) - K\right] \cdot \frac{1}{D(T_0, T_1)}. \end{aligned}$$

# Present value of FRA can be obtained via martingale property

Derivative pricing formula in $T_1$-terminal measure yields

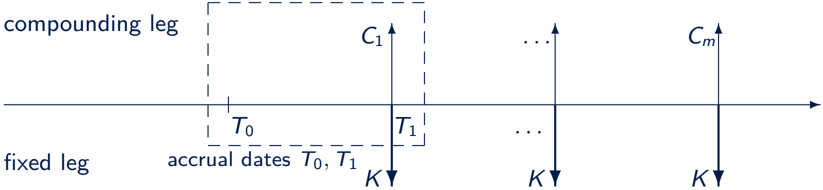
$$\frac{V(t)}{P(t,T_1)} = \mathbb{E}^{T_1}\left[\frac{P(T_F,T_1)}{P(T_F,T_1)}\cdot\tau\cdot\left[L^\delta(T_F,T_0)-K\right]\cdot\frac{1}{D(T_0,T_1)}\right]$$
$$= \tau\cdot\left[\mathbb{E}^{T_1}\left[L^\delta(T_F,T_0)\right]-K\right]\cdot\frac{1}{D(T_0,T_1)}$$
$$= \tau\cdot\left[L^\delta(t,T_0)-K\right]\cdot\frac{1}{D(T_0,T_1)}.$$

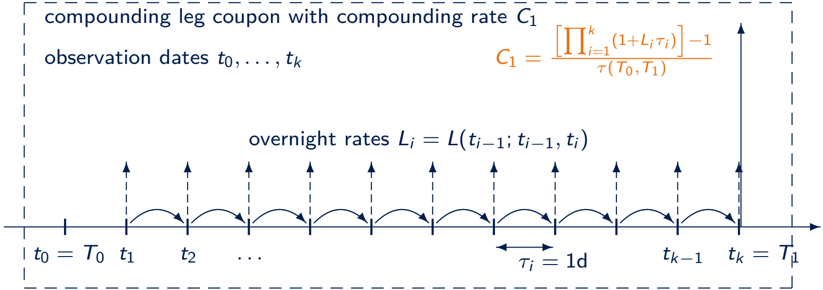
Using $1+\tau\cdot L^\delta(t,T_0) = \frac{P(t,T_0)}{P(t,T_1)}\cdot D(T_0,T_1)$ (deterministic spread assumption) yields

$$V(t) = P(t,T_0)\cdot\tau\cdot\left[L^\delta(t,T_0)-K\right]\cdot\left[\frac{P(t,T_0)}{P(t,T_1)}\cdot D(T_0,T_1)\right]^{-1}$$
$$= P(t,T_0)\cdot\frac{\left[L^\delta(t,T_0)-K\right]\cdot\tau}{1+\tau\cdot L^\delta(t,T_0)}.$$

# Outline

# Overnight index swap (OIS) instruments are further relevant instruments in the market



# We need to calculate the compounding leg coupon rate

- Assume overnight index swap (OIS) rate $L_i = L(t_{i-1}; t_{i-1}, t_i)$ is a credit-risk free Libor rate.
- Compounded rate (for a period $[T_0, T_1]$) is specified as

$$C_1 = \left\{ \left[ \prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)}.$$

- Coupon payment is at $T_1$.
- For pricing we need to calculate

$$\begin{aligned} \mathbb{E}^{T_1}\left[C_1 \mid \mathcal{F}_t\right] &= \mathbb{E}^{T_1}\left[ \left\{ \left[ \prod_{i=1}^{k} (1 + L_i \tau_i) \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)} \mid \mathcal{F}_t \right] \\ &= \left\{ \mathbb{E}^{T_1}\left[ \prod_{i=1}^{k} (1 + L_i \tau_i) \mid \mathcal{F}_t \right] - 1 \right\} \frac{1}{\tau(T_0, T_1)}. \end{aligned}$$

# How do we handle the compounding term?

Overall compounding term is

$$\prod_{i=1}^{k}(1+L_i\tau_i)=\prod_{i=1}^{k}\left[1+L(t_{i-1};t_{i-1},t_i)\tau_i\right].$$

Individual compounding term is

$$1+L(t_{i-1};t_{i-1},t_i)\tau_i=1+\left[\frac{P(t_{i-1},t_{i-1})}{P(t_{i-1},t_i)}-1\right]\frac{1}{\tau_i}\tau_i=\frac{P(t_{i-1},t_{i-1})}{P(t_{i-1},t_i)}.$$

We get

$$\prod_{i=1}^{k}(1+L_i\tau_i)=\prod_{i=1}^{k}\frac{P(t_{i-1},t_{i-1})}{P(t_{i-1},t_i)}=\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}.$$

We need to calculate the expectation of $\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}$.

# Expected compounding factor can be easily calculated

## Lemma (Compounding rate)

*Consider a compounding coupon period* $[T_0, T_1]$ *with overnight observation and maturity dates* $\{t_0, t_1, \ldots, t_k\}$, $t_0 = T_0$ *and* $t_k = T_1$*. Then*

$$\mathbb{E}^{T_1}\left[\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} \,|\, \mathcal{F}_{T_0}\right] = \frac{1}{P(T_0, T_1)}.$$

# We proof the result via Tower Law of conditional expectation

$$
\begin{aligned}
\mathbb{E}^{T_1}\left[\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}\,|\,\mathcal{F}_{T_0}\right] &= \mathbb{E}^{T_1}\left[\mathbb{E}^{T_1}\left[\prod_{i=1}^{k}\frac{1}{P(t_{i-1},t_i)}\,|\,\mathcal{F}_{t_{k-2}}\right]\,|\,\mathcal{F}_{T_0}\right] \\
&= \mathbb{E}^{T_1}\left[\prod_{i=1}^{k-1}\frac{1}{P(t_{i-1},t_i)}\;\mathbb{E}^{T_1}\left[\frac{P(t_{k-1},t_{k-1})}{P(t_{k-1},t_k)}\,|\,\mathcal{F}_{t_{k-2}}\right]\,|\,\mathcal{F}_{T_0}\right] \\
&= \mathbb{E}^{T_1}\left[\prod_{i=1}^{k-1}\frac{1}{P(t_{i-1},t_i)}\;\frac{P(t_{k-2},t_{k-1})}{P(t_{k-2},t_k)}\,|\,\mathcal{F}_{T_0}\right] \\
&= \mathbb{E}^{T_1}\left[\prod_{i=1}^{k-2}\frac{1}{P(t_{i-1},t_i)}\;\frac{1}{P(t_{k-2},t_k)}\,|\,\mathcal{F}_{T_0}\right] \\
\dots &= \mathbb{E}^{T_1}\left[\frac{1}{P(t_0,t_k)}\,|\,\mathcal{F}_{T_0}\right] \\
&= \frac{1}{P(T_0,T_1)}.
\end{aligned}
$$

# Expected compounding rate equals Libor rate

- Expected compounding rate as seen at start date $T_0$ becomes

$$\mathbb{E}^{T_1}\left[C_1 \,|\, \mathcal{F}_{T_0}\right] = \left[\frac{1}{P(T_0, T_1)} - 1\right] \frac{1}{\tau(T_0, T_1)} = L(T_0; T_0, T_1).$$

- Consequently, expected compounding rate equals Libor rate for full period.

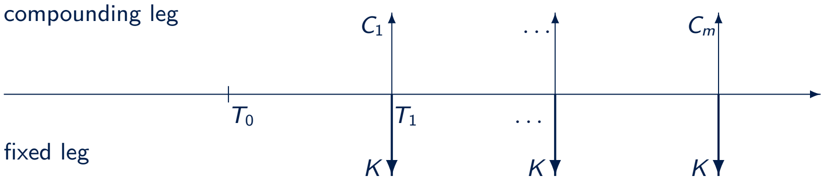
- Moreover, expectations as seen of time-$t$ are

$$\mathbb{E}^{T_1}\left[\prod_{i=1}^{k} \frac{1}{P(t_{i-1}, t_i)} \,|\, \mathcal{F}_t\right] = \frac{P(t, T_0)}{P(t, T_1)}$$

and

$$\mathbb{E}^{T_1}\left[C_1 \,|\, \mathcal{F}_t\right] = \left[\frac{P(t, T_0)}{P(t, T_1)} - 1\right] \frac{1}{\tau(T_0, T_1)} = L(t; T_0, T_1).$$

# Compounding swap pricing is analogous to Vanilla swap pricing




$$V(t) = \sum_{j=1}^{m} N \cdot \mathbb{E}^{T_j}\left[C_j \,|\, \mathcal{F}_t\right] \cdot \tau_j \cdot P(t, T_j) - \sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j)$$

$$= \sum_{j=1}^{m} N \cdot L(t; T_{j-1}, T_j) \cdot \tau_j \cdot P(t, T_j) - \sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j).$$

# Outline

# As a summary we give an overview of linear products pricing

## Vanilla (Payer) Swap

$$\text{Swap}(t) = \underbrace{\sum_{j=1}^{m} N \cdot L^{\delta}(t, \tilde{T}_{j-1}) \cdot \tilde{\tau}_j \cdot P(t, \tilde{T}_j)}_{\text{float leg}} - \underbrace{\sum_{i=1}^{n} N \cdot K \cdot \tau_i \cdot P(t, T_i)}_{\text{fixed Leg}}$$

## Market Forward Rate Agreement (FRA)

$$\text{FRA}(t) = \underbrace{P(t, T_0)}_{\text{discounting to } T_0} \cdot \underbrace{\left[L^{\delta}(t, T_0) - K\right] \cdot \tau}_{\text{payoff}} \cdot \underbrace{\frac{1}{1 + \tau \cdot L^{\delta}(t, T_0)}}_{\text{discounting from } T_0 \text{ to } T_0+\delta}$$

## Compounding Swap / OIS Swap

$$\text{CompSwap}(t) = \underbrace{\sum_{j=1}^{m} N \cdot L(t; T_{j-1}, T_j) \cdot \tau_j \cdot P(t, T_j)}_{\text{compounding leg}} - \underbrace{\sum_{j=1}^{m} N \cdot K \cdot \tau_j \cdot P(t, T_j)}_{\text{fixed leg}}$$

# Further reading on yield curves, conventions and linear products


- F. Ametrano and M. Bianchetti. Everything you always wanted to know about Multiple Interest Rate Curve Bootstrapping but were afraid to ask (April 2, 2013).
  Available at SSRN: http://ssrn.com/abstract=2219548 or http://dx.doi.org/10.2139/ssrn.2219548, 2013

- M. Henrard. Interest rate instruments and market conventions guide 2.0. Open Gamma Quantitative Research, 2013

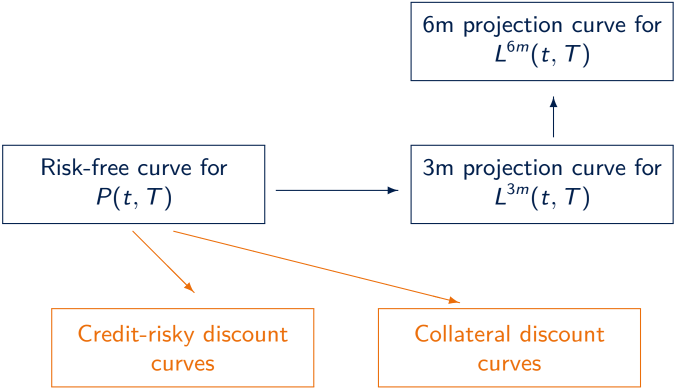
- P. Hagan and G. West. Interpolation methods for curve construction. *Applied Mathematical Finance*, 13(2):89–128, 2006


On current discussion of Libor alternatives, e.g.


- M. Henrard. A quant perspective on ibor fallback proposals. https://ssrn.com/abstract=3226183, 2018

# Outline

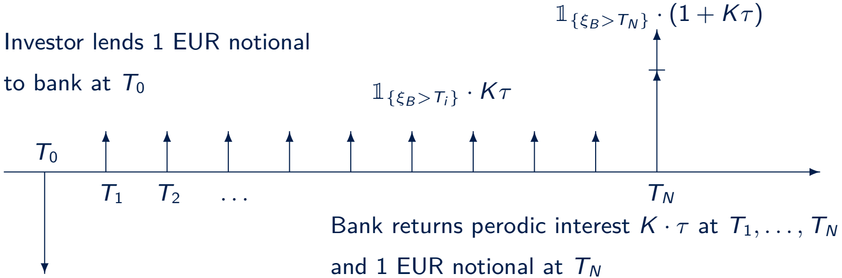
# So far we discussed risk-free discount curves and tenor forward curves - now it is getting a bit more complex




Specifying appropriate discount and projection curves for a financial instrument is an important task in practice.

# Outline

# Discounting of bond or loan cash flows is subject to credit risk




- Cash flows are paid only if no default occurs.
- Denote $\mathbb{1}_D$ the indicator function for an event $D$ and random variable $\xi_B$ the first time bank defaults.
- Assume independence of credit event $\{\xi_B > T\}$ and interest rate market

# We repeat credit-risky valuation from multi-curve pricing

Consider an observation time $t$ with $T_0 < t \le T_N$ then present value of bond cash flows becomes

$$\frac{V(t)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[ \mathbb{1}_{\{\xi_B > T_N\}} \frac{1}{B(T_N)} + \sum_{T_i \ge t} \mathbb{1}_{\{\xi_B > T_i\}} \frac{K\tau}{B(T_i)} \,|\, \mathcal{F}_t \right].$$

Independence of credit event $\{\xi_B > T\}$ and interest rate market yields (all expectations conditional on $\mathcal{F}_t$)

$$\frac{V(t)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_N\}}\right] \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T_N)}\right] + \sum_{T_i \ge t} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T_i\}}\right] \mathbb{E}^{\mathbb{Q}}\left[\frac{K\tau}{B(T_i)}\right].$$

Denote survival probability $Q(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\xi_B > T\}} \,|\, \mathcal{F}_t\right]$ and change to forward measure, then

$$V(t) = Q(t,T_N)P(t,T_N) + \sum_{T_i \ge t} Q(t,T_N)P(t,T_N)K\tau.$$

# Survival probabilities are parameterized in terms of spread curves - this leads to credit-risky discount curves

Assume survival probability $Q(t, T)$ is given in terms of a credit spread curve $s(t)$ and

$$Q(t, T) = \exp\left\{ -\int_t^T s(u)du \right\}.$$

Also recall that discount factors may be represented in terms of forward rates $f(t, T)$ and

$$P(t, T) = \exp\left\{ -\int_t^T f(t, u)du \right\}.$$

We may define a credit-risky discount curve $P^B(t, T)$ for a bond or loan as

$$P^B(t, T) = Q(t, T)P(t, T) = \exp\left\{ -\int_t^T \left[ f(t, u) + s(u) \right] du \right\}.$$

# We can adapt the discounted cash flow pricing method to cash flows subject to credit risk

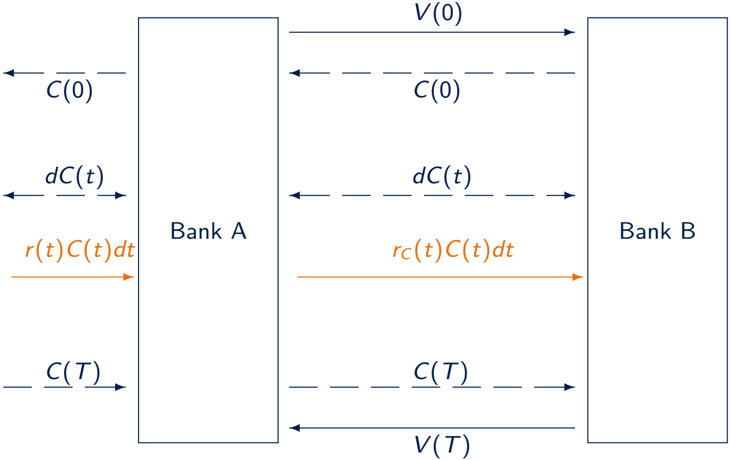
Present value of bond or loan cash flows become

$$V(t) = P^B(t, T_N) + \sum_{T_i \geq t} P^B(t, T_N) K\tau.$$

- Bonds are issued by many market participants (banks, corporates, governments, ...)
- Credit spread curves and credit-risky discount curves are specific to an issuer, e.g. Deutsche Bank has a different credit spread than Bundesrepublik Deutschland
- Many bonds are actively traded in the market. Then we may use market prices and infer credit spreads $s(t)$ and credit-risky discount curves $P^B(t, T)$

# Outline

# For derivative transactions credit risk is typically mitigated by posting collateral




Pricing needs to take into account interest payments on collateral.[2]

[2] Collateral amounts $C(t)$ and collateral rates are agreed in *Credit Support Annexes* (CSAs) between counterparties.

# Collateralized derivative pricing takes into account collateral cash flows

Collateralized derivative price is given by (expectation of) sum of discounted payoff

$$e^{-\int_t^T r(u)du} V(T)$$

plus sum discounted collateral interest payments

$$\int_t^T e^{-\int_t^s r(u)du} \left[ r(s) - r_C(s) \right] C(s) ds.$$

That gives

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r(u)du} V(T) + \int_t^T e^{-\int_t^s r(u)du} \left[ r(s) - r_C(s) \right] C(s) ds \,|\, \mathcal{F}_t \right].$$

# Pricing is reformulated to focus on collateral rate

## Theorem (Collateralized Discounting)

*Consider the price of an option $V(t)$ at time $t$ which pays an amount $V(T)$ at time $T \geq t$ (and no intermediate cash flows).*
*The option is assumed collateralized with cash amounts $C(s)$ (for $t \leq s \leq T$).*
*For the cash collateral a collateral rate $r_C(s)$ (for $t \leq s \leq T$) is applied.*
*Then the option price $V(t)$ becomes*

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[ e^{-\int_t^T r_C(u)du} V(T) \,|\, \mathcal{F}_t \right] - \mathbb{E}^{\mathbb{Q}}\left[ \int_t^T e^{-\int_t^s r_C(u)du} \left[r(s) - r_C(s)\right] \left[V(s) - C(s)\right] ds \,|\, \mathcal{F}_t \right]$$

For further details on collateralized discounting see, e.g.


- V. Piterbarg. Funding beyond discounting: collateral agreements and derivatives pricing. *Asia Risk*, pages 97–102, February 2010
- M. Fujii, Y. Shimada, and A. Takahashi. Collateral posting and choice of collateral currency - implications for derivative pricing and risk management (may 8, 2010). Available at SSRN: https://ssrn.com/abstract=1601866, May 2010

# Collateralized discounting result is proved in three steps

1. Define the discounted collateralized price process

$$X(t) = e^{-\int_0^t r(u)du} V(t) + \int_0^t e^{-\int_0^s r(u)du} \left[ r(s) - r_C(s) \right] C(s) ds$$

   and show that it is a martingale

2. Analyse the dynamics $dX(t)$ and deduce the dynamics for $dV(t)$

3. Solve the SDE for $dV(t)$ and calculate price via conditional expectation

# Step 1 - discounted collateralized price process (1/2)

Consider $T \geq t$, then

$$
\begin{aligned}
X(T) &= e^{-\int_0^T r(u)du} V(T) + \int_0^T e^{-\int_0^s r(u)du} \left[r(s) - r_C(s)\right] C(s) ds \\
&= e^{-\int_0^T r(u)du} V(T) + \int_0^t e^{-\int_0^s r(u)du} \left[r(s) - r_C(s)\right] C(s) ds + \\
&\quad \int_t^T e^{-\int_0^s r(u)du} \left[r(s) - r_C(s)\right] C(s) ds \\
&= e^{-\int_0^t r(u)du} \underbrace{\left[ e^{-\int_t^T r(u)du} V(T) + \int_t^T e^{-\int_t^s r(u)du} \left[r(s) - r_C(s)\right] C(s) ds \right]}_{K(t,T)} + \\
&\quad \int_0^t e^{-\int_0^s r(u)du} \left[r(s) - r_C(s)\right] C(s) ds.
\end{aligned}
$$

# Step 1 - discounted collateralized price process (2/2)

We have from collateralized derivative pricing that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[K(t,T)\,|\,\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}V(T) + \int_t^T e^{-\int_t^s r(u)du}\left[r(s) - r_C(s)\right]C(s)ds\,|\,\mathcal{F}_t\right] \\ &= V(t).\end{aligned}$$

This yields

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[X(T)\,|\,\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r(u)du}K(t,T) + \int_0^t e^{-\int_0^s r(u)du}\left[r(s) - r_C(s)\right]C(s)ds\,|\,\mathcal{F}_t\right] \\ &= e^{-\int_0^t r(u)du}\mathbb{E}^{\mathbb{Q}}\left[K(t,T)\,|\,\mathcal{F}_t\right] + \int_0^t e^{-\int_0^s r(u)du}\left[r(s) - r_C(s)\right]C(s)ds \\ &= e^{-\int_0^t r(u)du}V(t) + \int_0^t e^{-\int_0^s r(u)du}\left[r(s) - r_C(s)\right]C(s)ds \\ &= X(t).\end{aligned}$$

Thus, $X(t)$ is indeed a martingale.

# Step 2 - dynamics $dX(t)$ and $dV(t)$

From the definition $X(t) = e^{-\int_0^t r(u)du} V(t) + \int_0^t e^{-\int_0^s r(u)du} [r(s) - r_C(s)] C(s)ds$ follows

$$\begin{aligned} dX(t) &= -r(t)e^{-\int_0^t r(u)du} V(t)dt + e^{-\int_0^t r(u)du} dV(t) + e^{-\int_0^t r(u)du} [r(t) - r_C(t)] C(t)dt \\ &= e^{-\int_0^t r(u)du} [dV(t) - r(t)V(t)dt + [r(t) - r_C(t)] C(t)dt] \\ &= e^{-\int_0^t r(u)du} \underbrace{[dV(t) - r_C(t)V(t)dt + [r(t) - r_C(t)] [C(t) - V(t)] dt]}_{dM(t)}. \end{aligned}$$

Since $X(t)$ is a martingale we must have that $dM(t)$ are increments of a martingale. We get

$$dV(t) = r_C(t)V(t)dt - [r(t) - r_C(t)] [C(t) - V(t)] dt + dM(t).$$

# Step 3 - solution for $V(t)$ (1/2)

For the SDE $dV(t) = r_C(t)V(t)dt - [r(t) - r_C(t)]\,[C(t) - V(t)]\,dt + dM(t)$ we may guess a solution as

$$V(t) = e^{\int_{t_0}^{t} r_C(s)ds} V(t_0) - \int_{t_0}^{t} e^{\int_s^t r_C(u)du} \left\{[r(s) - r_C(s)]\,[C(s) - V(s)]\,ds - dM(s)\right\}$$

Differentiating confirms that

$$\begin{aligned}
dV(t) &= r_C(t)e^{\int_{t_0}^{t} r_C(s)ds} V(t_0) \\
&\quad - r_C(t)\int_{t_0}^{t} e^{\int_s^t r_C(u)du} \left\{[r(s) - r_C(s)]\,[C(s) - V(s)]\,ds - dM(s)\right\} \\
&\quad - e^{\int_t^t r_C(u)du} \left\{[r(t) - r_C(t)]\,[C(t) - V(t)]\,dt - dM(t)\right\} \\
&= r_C(t)\left[e^{\int_{t_0}^{t} r_C(s)ds} V(t_0) - \int_{t_0}^{t} e^{\int_s^t r_C(u)du} \left\{[r(s) - r_C(s)]\,[C(s) - V(s)]\,ds - dM(s)\right\}\right] \\
&\quad - [r(t) - r_C(t)]\,[C(t) - V(t)]\,dt + dM(t) \\
&= r_C(t)V(t) - [r(t) - r_C(t)]\,[C(t) - V(t)]\,dt + dM(t).
\end{aligned}$$

# Step 3 - solution for $V(t)$ (2/2)

Substituting $t \mapsto T$ and $t_0 \mapsto t$ yields the representation

$$V(T) = e^{\int_t^T r_C(s)ds} V(t) - \int_t^T e^{\int_s^T r_C(u)du} \left\{ [r(s) - r_C(s)] \, [C(s) - V(s)] \, ds - dM(s) \right\}$$

Solving for $V(t)$ gives

$$V(t) = e^{-\int_t^T r_C(s)ds} V(T) - \int_t^T e^{-\int_t^s r_C(u)du} \left\{ [r(s) - r_C(s)] \, [C(s) - V(s)] \, ds - dM(s) \right\}$$

The result follows now from taking conditional expectation

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T r_C(s)ds} V(T) - \int_t^T e^{-\int_t^s r_C(u)du} [r(s) - r_C(s)] \, [V(s) - C(s)] \, ds \,|\, \mathcal{F}_t \right] + \underbrace{\mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-\int_t^s r_C(u)du} dM(s) \,|\, \mathcal{F}_t \right]}_{0}$$

# A very important special case arises for full collateralization

## Corollary (Full collateralization)

*If the collateral amount $C(s)$ equals the full option price $V(s)$ for $t \leq s \leq T$ then the derivative price becomes*

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[ e^{-\int_t^T r_C(s)ds} V(T) \,|\, \mathcal{F}_t \right].$$

- Fully collateralized price is calculated analogous to uncollateralized price.
- Discount rate must equal to the collateral rate $r_C(s)$.
- Pricing is independent of the risk-free rate $r(t)$.
- Collateral bank account $B^C(t) = \exp\left\{ \int_0^t r_C(s)ds \right\}$ can be considered as numeraire in this setting

# The collateralized zero coupon bond can be used to adapt DCF method to collateralized derivative pricing

Consider a fully collateralized instrument that pays $V(T) = 1$ at some time horizon $T$. The price $V(t)$ for $t \leq T$ is given by
$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(s)ds} 1 \,|\, \mathcal{F}_t\right]$.

## Definition (Collateralized zero coupon bond)

The collateralized zero coupon bond price (or collateralized discount factor) for an observation time $t$ and maturity $T \geq t$ is given by

$$P^C(t, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(s)ds} \,|\, \mathcal{F}_t\right].$$

Consider a time horizon $T$ and the time-$t$ price process of a collateralized zero coupon bond $P^C(t, T)$:

- Collateralized zero coupon bond is an asset in our economy,
- price process $P^C(t, T) > 0$.

Thus collateralized zero coupon bond is a numeraire.

# The collateralized zero coupon bond can be used as numeraire for pricing

Define the collateralized forward measure $\mathbb{Q}^{T,C}$ as the equivalent martingale measure with $P^C(t,T)$ as numeraire and expectation $\mathbb{E}^{T,C}[\cdot]$.
The density process of $\mathbb{Q}^{T,C}$ (relative to risk-neutral measure $\mathbb{Q}$) is

$$\zeta(t) = \frac{P^C(t,T)}{B^C(t)} \cdot \frac{B^C(0)}{P^C(0,T)}.$$

This yields

$$\begin{aligned}
\mathbb{E}^{T,C}\left[V(T) \,|\, \mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\zeta(T)}{\zeta(t)} V(T) \,|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{P^C(T,T)}{B^C(T)} \cdot \frac{B^C(t)}{P^C(t,T)} V(T) \,|\, \mathcal{F}_t\right] \\
&= \frac{1}{P^C(t,T)} \mathbb{E}^{\mathbb{Q}}\left[\frac{B^C(t)}{B^C(T)} \cdot V(T) \,|\, \mathcal{F}_t\right] \\
&= \frac{1}{P^C(t,T)} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(s)ds} V(T) \,|\, \mathcal{F}_t\right] \\
&= \frac{V(t)}{P^C(t,T)}
\end{aligned}$$

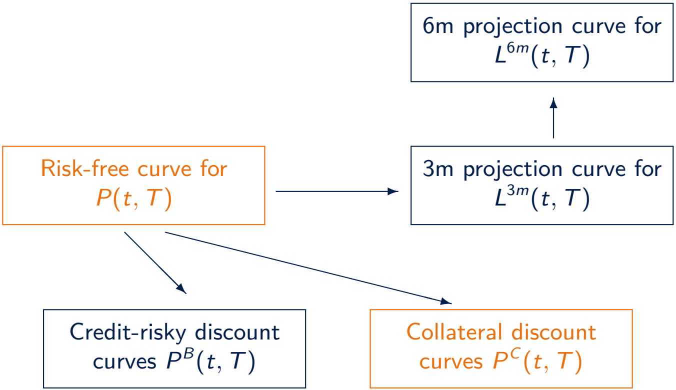
# Discounted cash flow method pricing requires to use the appropriate discount curve representing collateral rates

We have

$$V(t) = P^C(t,T) \cdot \mathbb{E}^{T,C}\left[V(T) \,|\, \mathcal{F}_t\right].$$

- Requires discounting curve $P^C(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_C(s)ds} \,|\, \mathcal{F}_t\right]$ capturing collateral costs and
- calculation of expected future payoffs $\mathbb{E}^{T,C}\left[V(T) \,|\, \mathcal{F}_t\right]$ in the collateralized forward measure.

# We summarise the multi-curve framework widely adopted in the market




- Standard collateral curve is typically also considered as the risk-free curve.
- Currently standard collateral curves move from Eonia to €STR collateral rate (EUR) and Fed Fund to SOFR collateral rate (USD).
- Projection curves are potentially not required anymore in the future if Libor and Euribor indices are decommissioned.

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