Modern Modeling and Pricing of Interest Rates Derivatives
Day 2 - Session 4: Inflation Derivatives

Vladimir Sankovich [1]
Qinghua Zhu [2]

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[1] vsankovich@drwholdings.com
[2] qzhu@drwholdings.com

Consumer price indices

Consumers care about what money can buy, not money itself.

A *Consumer Price Index* (CPI) is meant to quantify “what money can buy":

- A basket of goods and services is chosen to be representative of the purchases of the consumers.
- The nominal value of such basket (rebased to 100 at some point in time) is computed and released at regular intervals.

This is, in essence, what a *Consumer Price Index* (CPI) is.

CPI indices are widely used for

- economic inferences - providing guidance for monetary policy and insight about national consumption and living standards.
- indexing of contracts and wages.
- indexing of financial instruments.

Inflation is the change in the level of prices indicated by a consumer price index.

Example

Let's say an investor's assets are worth 100$, and that with that money he could buy 1 unit of the reference basket. The CPI index is at 100.

After a year's time, the *nominal* value of his assets has increased to 105$, with a *nominal* the rate of growth of

$$\frac{105 - 100}{100} = 5\% \, .$$

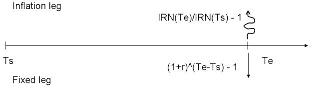
In the same year, however, the CPI index has increased to 110. As a result, in *real* terms, i.e. in terms of how many baskets of goods he can now purchase, the investor has suffered a loss of purchase power of

$$\frac{105/110 - 1}{1} = -4.5\% \, .$$

The inflation derivative's market has developed to exchange inflation exposure amongst investors.

Inflation-linked bonds

To have a guaranteed *real* return, rather than a guaranteed *nominal* one, an investor can use inflation-linked securities. The most basic example is the zero-coupon inflation-linked bond, that pays at maturity T the value $C(T)$ of the CPI index.



Denoting by $P_{IL}(0,T)$ the nominal value of the IL-ZC-bond as of today, we'll have

	today's value	value at maturity
in nominal units	$P_{IL}(0,T)$	$C(T)$
in real units	$P_r(0,T) = P_{IL}(0,T)/C(0)$	1

The real yield of this inflation-linked bond is

$$y_r = P_r(0,T)^{-1/T} - 1\,,$$

and it will be fully determined at the inception of the trade, whereas its nominal yield

$$y_n = [C(T)/C(0)]^{1/T}\; P_r(0,T)^{-1/T} - 1\,,$$

will be uncertain until maturity, when the value $C(T)$ of CPI index will be determined.

As for nominal bonds, issuers will typically issue coupon bearing inflation-linked bonds.

Inflation market - *The first steps*

In 1981, the UK was the first G7 issuer of inflation linked bonds ("linkers"). Their notional was linked to the UK-RPI index, with an 8-months indexation lag.

The intention of the issuance was to signal the credibility of the monetary policy.

Incidentally, however, they've been the first major step towards the developement of inflation as a tradable.

Very much later, in 1997, the US launched their "TIPS", Treasury Inflation Protected Securities.

Inflation linked bonds are now commonly issued by several countries, in addition to many corporate entities.

Inflation market - *Supply*

Natural supply of inflation-linked bonds comes from

- Market participants with income stream linked to inflation, either explicitly or implicitly, that want hedge their inflation risk.
- Issuers that want to avoid paying the inflation risk premium associated to the issuance of nominal bonds.
- Issuers that want to attract investors looking for risk diversification.

Typical suppliers are

- Governments
- Private Finance Initiative / Infrastructure projects - inflation linked revenues come either from construction contracts with explicit link to inflation, or from implicit inflation exposure.
- Real Estate - many contracts are inflation linked (for example, rents).
- Toll Roads - revenues often explicitly inflation linked, because of the contractual arrangements.
- Utilities - revenues directly linked to inflation.

Inflation market - *Demand*

Demand comes from entities that need to pay inflation-linked flows. For example

- Pension funds - often the exposure to inflation is explicit in the scheme, but even when it is not, there is a tacit link with inflation in the increases of wages and pensions.
- Inflation funds, asset managers.
- Insurers.
- Mutual funds (for portfolio diversification).
- Retail (not much).

There are also entities that drive both supply and demand

- banks - with the development of an interbank inflation market.
- hedge funds - with liquidity easing the possibility of running short term relative value trades.

Indexation lag

Time is required to collect and process consumer prices data, thus the inflation related to a particular month is typically announced two weeks after the end of that month.

To ensure that the payout of a coupon is known at payment date, and to allow the settling of accrued interests in between two payments, coupons are actually linked to the value of a so called *inflation reference number*, which is the value the CPI index had sometime before the payment date.

In some cases, products follow a *"month begin"* (MB) convention, whereby

$$IRN_{MB}(dd/mm/yyyy) = CPI(mm/yyyy - lag) \,.$$

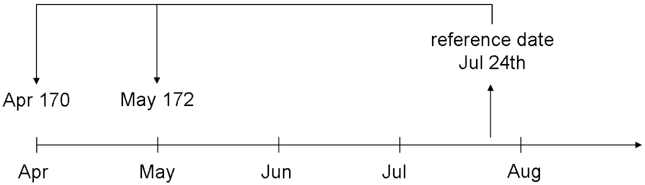
An alternative is to follow the *"interpolated fixings"* (IF) convention, whereby

$$IRN_{IF}(dd/mm/yyyy) = (1-\lambda)\, CPI(mm/yyyy - lag) + \lambda\, CPI(mm/yyyy - lag + 1m) \,,$$

with $\lambda = \frac{dd-1}{\text{days in mm}}$.

The lag between the CPI fixing date and the payment date is called inflation lag. Products on UK-RPI are usually MB with 2m-lag, whereas products on US-CPI are most of the time IF with 3m-lag.

Inflation Reference Number - example



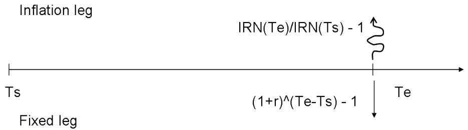
IRN = 170 + (24-1)/31 * (172-170)

Zero-coupon inflation swaps

The zero-coupon inflation swaps are the building blocks of the inflation derivatives market. In these transactions the inflation payer and receiver agree to swap on end date the inflation realised over the period going from start to end versus a compounded fixed rate flow

$$\frac{IRN(T_e)}{IRN(T_s)} - 1 \quad \text{vs} \quad (1+r)^{\tau} - 1\,,$$

where $\tau = T_e - T_s$ is the tenor of the swap.



Zero-coupon inflation swaps - term sheet & example

Notional:	\$1,000,000
Index:	US CPI-NSA
Trade date:	25 Feb 2013
Start date:	27 Feb 2013
End date:	27 Feb 2018
Base value:	$IRN_{FI}(27Feb2013) = \frac{2}{28} CPI(Nov12) + \frac{27-1}{28} CPI(Dec12) = 229.64$
Fixed leg:	$(1 + 2.52\%)^5 - 1$
Inflation leg:	$\frac{IRN_{FI}(27Feb2018)}{IRN_{FI}(27Feb2013)} - 1 = \frac{\frac{2}{28} CPI(Nov17) + \frac{27-1}{28} CPI(Dec17)}{\frac{2}{28} CPI(Nov12) + \frac{27-1}{28} CPI(Dec12)} - 1$

Zero-coupon inflation swaps - break-even rate & curve construction

The break-even rate is quoted for standard ZCIS that

- are spot starting, i.e. their start date is at at some standard settlement offset from the trade date such that $IRN(T_s)$ is known at inception,
- have a tenor that is a multiple of whole years,
- follow the same standard (index-dependent) convention.

These quotes are very liquid, and they are the starting point for building an inflation curve, or better, a curve for the IRN's: given the break-even rate $b(0, T_s, T_e)$ seen today for the period T_s–T_e, the forward value of the IRN at T_e will be

$$IRN(0, T_e) = [1 + b(0, T_s, T_e)]^{\tau} \cdot IRN(T_s)\,.$$

Because of seasonality, one can't simply interpolate these numbers.

Seasonality

Inflation indices exhibit a marked seasonal pattern. Things like sales periods, availability of types of food during the year, regular fiscal reviews, etc, create periodic patterns in the prices of constituents of the reference basket that are reflected by the index.

Seasonality can be thought as a multiplicative or additive pattern that is superimposed to the trend of inflation.

There are several ways to estimate such seasonality pattern, the "official" one being the X13-ARIMA-SEATS.

The same modulation pattern is assumed to apply to the inflation curve:

- the quotes for standard ZCIS are not sensitive to seasonality, because the swaps have whole year tenors,
- the interpolation of the IRN resulting from standard ZCIS would contain no seasonality effects, and it can be assumed to be representative of the inflation trend,
- the seasonality modulation can then be superimposed to it.

Inflation curve

One typically ends up building the curve of the forward values of inflation reference numbers (IRN's). More precisely, the curve at time t represents the value

$$I(T) = \mathbb{E}^{T+\delta}\left[(1-\alpha_T)\mathrm{CPI}_{[T]-\tau} + \alpha_T \mathrm{CPI}_{[T]-\tau+1}\,\middle|\, t=0\right],$$

where $[T]$ is the month at time T, τ is the indexation lag, α_T is the month fraction at T, and δ is a measure lag

$$\delta = \begin{cases} 0 & \textit{interpolated fixings (IF)} \text{ convention}, \\ \text{value date - [value date]} & \textit{month begin (MB)} \text{ convention}. \end{cases}$$

When dealing with *MB* indices, one will be interested in forwards on the 1st day of the months, i.e. $I([T])$, whereas for *IF* indices one will look for the quantity $I(T)$.

One should always keep in mind that the result corresponds to the expectation in a lagged measure. For MB indices, the measure lag changes during the month - instruments paying on different days of the month will bear a slightly different amount of convexity.

Inflation curve - II

This interpretation of the curve has the following advantages:

- the model represents more faithfully what the market is pricing (market prices already include the natural convexity);
- the reference measure is clearly stated;
- there's no more need to explicitly linearly interpolate the index fowards when working in the interpolated convention, as the curve models the interpolated object directly.
- one achieves the unification of the *month begin* and *interpolated fixings* conventions;
- even in the *interpolated fixings* convention, the curve can be directly calculated from the break-even of ZC inflation swaps (for simple interpolators, at least).

Seasonality

Seasonality can be represented by a 1y-periodic multiplicative modulation pattern. This is implemented as a periodic log-linear interpolator of monthly seasonality data, with the data anchored to the 1st day of their respective month.

Internally, the curve $I(T)$ is represented as the product of two functions, the trend $\mathrm{Trend}(T)$ and the seasonality $S(T)$

$$I(T) = \mathrm{Trend}(T) \cdot S(T)$$

The trend function $\mathrm{Trend}(T)$ is also implemented by an interpolator (typically, log-linear).
For any zero-coupon inflation swap in the calibration set, maturing at time T_i, there is an intepolator node, whose abscissa is $[T_i]$ for indices following the *month begin* convention, and T_i for indices following the *interpolated fixing* convention.

Seasonality Risk

In our setup, if $\mathbf{x}$ is the vector of the monthly seasonalities, and $\mathbf{z}$ is the vector of market-observed ZCIS prices, then the price P of any inflation-curve based product will satisfy the equation

$$P(\mathbf{x}; \mathbf{z}) = P(\alpha \mathbf{x}; \mathbf{z}).$$

The above relation holds because if one scales the seasonality inputs, the internal representation of the trend will scale inversely, in order to maintain the model calibrated to the input ZCIS quotes $\mathbf{z}$.

By Euler's theorem,

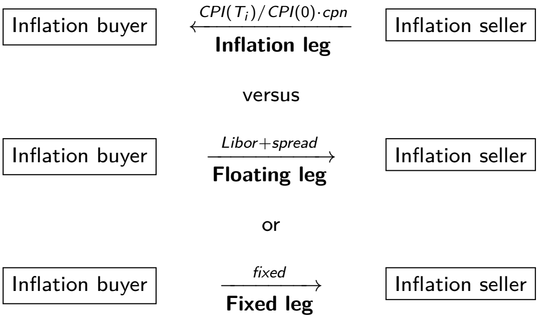
$$\sum_i x_i \partial_{x_i} P(\mathbf{x}; \mathbf{z}) = 0\,,$$

i.e. the seasonality risks are not independent, and that any one of them could be expressed as a linear combination of the remaining ones.

In other words, it's enough to hedge all but one seasonality risks.

Inflation Swaps

An inflation buyer pays a fixed or floating rate to an inflation seller, in exchange for inflation-linked payments.

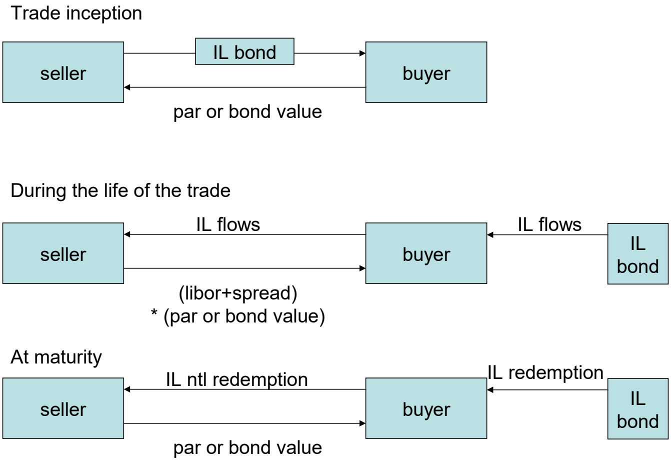


- Hedging inflation exposures - it would be impractical to do so through inflation-linked bonds (scarce liquidity, and heavy on balance sheet).
- Taking a view on inflation in an unfunded way.
- The swap can be tailored to suit the needs of the investor.

Inflation Asset Swaps

Asset swaps are meant to replicate the flows of an asset, without having direct exposure to the asset itself, thereby isolating the credit and liquidity component of the inflation-linked bond.

The schema below describes the mechanics of an inflation asset swap



YoY inflation swaps

In a YoY inflation swaps the inflation leg pays yearly the inflation rate realised over the payment period

$$YoY(T) = CPI(T)/CPI(T-1) - 1\,,$$

versus fixed or a floating leg. An example term-sheet could be

Notional:	ï£¡1,000,000
Index:	HICPxT
Trade date:	25 February 2013
Start date:	27 February 2013
End date:	27 February 2018
Rolls:	27th
Payment:	Annual, modified following
Day count:	30/360 unadjusted
Fixed rate:	1.73%
Fixed leg:	day count fraction * fixed rate
Inflation leg:	HICPxT(Nov yy)/HICPxT(Nov yy-1) - 1, yy=14,15,16,17,18.

Convexity

Let's take a closer look to a single YoY flow paying at T. It's forward value

$$\mathbb{E}^T\left[\frac{CPI(T)}{CPI(T-1)}|t=0\right]-1$$

clearly depends on the volatility and the correlation of the two inflation fixings, and it can't be valued off the inflation curve, which can only express expectations of the type

$$\mathbb{E}^T[CPI(T)|t=0].$$

Also, the curve provides the forward for $CPI(T-1)$ in the $(T-1)$-measure only. When valuing the YoY forward, we will have to move $CPI(T-1)$ to the T-forward measure. This will introduce a dependence of the YoY forward on the correlation inflation and interest rates, and on the interest rates volatility.

Analogous considerations apply when dealing with products with an indexation lag different from the one of the standard ZCIS's used to build the curve: the natural measure for these flows is not the one embedded in the curve, and one needs to perform a measure change to calculate their forward values. This change of measure introduces a dependence of the forward values of these flows on the correlation between interest rates and inflation, and their volatilities.

HJM model for for inflation

The first term structure model for inflation was the Jarrow-Yildirim model [2], based on the Inflation-FX analogy: the inflation index is a factor for converting the value of an asset in the real economy into the nominal economy. The instantaneous forward interest rates at T in the nominal and real economy, $f_n(t,T)$ and $f_r(t,T)$, together with the spot CPI $C(t)$ follow HJM-style relations.

Equivalently, let's model directly the inflation forward curve

$$C(t,T) = \mathbb{E}^T[C(T)|t] = C(t)\, I(t,T) = C(t)\, e^{\int_t^T \mathrm{d}u\, i(t,u)}\,, \tag{1}$$

where $i(t,T)$ is the the instantaneous forward inflation rate, which in the JY formulation would have been

$$i(t,T) = f_n(t,T) - f_r(t,T)\,. \tag{2}$$

By tower law, $C(t,T)$ will have to be a martingale in the T-forward measure; imposing this condition we'll find a set of HJM relations.

HJM model for for inflation

One starts as usual from

$$\begin{aligned} \mathrm{d}f_n(t,T) &= \sigma_n(t,T)\cdot\Sigma_n(t,T)\,\mathrm{d}t + \sigma_n(t,T)\cdot\mathrm{d}W_t\,, \\ \mathrm{d}i(t,T) &= \mu_i(t,T)\,\mathrm{d}t + \sigma_i(t,T)\cdot\mathrm{d}W_t\,, \\ \mathrm{d}C(t)/C(t) &= \mu_C(t)\,\mathrm{d}t + \sigma_C(t)\cdot\mathrm{d}W_t\,. \end{aligned} \tag{3}$$

- W_t is a multidimensional Wiener process in the measure associated to the cash account numeraire $B_n(t) = \exp(\int_0^t \mathrm{d}u\, r_n(u)) = \exp(\int_0^t \mathrm{d}u\, f_n(u,u))$,
- volatilities are intended as vectors which also express correlations,
- $\langle \mathrm{d}W_t, \mathrm{d}W_t\rangle = \mathbb{I}\mathrm{d}t$,
- $\Sigma(t,T) = \int_t^T \mathrm{d}u\, \sigma(t,u)$.

HJM model for for inflation - drifts

Then

$$\frac{\mathrm{d}C(t,T)}{C(t,T)} = \frac{\mathrm{d}C(t)}{C(t)} + \frac{\mathrm{d}I(t,T)}{I(t,T)} + \langle \frac{\mathrm{d}C(t)}{C(t)}, \frac{\mathrm{d}I(t,T)}{I(t,T)} \rangle =$$

$$= \mu_C(t)\,\mathrm{d}t + \sigma_C(t) \cdot \mathrm{d}W_t - i(t,t)\,\mathrm{d}t + \int_t^T \mathrm{d}u\, \mu_i(t,u)\,\mathrm{d}t + \Sigma_i(t,T) \cdot \mathrm{d}W_t$$

$$+ \frac{1}{2}\Sigma_i(t,T) \cdot \Sigma_i(t,T)\,\mathrm{d}t + \sigma_C(t) \cdot \Sigma_i(t,T)\,\mathrm{d}t\,. \qquad (4)$$

Remember that $\mathrm{d}W_t = \mathrm{d}W_t^T - \Sigma_n(t,T)\,\mathrm{d}t$, so imposing $C(t,T)$ to be a T-fwd martingale, i.e. imposing the drift in Eq. (4) above to be zero, leads to

$$\mathrm{d}C(t,T) = C(t,T)\,[\sigma_C(t) + \Sigma_i(t,T)] \cdot \mathrm{d}W_t^T\,. \qquad (5)$$

Imposing the limit $T \to t$ of the drift in the T-fwd measure to be zero

$$\mu_C(t) = i(t,t)\,. \qquad (6)$$

Imposing the derivative wrt to T of the drift in the T-fwd measure to be zero

$$\mu_i(t,T) = [\sigma_C(t) + \Sigma_i(t,T)] \cdot [\sigma_n(t,T) - \sigma_i(t,T)] + \sigma_i(t,T) \cdot \Sigma_n(t,T)\,. \qquad (7)$$

HJM model – deterministic vols, pricing ZC options

From Eq. (5), for deterministic vols

$$C(T) = C(T,T) = C(t,T)\,\mathcal{E}\left(\int_t^T \mathrm{d}W_s^T \cdot [\sigma_C(s) + \Sigma_i(s,T)]\right)$$
$$= C(t,T)\,\mathcal{E}\left(\int_t^T \mathrm{d}W_s^T \cdot \tilde{\Sigma}(s,T)\right), \qquad (8)$$

where $\mathcal{E}\left(\int_a^b \mathrm{d}W_s^T \cdot f(s)\right) \doteq e^{\int_a^b \mathrm{d}W_s^T \cdot f(s) - \frac{1}{2}\int_a^b \mathrm{d}s \|f(s)\|^2}$.

Thus, $C(T)$ is lognormal with volatility $\upsilon_{\mathrm{CPI}}(T)$ defined by

$$\upsilon_{\mathrm{CPI}}(T)^2\, T = \int_0^T \mathrm{d}s\, \tilde{\Sigma}(s,T) \cdot \tilde{\Sigma}(s,T)\,. \qquad (9)$$

The present value call option on the CPI index expiring and paying at T is then

$$\mathrm{Call}_{\mathrm{CPI}}^T(0) = P(0,T)\,\mathbb{E}^T\left[[C(T) - K]^+ | \mathcal{F}_t\right]$$
$$= P(0,T)\,\mathrm{BS}\left(C(0,T), K, \upsilon_{\mathrm{CPI}}(T), T\right). \qquad (10)$$

HJM model – deterministic vols, pricing ZC options with delayed pay

When the option pays at some other time T_p, its value will be

$$\text{Call}_{\text{CPI}}^{T_p}(0) = P(0, T_p)\, \mathbb{E}^{T_p} \left[[C(T) - K]^+ | \mathcal{F}_t \right] . \tag{11}$$

Changing the measure from the T-forward to the T_p-forward measure

$$\mathrm{d}W_t^T = \mathrm{d}W_t^{T_p} - [\Sigma_n(t, T_p) - \Sigma_n(t, T)]\mathrm{d}t = \mathrm{d}W_t^{T_p} - \Gamma(t, T, T_p)\mathrm{d}t\,, \tag{12}$$

one can write

$$\begin{aligned} C(T) &= C(t, T)\, \mathcal{E}\left(\int_t^T \mathrm{d}W_s^T \cdot \tilde{\Sigma}(s, T) \right) = \\ &= C(t, T)\, e^{-\int_t^T \mathrm{d}s\, \tilde{\Sigma}(s,T)\cdot\Gamma(s,T,T_p)}\, \mathcal{E}\left(\int_t^T \mathrm{d}W_s^{T_p} \cdot \tilde{\Sigma}(s, T) \right) , \end{aligned} \tag{13}$$

and then

$$\text{Call}_{\text{CPI}}^{T_p}(0) = P(0, T_p)\, \text{BS}\left(C(0, T)\, e^{\lambda(T;T_p)}, K, \upsilon_{\text{CPI}}(T), T \right) , \tag{14}$$

with the convexity adjustment $\lambda(T; T_p) = e^{-\int_0^T \mathrm{d}s\, \tilde{\Sigma}(s,T)\cdot\Gamma(s,T,T_p)}$.

HJM model – deterministic vols, pricing Inflation options

The value of an option paying at T_p the growth of the CPI index $C(T)/C(T')$ is

$$\mathrm{Call}_{\mathrm{infl}}^{T_p}(t) = P(t, T_p)\, \mathbb{E}^{T_p} \left[[C(T)/C(T') - K]^+ | \mathcal{F}_t \right] . \tag{15}$$

Measure-changing both $C(T)$ and $C(T')$ to the T_p-forward measure one can easily see that their ratio is lognormally distributed, and the present value of the option is

$$\mathrm{Call}_{\mathrm{infl}}^{T_p}(0) = P(0, T_p)\, \mathrm{BS} \left(C(0,T)/C(0,T')e^{\lambda(T,T';T_p)}, K, \upsilon_{\mathrm{infl}}(T,T'), T \right) , \tag{16}$$

where

$$\upsilon_{\mathrm{infl}}^2(T,T')\, T = \upsilon_{\mathrm{CPI}}^2(T) T + \upsilon_{\mathrm{CPI}}^2(T')T' - 2 \int_0^{T'} \mathrm{d}s\, \tilde{\Sigma}(s,T) \cdot \tilde{\Sigma}(s,T') , \tag{17}$$

$$\lambda(T,T';T_p) = \lambda(T;T_p) - \lambda(T';T_p) + \int_0^{T'} \mathrm{d}s\, \tilde{\Sigma}(s,T') \cdot \left[\tilde{\Sigma}(s,T') - \tilde{\Sigma}(s,T) \right] . \tag{18}$$

HJM model – deterministic vols. The triangle of volatilities

The above equations can be rewritten as

$$\upsilon^2_{\mathrm{CPI}}(T)\,T = \|\Sigma_T\|^2 \tag{19}$$

$$\upsilon^2_{\mathrm{infl}}(T,T')\,T = \|\Sigma_T - \Sigma_{T'}\|^2 = \|\Sigma_T\|^2 + \|\Sigma_{T'}\|^2 - 2\langle\Sigma_T,\Sigma_{T'}\rangle \tag{20}$$

$$\lambda_t(T,T';T_p) = -\langle\Sigma_T,\Gamma_{T,T_p}\rangle + \langle\Sigma_{T'},\Gamma_{T',T_p}\rangle + \|\Sigma_{T'}\|^2 - \langle\Sigma_{T'},\Sigma_T\rangle \tag{21}$$

One can introduce the effective CPI correlation

$$\xi(T',T) = \frac{\langle\Sigma_{T'},\Sigma_T\rangle}{\|\Sigma_{T'}\|\|\Sigma_T\|} = \frac{\|\Sigma_{T'}\|^2 + \|\Sigma_T\|^2 - \|\Sigma_T - \Sigma_{T'}\|^2}{2\|\Sigma_{T'}\|\|\Sigma_T\|} \tag{22}$$

and the effective Inflation-IR correlation

$$\xi^{\mathrm{IR}}(T,T_p) = \frac{\langle\Sigma_T,\Gamma_{T,T_p}\rangle}{\|\Sigma_T\|\|\Gamma_{T,T_p}\|}\,, \tag{23}$$

which corresponds to **minus** the correlation between the inflation forward at T_i and the forward starting zero-coupon bond $P(t,T,T_p)$.

A very practical approach when trading exclusively ZC and YoY options is to mark separately ZC and YoY smiles, mark the above correlations, and make sure that the relations (21) hold at least ATM. The bond volatilities required for the cvx adj can be proxied from those of caplets/swaptions.

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