Modern Modeling and Pricing of Interest Rates Derivatives
Day 1 - Session 3-1: An Introduction to Convexities

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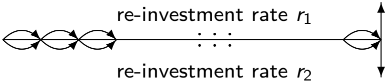
Convexity is Relative

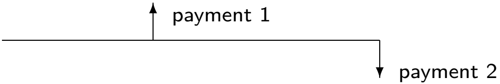
- *Convexity* and *Linearity* are relative terms.
- Relative between the product we are pricing, and the hedging product. When exposure of the pricing product cannot be statically hedged with hedging product, there is convexity.
- Claim 1: Eurodollar futures are linear, because every basis point change in rates is worth $25.
  Claim 2: FRAs are linear, because one calculates forward rates directly from a yield curve.
- However, there is convexity *between* futures and FRAs, because they have different settlement rules - details to follow.
- Pricing product: FRA/swap; Hedging instrument: Futures. $\Rightarrow$ Add convexity on futures prices when constructing yield curve, which is then used to price FRAs and swaps.

Mis-match of Cashflows

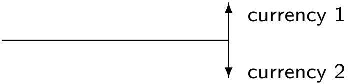
- Convexities come from the mis-match of cashflows that have the same underlying but different
  - **Funding:** CSA terms, un-collateralized, daily settled(futures), etc



  - **Timing:** Different payment dates



  - **Payment currency:** Quanto

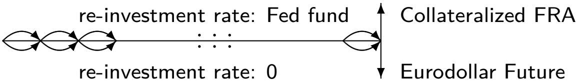


- Although old concept, multi-curve framework brings more complexity.

Funding Convexity

Example 1: Eurodollar Future vs Collateralized FRA

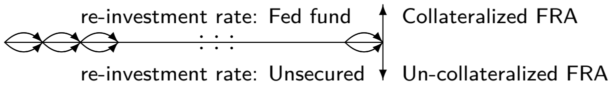
- Long FRA, hedge with futures;
- Collateral posted/received from the FRA offsets the variation margin received/posted from the EDF;
- FRA collateral pays Fed fund rates, EDF variation margin pays no interest;
- Net: Fed fund rates;
- Assume positive correlation between Libor rate and Fed fund rate;
- Rates go up: get collateral on FRA, post to CME, pay counterparty Fed fund rate, which go up too;
- Rates go down: post collateral, receive interest from counterparty on a lower rate;
- Portfolio has negative PV, FRA rate is lower than futures rate.
- Convexity depends on the correlation between underlying Libor and Fed fund rates.



Funding Convexity

Example 2: Collateralized FRA vs Un-collateralized FRA

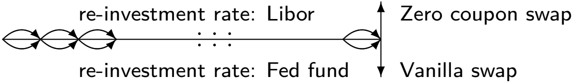
- Long un-collateralized FRA, hedge with collateralized FRA;
- Post/receive collateral on collateralized FRA (earn/pay Fed fund rate) and borrow/lend at un-secured funding rate;
- Net is *Funding Spread*: Un-secured funding - Fed fund rate;
- Assume positive correlation between Libor rate and funding spread;
- Rates go up: net interest to pay on collateral is higher;
- Rates go down: net interest to receive on collateral is lower;
- Portfolio has negative PV, un-collateralized FRA rate is lower than collateralized FRA rate.
- Convexity depends on the correlation between underlying Libor and funding spread.



Funding Convexity

Example 3: Zero Coupon Swaps

- Floating leg pays compounded Libor rates
- Hedge with vanilla swap
- "Compounding" means each Libor reset gets re-invested at next Libor
- If funded at Libor, no convexity
- If funded at OIS, net is Libor-OIS basis. If none-zero correlation between Libor and Libor-OIS basis, there is convexity.



Funding Convexity

Summary

- Let r_F: funding rate of the cashflow to be priced
  r_H: funding rate of the hedging cashflow.
- r_F and r_H could be
  1) c if domestically collateralized
  2) r_{bank} if un-secured
  3) $r - (r_f - c_f)$ is foreign collateralized
  4) 0 is exchange traded
  5) $\max_{i=1,\ldots,n}(r - r_i + c_i)$ if CTD.
- Current market: $r_H = c$.
- $V_F(t) = \mathbb{E}_t^Q[e^{-\int_t^T r_F(s)ds} V(T)] = P_F(t,T)\mathbb{E}_t^{T_F}[V(T)]$
  $V_H(t) = \mathbb{E}_t^Q[e^{-\int_t^T r_H(s)ds} V(T)] = P_H(t,T)\mathbb{E}_t^{T_H}[V(T)]$
- Need $\mathbb{E}_t^{T_F}[V(T)]$, have $\mathbb{E}_t^{T_H}[V(T)]$. Convexity is the difference.
- $\mathbb{E}_t^{T_F}[V(T)] = \frac{P_F(t,T)}{P_H(t,T)}(\mathbb{E}_t^{T_H}[V(T)e^{-\int_t^T [r_F(s)-r_H(s)]ds}] = \mathbb{E}_t^{T_H}[V(T)] + C$, where
  $C = \frac{P_F(t,T)}{P_H(t,T)} Cov(V(T), e^{-\int_t^T [r_F(s)-r_H(s)]ds})$.
- Funding convexity is embedded in non-USD funding curve during curve construction

Timing Convexity and Quanto Convexity

Timing Convexity

- Need $\mathbb{E}_t^{T'}[V(T)]$, have $\mathbb{E}_t^{T}[V(T)]$.
- $\mathbb{E}_t^{T'}[V(T)] = \frac{P(t,T)}{P(t,T')}\mathbb{E}_t^{T}[V(T)e^{-\int_T^{T'} r(s)ds}]$
- Convexity: $\frac{P(t,T)}{P(t,T')}Cov(V(T), e^{-\int_T^{T'} r(s)ds})$

Quanto Convexity

- Need $\mathbb{E}_t^{T_f}[V(T)]$, have $\mathbb{E}_t^{T_d}[V(T)]$.
- $\mathbb{E}_t^{T_f}[V(T)] = \frac{P_d(t,T)}{P_f(t,T)X(t)}\mathbb{E}_t^{T}[V(T)X(T)]$
- Convexity: $\frac{P_d(t,T)}{P_f(t,T)X(t)}Cov(V(T), X(T))$

Example: MtM Xccy Swap

- Liquid; widely used in curve calibration.
- A strip of one-period xccy swaps, where the notional of the USD leg is reset at the beginning of each period, $N_\$(T_i) = N_d \cdot X(T_i)$
- Assume OIS discounting
- Each period:
  $PV_\$ = \mathbb{E}_t^Q \left(X(T-\tau) \left[-e^{-\int_t^{T-\tau} c(s)ds} + e^{-\int_t^{T} c(s)ds} (1 + \tau L(T-\tau, T)) \right] \right)$
  $= P(t,T)\mathbb{E}_t^T[X(T-\tau)S(T)]$, where $S(t)$ is the FRA-OIS spread.
- Hedge with constant notional xccy swap: each period
  $PV_\$ = P(t,T)X(0)\mathbb{E}_t^T[S(T)]$
- Net: $S(T) \cdot (X(T-\tau) - X(0))$
- **Quanto convexity** depends on the convariance between FX rate and FRA-OIS spread.
- There is also **timing convexity** since the FX rate is fixed at the *beginning* of the payment period.
- Non-USD leg is funded at USD OIS, but non-USD Libor is calibrated under domestic OIS funding assumption $\Rightarrow$ **funding convexity!**
- Calibrated non-USD funding curve has all of funding, timing and quanto convexities embedded.

Summary

- Convexities comes from mis-match of cashflows, equivalently, change of measure
- Covariance between the payoff and the Radon-Nikodym derivative
- Radon-Nikodym derivative
  - Funding: funding spread
  - Timing: funding rate between the two payment dates
  - Quanto: FX spot rate

Modern Modeling and Pricing of Interest Rates Derivatives
Day 1 - Session 3-2: Cheapest To Deliver Funding Option

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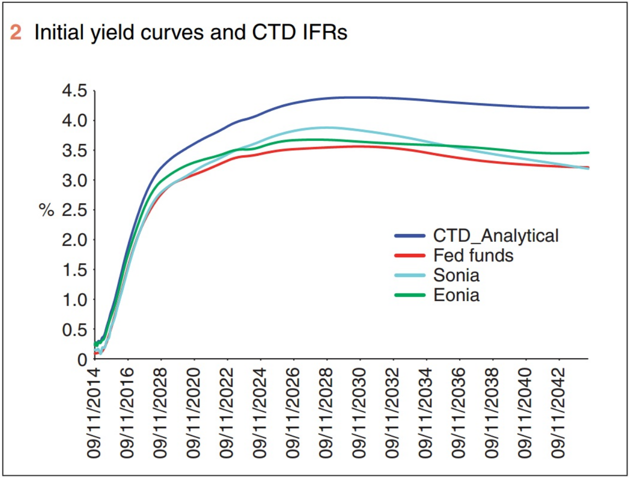
The Problem

- **Definition:** Some CSA allow several currencies as eligible collateral. This gives the counterparties to the contract the option to choose the currency in which they will post collateral.
- **Two flavors:** with and without substitution of collateral (New York Law vs UK Law)
  - Example: Original MtM \$50K, was posted in dollar; New MtM \$60K, need to post \$10K more, EUR is the cheapest.
  - with substitution: get \$50K USD cash back, post \$60k worth of EUR.
  - without substitution: cannot get USD cash back, post incremental collateral \$10K in EUR.
- **Posting Strategy (with substitution):** Choose the "cheapest" collateral currency, i.e., one that maximized return on collateral.
- **Discount Factors:**
  - No CTD, collateralized domestically, earning overnight rate of c(t) on posted collateral: $DF = \mathbb{E}_t^Q \left[e^{-\int_t^T c(s)ds} \right]$
  - No CTD, collateralized in a foreign currency, must take into account of cross currency basis: $DF = \mathbb{E}_t^Q \left[e^{-\int_t^T [c^f(s) - R^f(s) + R^d(s)]ds} \right]$
  - CTD, USD trade, three eligible currencies USD, EUR, GBP:
    $$DF = \mathbb{E}_t^Q \left[e^{-\int_t^T \max\{c^\$(s), c^€(s) - R^€(s) + R^\$(s), c^£(s) - R^£(s) + R^\$(s)\} ds} \right]$$

Modeling Choices

- **Ignore the option completely** - price to, say, domestic OIS
- **"Today's CTD" approach** - discount using the curve with the highest overnight rate today
- **"Intrinsic CTD"** - ignore the volatilities of all rates and spreads (i.e. the probability of the CTD changing over the life of the trade) and discount at the "composite curve" which traces the highest implied funding rates
- **"Complete CTD Model"** - build a stochastic model of rates/spreads, identify its parameters, price trades with CTD in the CSA using it.

Instantaneous Forward Rates



Model Motivation

- **Motivation:** We need a model for CTD which (ideally) integrates with the yield curve.
- **Why?** Because we want a single model to be the source of all discount factors rather than decide at pricing time which model to invoke.
- **Consequence:** It's desirable to avoid Monte Carlo, hence need fast and accurate approximations. This is the subject of this presentation.

Summary of Model

- Goal: Find a good approximation of the integral $I = \int_t^T \max_{i=1,\dots,N} r_i(s)ds$, where r_i is the cross currency adjusted short rate of the ith currency in the basket, $r_i = c^i - (R^i - R^d)$.
- Use a functional form to capture higher moments - quadratic function of a standard normal.
- Determine the coefficients by moment matching - need first three moments of the time integral I.
- Use LGM dynamics $dr_i(t) = \kappa_i(t)[\theta_i(t) - r_i(t)]dt + \sigma_i(t)dW_i(t)$
- Derive a good approximation for the distribution of the maximum of N Gaussian variables.

Summary of Model

- Find a process X_t that approximates the dynamics of the max of N Gaussian processes and matches the instantaneous distributions found.
- Calculate moments of the integral $Y_t \equiv \int_0^t X_s ds$.
- Fit a quadratic Gaussian distribution $Y_t = a(t)z^2 + b(t)z + c(t)$, where $z \sim N(0,1)$ to the first three moments of Y_t.
- Calculate discount factor: $DF(t) = \mathbb{E}[e^{-Y_t}] = \dfrac{\exp\left(\frac{b^2(t)}{2(1+2a(t))} - c(t)\right)}{\sqrt{1+2a(t)}}$

Parameter Estimation

- Requires the volatility and the mean reversion rate of each funding rate, and the correlations between them.
- No liquidly traded products to hedge vol/correlation/mean reversion exposure.
- Estimate parameters from historical data.
- Cannot estimate from short rates directly, as they are not observable. Instead, use IFRs implied from historical yield curves.
- $\kappa_i = -\dfrac{\ln \sigma_{f_i(t,T_1)} - \ln \sigma_{f_i(t,T_2)}}{T_1 - T_2}, \sigma_i = e^{\kappa_i(T_1-t)}\sigma_{f_i(t,T_1)}$
- Estimate $\sigma_{f_i(t,T_1)}$ and $\sigma_{f_i(t,T_2)}$ from time series of IFRs
- Estimate correlations from the same time series.

D. Estimated parameters							
$\sigma_\$$	0.73%	$\kappa_\$$	0.72%	$\rho_{\$,€}$	97%	$r_\$(0)$	0.0845%
$\sigma_€$	0.73%	$\kappa_€$	0.83%	$\rho_{\$,£}$	95%	$r_£(0)$	0.1514%
$\sigma_£$	0.74%	$\kappa_£$	0.80%	$\rho_{€,£}$	95%	$r_€(0)$	0.2265%

Results

E. Par rates of 10Y and 30Y USD swaps		
Funding	10Y	30Y
Fed Fund	2.667%	3.243%
Intrinsic	2.663%	3.234%
Stochastic	2.659%	3.222%

F. Price differential of 10Y and 30Y USD swaps				
Funding	ATM 10Y	OTM 10Y	ATM 30Y	OTM 30Y
Intrinsic	−0.51	−1.45	−0.89	−3.59
Stochastic	−0.92	−2.41	−1.95	−7.63

OTM swaps are 100 bps below par

Other Considerations

- **Further Simplification:**
  - Funding rate r_i has the payment currency embedded $r_i = r^R - (r_i^R - c_i)$. Need to build another model, if we have a different payment currency with the same basket.
  - Simplification: assume r^R and $r_i^R - c_i$ are uncorrelated.
  - $DF = \mathbb{E}_t \left[\exp\left(- \int_t^T r^R(s) ds \right) \right] \times \mathbb{E}_t \left[\exp\left(- \int_t^T \max_{i=1,\dots,N} (r_i^R(s) - c_i(s) ds \right) \right]$
- **Floored CSA:** Set $r_2(t) = 0, \sigma_2 = 0$.
- **An Alternative Way of Model Parameter Estimation**
  - Assume the spread between Libor and funding rate is deterministic
  - Calibrate Libor vols to market (hedging instruments)
  - Particularly useful for floored CSA

Is the Collateral Option Relevant?

- **Bi-lateral (uncleared) Swaps:** Development of Standard CSA terms will remove optionality from CSAs
- **Cleared Swaps:** Standard collateralization rules apply to cleared swaps (domestic cash, zero threashods, no MTA, no optionality)
- **So why should we bother with the CTD?**
- **Legacy CSA**
  - Existing and new transactions executed under legacy CSAs with optionality $\Rightarrow$ pricing and valuation should take CTD option into account
  - **Re-negotiating CSA** Moving to simpler CSA terms (e.g. domestic cash collateral only) $\Rightarrow$ we want to *at least* have an idea of the value of the option we're leaving behind (if not get paid for it)
  - **LCH backloads** Same as above. Cleared swaps will be collateralized with domestic cash, so backloads need to be executed so that we don't leave the value of the option on the table
- **Several houses take it into account when quoting and unwinding swaps; some software vendors are giving it incorporated too.**

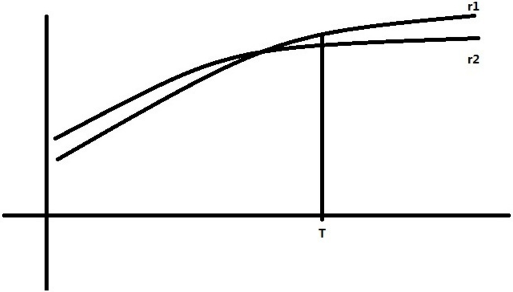
No-substitution CTD

Rules:

- Case 1: PV changes sign.
  Original collateral fully returned. New posting party choose currency to post.
- Case 2: PV becomes more negative to posting party.
  Can only add additional collateral, cannot substitute existing collateral with new currency.
- Case 3: PV becomes less negative to posting party.
  Receiving party has the optionality to choose which currency to return.

Simplest Case

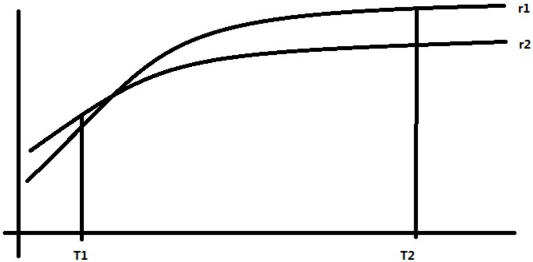
- Two currencies, deterministic rates, one cashflow, MtM doesn't change, no initial collateral in the account



- To maximize interest on collateral, choose Currency 2
- What matters? The max of term rates, instead of max of IFRs

Portfolio Effect

- Two cashflows: $C(T1) = \$1$, $C(T2) = -\$2$.
- Collateral amount = \$1 in [0, T1] and \$2 in [T1, T2].



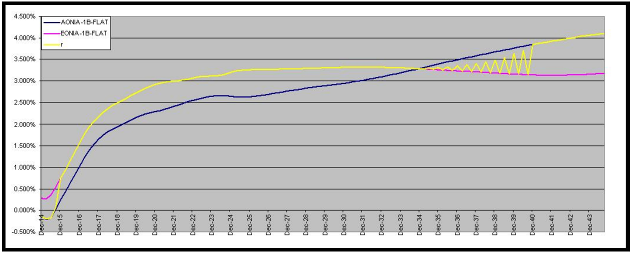
- If treated as 2 single cashflows, discount C(T1) at r2, and C(T2) at r1.
- Posting strategy: At T1, addition \$1 to post, choose currency r1.
  At T=0, \$1 to post, interest accrues from 0 to T2
  $\rightarrow$ choose currency r1
  Discount both cashflows at r1!

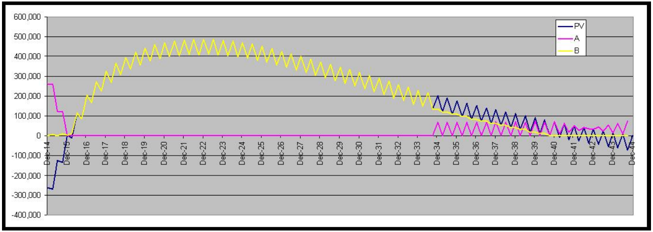
Dependence on Initial Collateral

- *Initial Collateral*: The amount of money in each currency already in the collateral account
- Example: Current collateral account has \$2, add a new trade with single cashflow \$1 at T, MtM changes to \$1.
- If IC(ccy2) $=$ \$2, IC(ccy1) $=$ \$0, receiving party returns \$1 in ccy 2. New cashflow is discounted at r2.
- If IC(ccy1) $=$ \$2, IC(ccy2) $=$ \$0, receiving party returns \$1 in ccy 1. New cashflow is discounted at r1.

Example

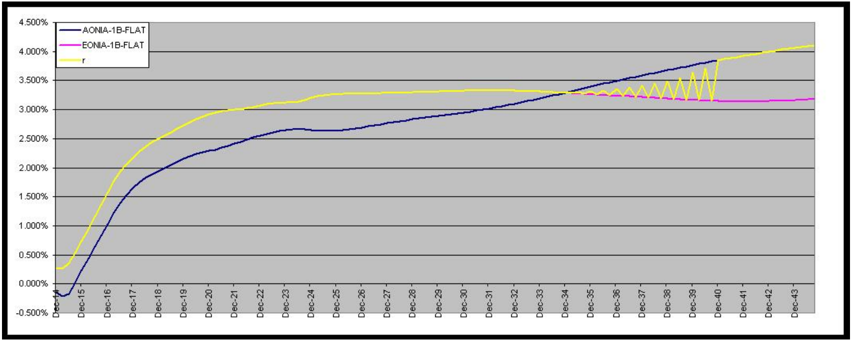
Basket consists two currencies
Only consider intrinsic case
Swap: 30y, USD semi bond, 3%, {EUR, AUD} basket
Initial collateral: \$500k in AUD





Example

Swap: 30y, USD semi bond, 3%, {EUR, AUD} basket
Initial collateral: $500k in EUR



Future Work

- More than two currencies in the basket
- Stochastic
- Similarity to Bermudan style options
- American-style Monte Carlo
  - Simulate underlying the funding rate together
  - Or, assume no correlation between V and funding rates
- HJB Equation
  - Deal with δ_i and the MinOrMax operator in continuous time.
  - Boundary condition on A, B and $A'(t), B'(t)$.
  - Solve an optimization problem at each time step.

Appendix: S-Z CTD model

Maximum of N Gaussian Variables

- **Clark's Approach:**
  - Distribution of max of a pair of Gaussian variables is known exactly.
  - Choose 2 variables from the set, compute the moments of their max:

    $$\mathbb{E}[(\max(X,Y))^i] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\max(X,Y))^i \phi^{(2)}_{x,y}(x,y)dxdy,$$

    then approximate it with a Gaussian variable which matches the first 2 moments.
  - Replace the original variables with this new Gaussian variable, which is jointly normal with the rest of the set.
  - Clark's procedure becomes progressively less accurate as the number of random variables in the set increases, due to substantial skew of the max.
- **Gram-Clarlier Expansion:**
  - $G(x) \equiv \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(x-\mu)^2}{2v}\right)\left[1 + \frac{k}{3!v^{3/2}} H_3\left(\frac{x-\mu}{\sqrt{v}}\right)\right]$, where μ, v, k are the mean, variance, and 3rd cumulant of the distribution; $H_3(x) = x^3 - 3x$ is the 3rd order Hermite polynomial.
  - Joint density of $X_1 = \max(r_1, r_2)$ and r_3:

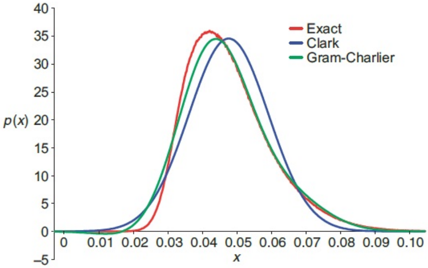
    $$p_{X_1,r_3}(x,y) = \phi^{(2)}_{X_1,r_3}(x,y) + \phi_{r_3}(y)(G_{X_1}(x) - \phi_{X_1}(x))$$

    .
  - Adjustments to Clark's formulas are derived.

Appendix: S-Z CTD model

Maximum of N Gaussian Variables

1 Pdf of $\max(X_1, X_2, X_3, X_4)$, where $X_1 \sim N(0.036, 0.01)$, $X_2 \sim N(0.03, 0.005)$, $X_3 \sim N(0.035, 0.02)$, $X_4 \sim N(0.025, 0.02)$, $\vec{\rho} = 0$



B. Moments of $\max(r_1, \ldots, r_n)$

n	Variance			Skewness		
	Exact	Clark's	Gram-Charlier	Exact	Clark's	Gram-Charlier
2	0.000061	0.000061	0.000061	0.5988	0	0.5995
3	0.000142	0.000139	0.000142	0.9941	0	0.9862
4	0.000147	0.000133	0.000149	0.8498	0	0.7149

Appendix: S-Z CTD model

The Dynamics of the Maximum

- Knowledge of terminal distributions is not sufficient; need knowledge of the process followed by $X_t := \max(r_i(t))$
- Why? Consider the approximation $\displaystyle\int_0^t X_s ds = w_1 X_{t_1} + w_2 X_{t_2}$, where w_1 and w_2 are some weighting factors. Then moments of the integral depend on how X_{t_2} and X_{t_2} are correlated.
- Ito-Tanaka formula: $\displaystyle dX_t = \sum_{i=1}^N 1_{X(t)=r_i(t)} dr_i(t) + \frac{1}{2} dL^0_{t,N}(\vec{r})$
- Write $dL^0_{t,N}(\vec{r}) = 2\alpha(t,\vec{r})dt$, then

$$dX_t = \left[\sum_{i=1}^N 1_{X(t)=r_i(t)}[\kappa_i(\theta_i(t) - X_t)] + \alpha(t,\vec{r})\right] dt + \sum_{i=1}^N 1_{X(t)=r_i(t)} \sigma_i dW_i(t)$$

- "Freeze" the indicators, $1_{X(t)=r_i(t)} \to \mathbb{E}[1_{X(t)=r_i(t)}]$
- Introduce $\displaystyle\kappa(t) = \sum_i \mathbb{E}[1_{X(t)=r_i(t)}]\kappa_i(t)$ and $Z(t) = \exp(\int_0^t \kappa(s)ds)X(t)$, and write $dZ(t) = \Lambda(t)[\Theta(t) - \alpha(t,\vec{r})]dt + \Gamma(t,\vec{r})d\vec{W}(t)$
- Assume $Z(t)$ has independent increments, i.e., ignore correlation between $Z(s)$ and $\alpha(t,\vec{r})$ for $t > s$.

Integral of the Maximum

- Calculate moments of $Y_t \equiv \int_0^t X_s ds$:

$$\mathrm{Var}[Y_t] = 2\int_0^t \int_0^\tau \mathrm{Cov}(X_s, X_\tau) ds d\tau = 2\int_0^t e^{-\int_0^\tau \kappa(u)du} \int_0^\tau e^{-\int_0^s \kappa(u)du} \mathrm{Cov}(Z_s, Z_\tau) ds d\tau$$

$$= 2\int_0^t e^{-\int_0^\tau \kappa(u)du} \int_0^\tau e^{-\int_0^s \kappa(u)du} \mathrm{Var}[Z_s] ds d\tau = 2\int_0^t e^{-\int_0^\tau \kappa(u)du} \int_0^\tau e^{\int_0^s \kappa(u)du} \mathrm{Var}[X_s] ds d\tau$$

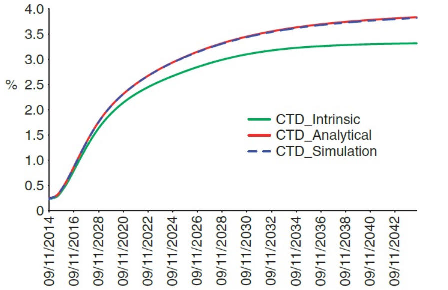
$$\mathbb{E}[(Y_t - \bar{Y}_t)^3] = 6\int_0^t e^{-\int_0^u \kappa(v)dv} \int_0^u e^{-\int_0^\tau \kappa(v)dv} \int_0^\tau e^{2\int_0^s \kappa(v)dv} \mathbb{E}[(X_s - \bar{X}_s)^3] ds d\tau du$$

- Fit a quadratic Gaussian distribution $Y_t = a(t)z^2 + b(t)z + c(t)$, where $z \sim N(0,1)$
- Calculate discount factor: $DF(t) = \mathbb{E}[e^{-Y_t}] = \dfrac{\exp\left(\frac{b^2(t)}{2(1+2a(t))} - c(t)\right)}{\sqrt{1+2a(t)}}$

Appendix: S-Z CTD model

Accuracy

3 Analytic versus simulation on zero rates



G. Maximum of differences in zero rates and IFRs between semi-analytical approximation and Monte-Carlo simulation

Volatility scenario	Zero rate max difference (bp)	IFR max difference (bp)
0.25%	0.5	0.7
0.50%	0.7	1.1
0.75%	1.3	3.0
1.00%	1.5	3.1
1.50%	1.5	8.7

Appendix: S-Z CTD model

Performance

- Choose a time discretization $\{t_i\}$; for each t_i, compute CTD discount factor; for $t_i < t < t_{i+1}$, apply interpolation.
- Calculation of the moments of Y_t reduces to computing repeated time integrals of moments of X_t, which contain only simple univariate normal CDF and PDF functions.
- Necessary integrals can be efficiently computed numerically using Gaussian quadratures.
- Once model is constructed, pricing with and without CTD options is done in the same amount of time.

H. Time to build CTD funding models as a ratio of time to build multi-currency yield curves

Number of currencies in basket	Time ratio
2	1.1
3	1.6
4	1.9
5	2.6

Appendix

Parameter calibration to Totem

Base CCY	Tenor	CollateralType	Collateral	Trade	FixedRate	FixedSpread	A	B	C	D	E	F	G	H	I
USD	10y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	Payer	Par	0	-0.2	-0.7	-0.7	-0.6	-0.8	-0.7	-0.6	-0.7	-0.7
USD	10y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	Payer	Par-100bps	-100	1.1	-2.7	-2.7	-2.7	-2.8	-2.9	-2.7	-2.8	-2.9
USD	10y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	Payer	Par+100bps	100	0.6	1.4	1.4	1.4	1.2	1.4	1.4	1.4	1.5
USD	10y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	Receiver	Par	0	0.2	0.7	0.7	0.6	0.8	0.7	0.6	0.7	0.7
USD	10y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	Receiver	Par-100bps	-100	1.1	2.7	2.7	2.7	2.8	2.9	2.7	2.8	2.9
USD	10y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	Receiver	Par+100bps	100	-0.6	-1.4	-1.4	-1.4	-1.2	-1.4	-1.4	-1.4	-1.5
USD	30y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	Payer	Par-100bps	-100	-3.7	-5.4	-6.3	-6		-6.6	-5.5	-5.7	-6.8
USD	30y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	PayerNotional	Bank pays X% notional	SingleCF					-452.9			-467	-600.1
USD	50y	MultiCash	USD FedFunds, EUR EONIA and GBP SONIA	PayerNotional	Bank pays X% notional	SingleCF					-519.4			-513.7	-626.2
USD	10y	MultiCash	USD FedFunds, GBP SONIA	Payer	Par-100bps	-100	-0.6		-1.2		-0.9			-0.8	-1.1
USD	10y	MultiCash	USD FedFunds, EUR EONIA	Payer	Par-100bps	-100	-1		-2.6		-2.8	-2.9		-2.8	-2.8
USD	10y	MultiCash	USD FedFunds, CAD OIS	Payer	Par-100bps	-100	-1.6		0		0	-0.6		0	-0.3

Appendix: No-substitution CTD

Model

- Model is counterparty specific
- Input
  - Yield Curve (multi curve)
  - Vol/Correlation info if stochastic
  - Exposure profile
  - Initial collateral
- Output
  - Discount curve

Appendix: No-substitution CTD

Model Setup

- Assume there are only 2 currencies in the basket A and B.
- Let $t_i, i = 1, ..., n$ be collateral call date; $\tau_i = t_{i+1} - t_i$.
- Let a_i and b_i be the term rates applied on $[t_i, t_{i+1}]$ of A and B, respectively.
- Let V_i be PV on t_i, A_i and B_i be the absolute amount of A and B in collateral account at t_i; $A_i \geq 0, B_i \geq 0, A_i + B_i = |V_i|$.
- Let $\delta_i = \text{sgn}(V_i \cdot V_{i-1})$.
- Admissible posting strategies:
  - Case 1: $\delta_i = -1$, no constraint;
  - Case 2: $\delta_i = 1, |V_i| \geq |V_{i-1}|$, then $A_i \geq A_{i-1}, B_i \geq B_{i-1}$;
  - Case 3: $\delta_i = 1, |V_i| < |V_{i-1}|$, then $A_i \leq A_{i-1}, B_i \leq B_{i-1}$;

Appendix: No-substitution CTD

Dynamic Programming

- Define operator

$$\text{MinOrMax}_i = \begin{cases} \min, & \text{if } V_i \cdot V_{i-1} \geq 0 \text{ and } |V_i| < |V_{i-1}| \\ \max, & \text{otherwise} \end{cases}$$

- Starting at $i = n$. Find A_n to $\text{MinOrMax}_n f_n$, where $f_n(A_n) = \tau_n(a_n A_n + b_n B_n)$ is the interest on collateral
- Going backward to $n-1$. Net interest on collateral $f_{n-1}(A_{n-1}) = \tau_{n-1}(a_{n-1}A_{n-1} + b_{n-1}B_{n-1}) + \mathbb{E}_{n-1}[DF(t_n, T)\delta_n \text{MinOrMax}_n f_n(A_n)]$
- Bellman Equation:

  $f_i(A_i) = \tau_i(a_i A_i + b_i B_i) + \mathbb{E}_i[DF(t_{i+1}, t_{i+2})\delta_{i+1} \text{MinOrMax}_{A_{i+1} \in \theta(A_i)} f_{i+1}(A_{i+1})]$

  where $\theta(A_i)$ is the allowed set of A_{i+1} given A_i.

Appendix: No-substitution CTD

Intrinsic Case

- A recursive algorithm
  - If $f_{i+1}(A_{i+1})$ is piecewise linear in A_{i+1}, then f_i is piecewise linear in A_i
  - Why? f_{i+1} can only achieve its MinOrMax at one of its turning points c_k or boundaries $A_{i+1,\min}$ and $A_{i+1,\max}$.
  - $A_{i+1,\max}$ and $A_{i+1,\min}$ are linear in A_i.
  - Compare $f_{i+1}(c_k)$, $f_{i+1}(A_{i+1,\min})$, $f_{i+1}(A_{i+1,\max})$.
  - $\text{MinOrMax} f_{i+1}(A_{i+1})$ is piecewise linear in A_i, so is f_i.
  - Backward induction.
- Discount curve:

$$r_i = \frac{A_i}{|V_i|} a_i + \frac{B_i}{|V_i|} b_i$$

Appendix: No-substitution CTD

Piterbarg's approach Piterbarg, *Stuck with Collateral*, Risk, November 2013
Piterbarg, *Optimal Posting of Sticky Collateral*, SSRN, January, 2013

- Assume posting party never changes.
- Maximize total excess accrued interest $\mathbb{E} \sum q_i A_i$, where $q_i = a_i - b_i$. Not consistent with rules.
- Bellman equation: $J_k(a) = q_k a + \mathbb{E}_k[\max_{\tilde{a} \in I_k(a)} J_{k+1}(\tilde{a})]$
- Simplifications
  - Constant portfolio MTM: reduce dimensionality; normalize total collateral to 1, interpret the amount of A as proportion.
  - Infinite time horizon
  - Disappearing optionality: optionality exists only until collateral is standardized; discount future collateral gains at hazard rate.
  - Continuous time: only allow for collateral posting to change at a fixed rate.
  - Homogeneous collateral rate: $dq(t) = m(q)dt + v(q)dW_t$. (No respect of today's term structure.)

Appendix: No-substitution CTD

Piterbarg's approach

- Value function: $J(x,y) = \sup_{a(\cdot,x)\in\mathcal{A}} \mathbb{E}[\int_0^\infty e^{-\lambda t} q(t;y)a(t;x)dt]$
- Admissible strategies:

$$
\begin{aligned}
&\frac{d}{dt}a(t;x) \in \{-k,k\}, a(t;x) \in (0,1) \\
&\frac{d}{dt}a(t;x) \in \{-k,0\}, a(t;x) = 1 \\
&\frac{d}{dt}a(t;x) \in \{k,0\}, a(t;x) = 0
\end{aligned}
$$

- HJB PDE can be derived.
- Optimal strategy: $A'(0,x) = k\mathrm{sgn}(R(x,y))$, where $R(x,y) = \int_0^{\tau_x} e^{-\lambda t} q(t;y)dt$ is the term rate until collateral hits 1.
- Switch boundary: for each y, there is exists $x \in [0,1]$ s.t. $R(x,y) = 0$, call it $\beta(y)$. Above $\beta(y)$, increase collateral; below $\beta(y)$, decrease collateral.
- Key insight of the article: term rate $R(x,y)$ determines the optimal strategy.