Modern Modeling and Pricing of Interest Rates Derivatives
Day 1 - Session 2: Yield Curve Model

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London, June 6-7, 2018

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Yield Curve Construction

- Pricing a derivative:

$$
\begin{aligned}
V(t) &= \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds} V(T)\right] \\
&= P(t,T)\mathbb{E}_t^T\left[V(T)\right], \text{where } P(t,T) = \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds}\right] \\
&= \text{DiscountFactor} * \text{ForwardValue}
\end{aligned}
$$

- What is a Yield Curve Model? It provides
  - Discount factors (Fed Fund, EONIA, etc.)
  - Forward rates (Libor, OIS, FX forward, etc.)
- What is a Yield Curve Model used for?
  - Price *linear* products (Fixed-floating basis swaps, tenor basis swaps, cross currency basis swaps, OIS swaps, Libor-OIS basis swaps, FRAs, Futures, etc)
  - Foundation of upstream models.

Pre-OIS, Libor Discounting, Single Curve

- Historically, Libor rates are used to approximate risk-free rates.
- Build curve with Libor swaps:

$$
\begin{aligned}
C \sum P(t, T_i)\delta_i &= \sum P(t, T_i)\tau_i L_i, \\
L_i &= \mathbb{E}_t^{T_i}[L(T_{i-1}, T_i)] \\
&= \frac{1}{\tau_i}\left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right) \\
\implies C \sum P(t, T_i)\delta_i &= P(t, T_0) - P(t, T_n)
\end{aligned}
$$

- One set of unknowns $\{P(t, T_i)\}$, single curve for both discounting and Libor forwards.

OIS Discounting

- OIS Discounting

$$D(t, T) = \mathbb{E}_t^Q[e^{-\int_t^T c(s)ds}]$$
$$V(t) = D(t, T)\mathbb{E}_t^{T_c}[V(T)]$$

- E.g., Libor Swaps:

$$PV = C\sum D(t, T_i)\delta_i - \sum D(t, T_i)\tau_i L_i$$
$$L_i = \mathbb{E}_t^{T_{i,c}}[L(T_{i-1}, T_i)]$$

- Need two curves
  - OIS curve, to calculate discount factors $D(t, T_i)$.
  - Libor curve, to calculate Libor forwards L_i.
- Two sets of unknowns $\{D(t, T_i), L_i\}$; separate forward and discounting curves.

A Wrong Approach

- Build OIS curve with OIS swaps:
- OIS Swaps:

$$C \sum D(t, T_i)\delta_i = D(t, T_0) - D(t, T_n)$$

- Build Libor curve in the same way as before:

$$C \sum P(t, T_i)\delta_i = P(t, T_0) - P(t, T_n)$$

- To price collateralized trades, calculate discount factors and Libor forwards from the two curves respectively.
- Assumption with this approach? Libor swaps are self-funded.
- Not a valid assumption anymore! Most swaps traded in the market are cleared, market quotes are OIS-discounted swap rates.

Right Approach

- Build OIS curve with OIS swaps.
- Build Libor curve using DFs from OIS curve:

$$C \sum D(t, T_i)\delta_i = \sum D(t, T_i)\tau_i L_i,$$

  i.e, knowing $D(t, T_i)$, solve for L_i.
- In reality, long term OIS swaps are not liquid; Libor-OIS basis swaps are.

$$\sum D(t, T_i)\delta_i(OIS_i + S) = \sum D(t, T_i)\tau_i L_i, \text{where } OIS_i = \frac{D(t, T_{i-1})}{D(t, T_i)}.$$

- Dual curve construction, Libor and OIS united, solve for $D(t, T_i)$ and L_i together.

Tenor Swaps

- Two floating legs which pay Libor with different tenors.
- E.g. 3m x 6m basis swaps

$$\sum D(t, T_i)\delta_i(L_i^{3M} + S) = \sum D(t, T_i)\tau_i L_i^{6M}$$

- Given $D(t, T_i)$ and L_i^{3M}, solve for L_i^{6M}.
- Note: standard USD swaps are on 3M Libor; EUR and GBP swaps are on 6M Libor, simultaneously solving for L_i^{3M} and L_i^{6M} is required.

Cross Currency Curve

- Why are cross currency curves needed?
- Pricing of foreign collateralized trades. E.g. EUR trade collateralized in USD.
- No liquid Euribor swap market collateralized in USD.
- What do we have?
  - USD OIS swaps
  - USD Libor swaps collateralized in USD at Fed fund rates
  - EUR OIS swaps
  - Euribor swaps collateralized in EUR at EONIA rates
  - EUR/USD cross currency swaps collateralized in USD

Cross Currency Curve

- $$\left.\begin{array}{l}\text{USD OIS swaps}\\ \text{USD Libor swaps collateralized in USD}\end{array}\right\} \Longrightarrow \{D^{\$}(t, T_i), L_i^{\$}\}$$

  $$\left.\begin{array}{l}\text{EUR OIS swaps}\\ \text{EUR Libor swaps collateralized in EUR}\end{array}\right\} \Longrightarrow \{D^{€}(t, T_i), L_i^{€}\}$$

- EUR/USD cross currency swaps collateralized in USD:

  $$FX(t) \sum \delta_i D^{\$}(t, T) L_i^{\$} = \sum \tau_i P^{€,\$}(t, T) \mathbb{E}_t^{T_i^{€,\$}} [L^{€}(T_{i-1}, T_i) + s]$$

  Note that $L_i^{€} = \mathbb{E}_t^{T_{i,c}^{€}} [L^{€}(T_{i-1}, T_i)]$
  Ignore convexity

  $$\mathbb{E}_t^{T_i^{€,\$}} [L^{€}(T_{i-1}, T_i)] = L_i^{€}$$

- Calibrate $P^{€,\$}(t, T)$ to EUR/USD cross currency swaps collateralized in USD.
- EUR has two discounting curves: $P^{€,\$}(t, T)$ and $D^{€}(t, T)$.

Currency Chain

- How to calculate FX forwards?
- Consider a EUR/USD FX forward contract collateralized in USD:

$$PV = \mathbb{E}^{Q_\$}\left[e^{-\int_t^T c^\$(s)ds}(FX(T) - K)\right]$$

- Forward FX rate is the par rate:

$$PV = 0 \Rightarrow \mathbb{E}^{Q_\$}\left[e^{-\int_t^T c^\$(s)ds}\right] K = \mathbb{E}^{Q_\$}\left[e^{-\int_t^T c^\$(s)ds} FX(T)\right]$$

$$\text{(change measure)} \Rightarrow \mathbb{E}^{Q_\$}\left[e^{-\int_t^T c^\$(s)ds}\right] K = FX(t)\mathbb{E}^{Q_€}\left[e^{-\int_t^T [r^€(s) - r^\$(s) + c^\$(s)]ds}\right]$$

$$\Rightarrow K = FX(t)\frac{P^{€,\$}(t,T)}{D^\$(t,T)}$$

- Note that $K \neq FX(t)\frac{D^€(t,T)}{D^\$(t,T)}$, i.e, cannot use OIS curves of respective currencies to calculate FX forwards.
- FX forwards are implied from cross currency swap market.

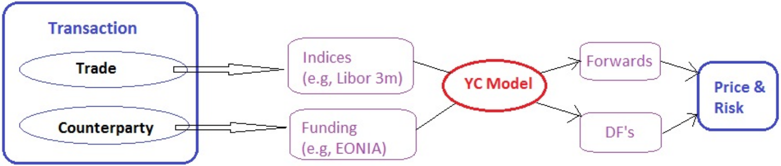
Currency Chain

- Notation: $P^{p,c}$ is the discount factor if the payment currency is p and collateral currency is c.
- Forward FX rate of DOM/FOR collateralized in currency c: $FwdFX(DOM/FOR) = FX(t)\frac{P^{DOM,c}}{P^{FOR,c}}$
- Problem: price a EUR trade collateralized in GBP.
- Need $P^{€,£}$.
- No liquid cross currency market of EUR/GBP xccy swaps collateralized in GBP.
- But there are liquid EUR/USD and GBP/USD xccy swaps collateralized in USD.
  $\Longrightarrow$ Build a EUR-USD-GBP chain!
- Assuming FX forwards are independent of collateral currency,

$$FwdFX(EUR/GBP) = SpotFX\frac{P^{€,£}}{P^{£,£}} \quad = SpotFX\frac{P^{€,\$}}{P^{£,\$}}$$

$$\Longrightarrow P^{€,£} = \frac{P^{€,\$}}{P^{£,\$}}P^{£,£} \quad = \frac{P^{€,\$}}{D^{£,\$}}D^{£}$$

Multi-curve framework: Pricing workflow



Example: 10Y GBP swap effective Jan 2016, notional £10mm, coupon 2%.

Curve	Collateral Currency	PV	Par Rate
Multi-Curve	GBP	-832,703	1.890%
Multi-Curve	USD	-858,063	1.911%
Multi-Curve	EUR	-829,834	1.904%
Single-Curve	N/A	-823,874	1.890%

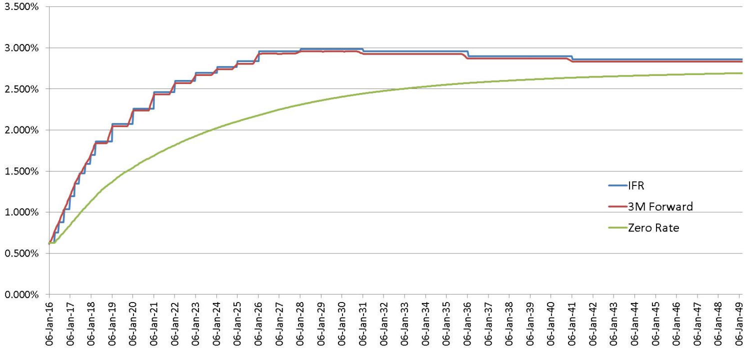
Curve	CollateralCcy	GBP/USD	EUR/USD	GBP Libor	FedFund	EONIA	SONIA
Multi-Curve	GBP	0	0	-8,916	0	0	-422
Multi-Curve	USD	576	0	-8,957	-586	0	150
Multi-Curve	EUR	555	-565	-8,738	0	-562	145
Single-Curve	N/A	0	0	-8,830	0	0	0

Interpolation

- **Why?**
  E.g., price swaps with non-standard tenors; price seasoned swaps.
- **What?**
  - Discount factor: $DF(t, T)$.
  - Time-weighted zero rate: $Z(t, T) = -\ln DF(t, T)$.
  - Zero rate: $z(t, T) = Z(t, T)/(T - t)$.
  - Instantaneous forward rate: $f(t, T) = -\dfrac{d \ln DF(t, T)}{dT} = \dfrac{dZ(t, T)}{dT}$.
- **How?**
  - Two common interpolation methods: flat forwards and cubic spline.
  - Tradeoff between smoothness and locality.
  - Tension spline.
  - We can have one curve (e.g, 6m Libor) spread to another (e.g, 3m Libor); interpolators can be applied to spread quantities, e.g., $\frac{DF^{6m}}{DF^{3m}}, Z^{6m} - Z^{3m}, Z^{6m} - Z^{3m}, f^{6m} - f^{3m}$.
- Interpolation and curve bootstraping are not two separated processes!
  More cash flow dates than benchmark security prices.

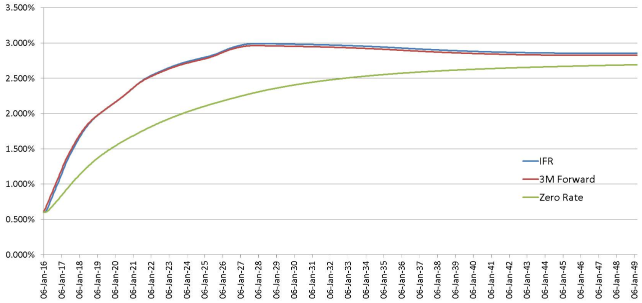
Flat Forwards Interpolation

- Instantaneous forward rates are piecewise flat: $f(t,T) = c_i$, for $T \in [T_{i-1}, T_i)$.
- Equivalent to linear on $Z(t,T)$ and log-linear on $DF(t,T)$.
- Easy to implement, often used as a benchmark to compare to more sophisticated methods.
- Local, not smooth.



Cubic Spline Interpolation

- Often applied on $Z(t, T)$.
- $Z(t, T)$ is piecewise cubic:
  $Z(t, T) = a_i + b_i(T - T_i) + c_i(T - T_i)^2 + d_i(T - T_i)^3$ for $T \in [T_i, T_{i+1})$,
  $i = 1, ..., n - 1$.
- $4n - 4$ coefficients.
- *Natural Cubic Spline*: Requires twice differentiable. Constraints: Reprice n benchmark instruments; $Z(t, T), Z'(t, T)$ and $Z''(t, T)$ continuous at each $T_i, i = 2, ..., n - 1$; boundary condition $Z''(t, T_1) = Z''(t, T_n) = 0$.
- Smooth but not local.



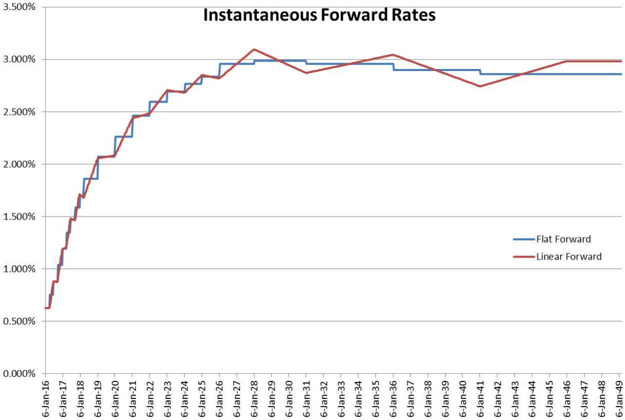
Hedging Locality

E.g., curves built from one spot FRA, 8 Eurodollar futures, and 3y, 4y, 5y, 6y, 7y, 8y, 9y, 10y, 12y, 15y, 20y, 25y, 30y swaps.
Price a 11y swap, coupon 2%.

Swap Tenor	Flat Forward	Cubic Spline
3Y	-5	-6
4Y	-7	-6
5Y	-9	-17
6Y	-11	24
7Y	-13	-169
8Y	-15	652
9Y	-17	-2,838
10Y	4,432	7,397
12Y	5,406	5,334
15Y	0	-716
20Y	0	145
25Y	0	-47
30Y	0	10

Other Interpolation Methods

- References:
  - P. Hagan, G. West, *Methods for Constructing a Yield Curve*, WILMOTT Magazine, 2008
  - L. Andersen, V. Piterbarg, *Interest Rate Modeling*, 2010
- Piecewise linear on $f(t, T)$ or $z(t, T)$: zig-zaging behavior.

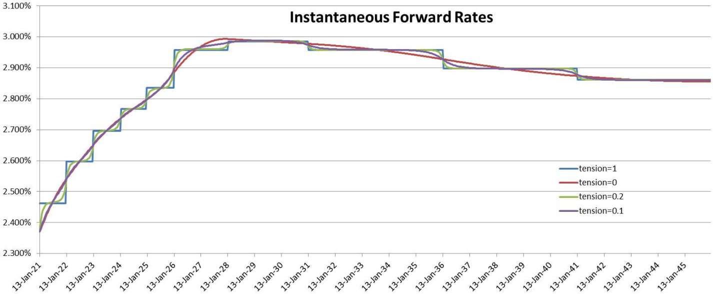


Other Interpolation Methods

- Hermite spline on $Z(t, T)$: piecewise cubic; $Z''(t, T)$ not necessarily continuous; $Z'(t, T_i)$ specified through finite difference; sacrifice smoothness for locality; excess convexity; doesn't preserve monotonicity.
- Monotone convex: Sacrifice smoothness to preserve monotonicity; relatively local.
- Penalty function:
  - Minimize $\int_{T_1}^{T_N} [f'(t, T)]^2 dT$ or $\int_{T_1}^{T_N} \sqrt{1 + [f'(t, T)]^2} dT$; similar to cubic spline on $Z(t, T)$.
  - Tension Spline:
    - Minimize $\int_{T_1}^{T_N} \left([Z''(t, T)]^2 + \lambda^2 [Z'(t, T)]^2 \right) dT$.
    - λ is the *tension factor*.
    - $\int_{T_1}^{T_N} [Z''(t, T)]^2 dZ$ penalizes high 2nd order derivative of Z to avoid kinks and discontinuities.
    - $\int_{T_1}^{T_N} [Z'(t, T)]^2 dZ$ penalizes oscillations and excess convexity.
    - $\lambda \to 0 \implies$ cubic spline; $\lambda \to \infty \implies$ flat forwards.
    - λ represents the tradeoff between smoothness and locality.

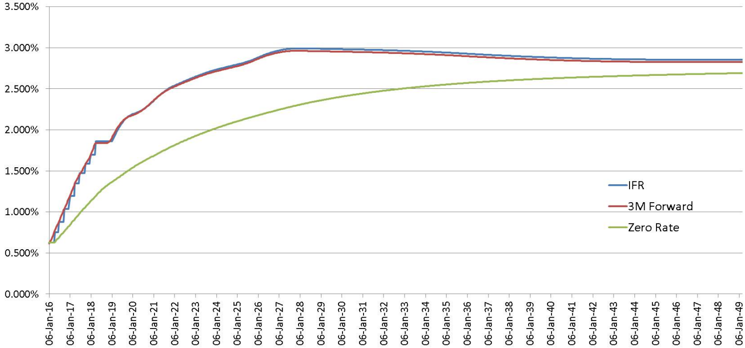
Tension Spline

E.g, IFR graph with different tensions. Note that λ has been rescaled to be between 0 and 1.



Tension Spline

Blend tensions. E.g. set tension=1 in future strips; tension=0 in swaps region.



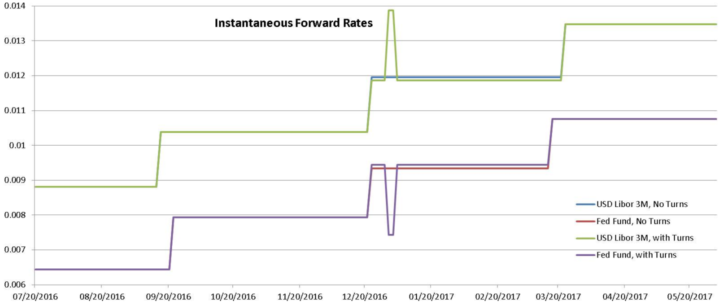
Tension Spline

Hedging locality

Swap Tenor	tension=0	tension=0.1	tension=0.2	tension=1
3Y	-6	-5	-5	-5
4Y	-6	-7	-7	-7
5Y	-17	-11	-9	-9
6Y	24	-3	-11	-11
7Y	-169	-58	-13	-13
8Y	652	231	-5	-15
9Y	-2,838	-1,370	-240	-17
10Y	7,397	5,791	4,673	4,432
12Y	5,334	5,648	5,494	5,406
15Y	-716	-484	-116	0
20Y	145	31	2	0
25Y	-47	-2	0	0
30Y	10	0	0	0

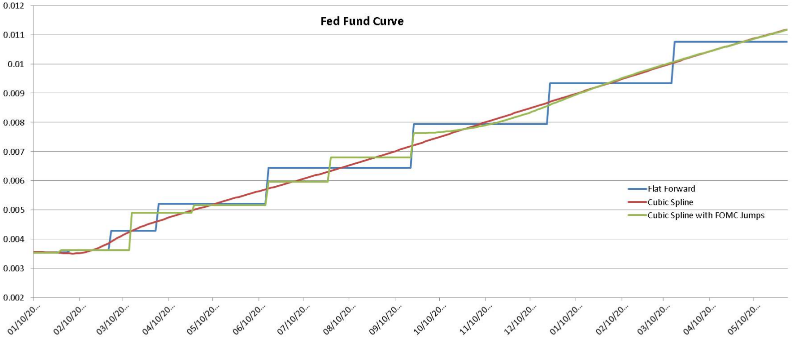
Incorporating Market Expectations - Turns

- At year end, banks look to bolster their cash reserves. Borrowing rate hikes between last business day of year and first business day of the next year.
- Fed provides liquidity at year end.
- Overlaid interpolator: $f(t, T) = f^*(t, T) + \epsilon \cdot 1_{T_s \leq T \leq T_e}$, where $f^*(t, T)$ is the usual interpolator and ϵ is the pre-specified turn amount.



Incorporating Market Expectations - Central Bank Meetings

Central banks set target rates in their meetings. Overnight rate may jump on meeting dates.



Bucketed Risk

- Pricing, risk and hedging in a consistent framework
  - A yield curve model is built from benchmark securities.
  - When pricing a trade/portfolio with a yield curve, it has direct sensitivities on the benchmark securities.
  - Benchmark securities are also used for hedging.
- *Bucketed risk (par-point delta):* $\frac{d\ PV}{d\ R_i}$, where R_i are par rates of benchmark securities, e.g, par swap rates.
- How to calculate?
  - Bump and re-value: Bump benchmark rates one by one, build new yield curve, and re-value the portfolio.
  - Jacobian approach: build a Jacobian matrix $\frac{d\ R_i}{d\ MP_j}$ during curve construction, where MP_j are the *model parameters* (interpolation quantities, i.e, DFs, zero rates, IFRs, etc). Calculate bucketed risk using $\frac{d\ PV}{d\ R_i} = \left[\frac{d\ R_i}{d\ MP_j}\right]^{-1} \frac{d\ PV}{d\ MP_j}$.

More on Risk and Hedging

- Hedging each bucket can be expensive, especially if the interpolation is not local and exhibits oscillating hedging pattern.
- Benchmark instruments do not move independently.
- PV01: Change of PV when parallel shifting the curve by 1 bp.
- Perturbing forward rate / zero rates: gives detailed exposure to each forward rate and discount factor; does not directly suggest hedging instruments.
- PCA hedging: choose a subset of benchmark instruments; hedge the portfolio such it's neutral to first 3 principal components.