Modern Modeling and Pricing of Interest Rates Derivatives
Day 1 - Session 1: Framework

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An Overview of Market Practice

- How to price a derivative? Refer to the existing market instruments:

$$\text{fair value of a derivative} \Leftarrow \text{fair value of its replication portfolio}$$

- Replication: a combination of instruments that has indistinguishable payoff as the original trade, under any market scenarios.
- We borrow money to purchase the replication instruments at certain cost, unwind the portfolio at a future date after receiving the final payoff. Different funding strategies imply different present values.

$$\text{fair value of the trade} \Leftarrow \text{cost of replication} \Leftarrow \text{cost of funding}$$

- Fair value of the trade is therefore the average (expectation) of the combination of the trade payoff and the funding cost, across all possible market scenarios.
  How to assign probabilities to the random scenarios?

Traditional Risk-Neutral Pricing Framework

Fundamental Theorem of Arbitrage-Free Pricing

- **Complete market:** “the complete set of possible bets on future states-of-the-world can be constructed with existing assets without friction.“
- **Risk-neutral measure Q:** risky assets have the same expected return as riskless money market account.
- In absence of arbitrage, **complete market** $\Longleftrightarrow$ **unique risk-neutral measure**.
- Contingent claims can be replicated by long/short underlyings and borrowing/lending at risk-free rate.
- Discounted payoff is a martingale under Q; the present value equals to the expectation of discounted payoff:

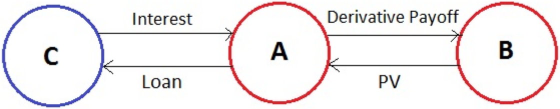
$$V(t) = \mathbb{E}_t^Q[e^{-\int_t^T r(s)ds} V(T)],$$

- Risk free rate r is used for discounting.

Traditional Risk-Neutral Pricing Framework

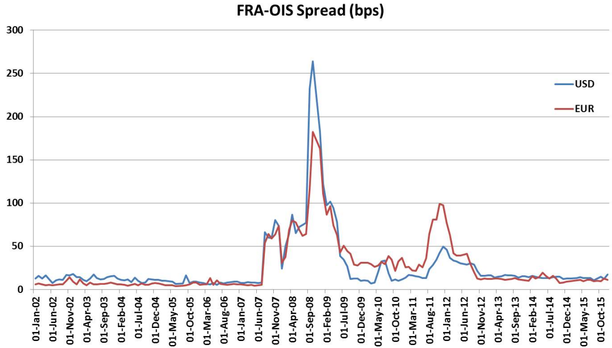
Practice - Funding and Discounting

- Discounting rate is determined by cost of funding.
- $V(t) = \mathbb{E}_t^Q[e^{-\int_t^T r_F(s)ds} V(T)]$.
- What is r_F? Bank funding rate $\approx$ Libor $\approx$ Fed Fund rate $\approx$ Government bond yield $\approx$ repo rate...
- **Libor was widely used for discounting.**
- Banks used to fund their positions by borrowing/lending in the interbank market.



Breakdowns of Libor Discounting

- Spreads between different rates widen.



- OIS (Overnight Index Swap) rate: compounded overnight rates.
- Most interest rate derivatives are now collateralized/cleared.
- Cannot use Libor to discount everything anymore!

Introduction to CSA

- CSA = "Credit Support Annex". A part of the ISDA Master Agreement
- CSA is not mandatory, i.e. it is possible to have an ISDA without a CSA (the reverse is generally not true)
- CSA stipulates the rules of collateral posting such as
  1. Eligible collateral (e.g. cash, bonds, etc.)
  2. Minimum Transfer Amounts (MTA)
  3. Thresholds
  4. Interest earned on posted collateral
  5. ...and many others...
- Originally collateral was viewed as primarily a credit risk mitigant
- Post crisis collateral posting was recognized as a source of funding$\Rightarrow$discounting
- Two main types of CSA: UK Law, New York Law
  - Under UK Law collateral is delivered as transfer of title (i.e. becomes property of the party being posted to)
  - Under NY Law collateral remains property of the posting party
- Specifics of the posting process are detailed in Paragraph 13 of NY Law CSA and Paragraph 11 of UK Law CSA
- The rules of collateral posting have profound impact on the discounting and therefore valuation of derivatives

Pricing Under Domestic Cash Collateral

- Fujii, Takahashi, *Choice of collateral currency*, Jan 2011, Risk
- Risk-neutral measure still exists.
- $V(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(s)ds} V(T) + \int_t^T e^{-\int_t^s r(u)du} [r(s) - c(s)] V(s) ds \right]$
- $X(t) := e^{-\int_0^t r(s)ds} V(t) + \int_0^t e^{-\int_0^s r(u)du} [r(s) - c(s)] V(s) ds$ is a Q-martingale [3], which implies
- $dV(t) = c(t)V(t)dt + dM(t)$.
- $V(t) = \mathbb{E}_t^Q [e^{-\int_t^T c(s)ds} V(T)]$.

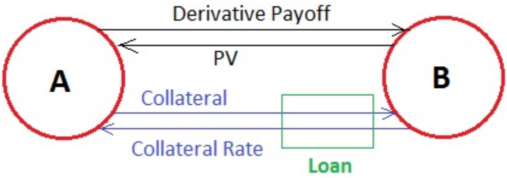
3

$$
\begin{aligned}
\mathbb{E}_\tau^Q[X(t)] &= \mathbb{E}_\tau^Q \left[e^{-\int_0^t r(s)ds} V(t) + \int_0^t e^{-\int_0^s r(u)du} [r(s) - c(s)] V(s) ds \right] \\
&= \mathbb{E}_\tau^Q \left[e^{-\int_0^T r(s)ds} V(T) + \int_0^T e^{-\int_0^s r(u)du} [r(s) - c(s)] V(s) ds \right] \\
&= e^{-\int_0^\tau r(s)ds} \mathbb{E}_\tau^Q \left[e^{-\int_\tau^T r(s)ds} V(T) + \int_\tau^T e^{-\int_\tau^s r(u)du} [r(s) - c(s)] V(s) ds \right] + \int_0^\tau e^{-\int_0^s r(u)du} [r(s) - c(s)] V(s) ds \\
&= e^{-\int_0^\tau r(s)ds} V(\tau) + \int_0^\tau e^{-\int_0^s r(u)du} [r(s) - c(s)] V(s) ds = X(\tau)
\end{aligned}
$$

Modern Pricing Framework

Pricing Under Domestic Cash Collateral - Discounting and Funding

- Discount at collateral rate. Risk free rate r disappears in the valuation formula.



- No external funding is needed.
- Fund through counterparty.
- Cost of funding: collateral rate.
- Futures pricing is a special case with $c = 0$. $\Rightarrow F(t) = \mathbb{E}_t^Q[L(T)]$.

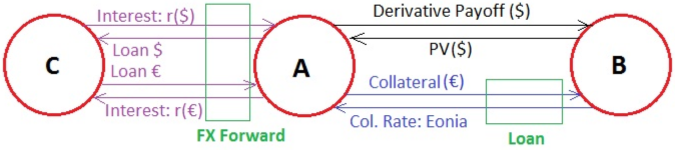
Modern Pricing Framework

Pricing Under Foreign Cash Collateral

- $V(t) =$
  $$\mathbb{E}_t^{Q_d}[e^{-\int_t^T r_d(s)ds}V(T)] + f_{d,f}(t)\mathbb{E}_t^{Q_f}\left[\int_t^T e^{-\int_t^s r_f(u)du}(r_f(s) - c_f(s))\frac{V(s)}{f_{d,f}(s)}ds\right]$$
- Change to domestic risk-neutral measure:
  $$V(t) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T r_d(s)ds}V(T) + \int_t^T e^{-\int_t^s r_d(u)du}[r_f(s) - c_f(s)]V(s)ds\right]$$
- $V(t) = \mathbb{E}_t^{Q_d}[e^{-\int_t^T [r_d(s)-r_f(s)+c_f(s)]ds}V(T)]$. [4]
- Discount rate: $c_f + (r_d - r_f)$.

[4] $r_f - c_f$ can be written as $r_d - (r_d - r_f + c_f)$, plugging into previous formula, discount rate becomes $r_d - r_f + c_f$.

Pricing Under Foreign Cash Collateral - Discounting and Funding

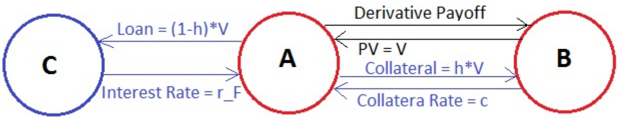


- Discount at FX-adjusted foreign collateral rate: c_f is the foreign collateral rate; $r_d - r_f$ is the FX adjustment.
- Interest rate parity $\Rightarrow$ drift of FX rate is $r_d - r_f$.
- $r_d - r_f$ is implied from cross currency market, no relationship with c_d or c_f.

Modern Pricing Framework

Over- and Under- Collateral

- Collateral amount: $C(t)$. (if $C(t) < V(t)$, under-collateral; if $C(t) > V(t)$, over-collateral)
- $V(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(s)ds} V(T) + \int_t^T e^{-\int_t^s r(u)du} [r(s) - c(s)] C(s) ds \right]$
- $C(t) = h \cdot V(t) \Rightarrow V(t) = \mathbb{E}_t^Q [e^{-\int_t^T [(1-h)r(s) + h \cdot c(s)]ds} V(T)]$. [5]
- Bank's unsecured funding rate: r_F.
- Discount rate: $(1-h)r_F + h \cdot c$.
- Blending collateral rate and unsecured funding rate.

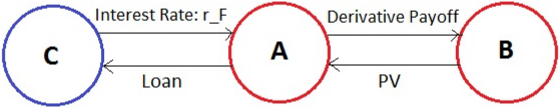


[5] $h \cdot [r(s) - c(s)]$ can be written as $r(s) - [(1-h)r(s) + h \cdot c(s)]$.

Modern Pricing Framework

Uncollateralized Trades

- $C(t) = 0$; special case of under-collateral.
- $V(t) = \mathbb{E}_t^Q[e^{-\int_t^T r_F(s)ds} V(T)]$
- Discount at unsecured funding rate.



Modern Pricing Framework

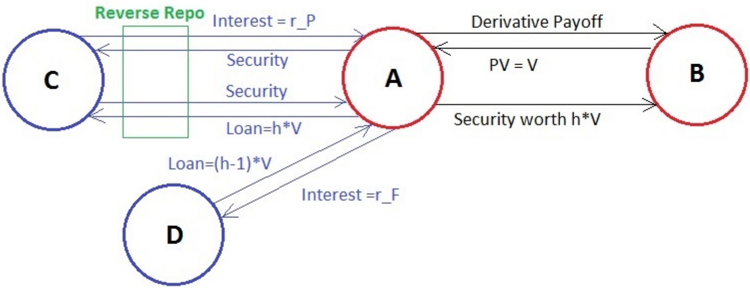
Physical Collateral

- Government bonds, MBS, corporate bonds, etc.
- Assumption: collateral is fully rehypothecable, i.e, can be reused by the receiver for its own borrowing.
- Haircut (valuation percentage) h': MtM(trade) = MtM(collateral)$\cdot h'$.
- Let $h = 1/h'$, then $C(t) = h \cdot V(t)$. Note that $h > 1$, this is over-collateralization.
- Why haircut? Risk of underlying asset, liquidity.
- Unsecured funding rate: r_F; Security's repo rate: r_P.
- Receiver of collateral enters a repo contract, borrowing $C(t)$ at rate r_P; immediately lend it out at r_F. Net return: $r_F - r_P$.
- $V(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T r_F(s)ds} V(T) + \int_t^T e^{-\int_t^s r_F(u)du} [r_F(s) - r_P(s)] h \cdot V(s) ds \right]$
- $V(t) = \mathbb{E}_t^Q [e^{-\int_t^T [(1-h) r_F(s) + h \cdot r_P(s)] ds} V(T)]$

Modern Pricing Framework

Physical Collateral - Discounting and Funding

- Discount rate: $(1-h)r_F(s) + h \cdot r_P(s)$
- Blending security's repo rate and unsecured funding rate.



Modern Pricing Framework

CTD Collateral Option

- Collateral can be chosen from a basket of currencies and/or physical securities.
- NY Law: collateral belongs to the posting party.
  - Collateral can be fully substituted at any time.
  - $V(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(s)ds} V(T) + \int_t^T e^{-\int_t^s r(u)du} [r(s) - \max_i r_i(s)] h \cdot V(s) ds \right]$, where $r_i = c_i + (r_d - r_i)$ is cross-currency adjusted collateral rate.
  - $V(t) = \mathbb{E}_t^Q [e^{-\int_t^T \max_i r_i(s)ds} V(T)]$
- UK Law: collateral belongs to the receiving party.
  - If PV changes sign, original collateral fully returned, new posting party chooses to post new PV.
  - If PV becomes more negative to posting party, can only choose currency/security for additional collateral amount, cannot substitute existing collateral.
  - If PV becomes less negative to posting party, receiving party chooses what currency/security to return.
  - Non-linear, portfolio dependent, optionality cannot be factored into discount factors. – Details to follow.