ISMA CENTRE - THE BUSINESS SCHOOL
OF THE FINANCIAL MARKETS

UNIVERSITY OF READING
ENGLAND

IFID Certificate Programme

Fixed Income Analysis

Interest Rate Risk

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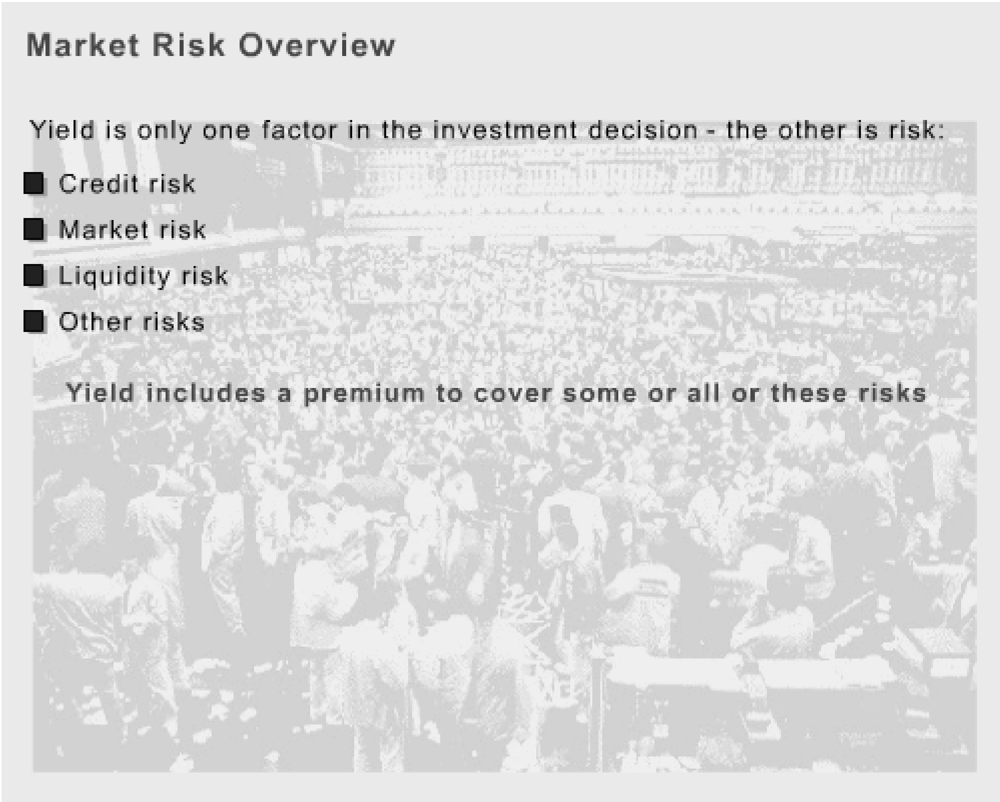
1. Overview



Yield is only one factor in the investment decision - the other factor is risk:

- **Credit risk** - the risk of default or delays in payments of coupons or principal on a fixed income security
- **Market risk** - the potential loss that could arise from an adverse change in the price of a security
- **Liquidity risk** - the ability to trade in large size without adversely affecting the market price
- **Other risks**

The yield that an investor requires on a security must include a risk premium to reflect some or all of these risks. In this module we focus on market risk. We discuss various ways of quantifying this risk, and we explore how investors use them to manage fixed income positions.

Learning Objectives

By the end of this module, you will be able to:

1. Define the concept of Macaulay duration and describe its application in asset and liability management

2. Explain how the coupon rate, the maturity of the bond and its yield affect its duration

3. Calculate the Macaulay duration on a floating rate note

4. Calculate the Macaulay duration of:
   - A fixed income portfolio
   - A liability structure
   - A combination of the two

5. Distinguish between duration matching and cash flow matching

6. Explain why a duration-immunised portfolio has to be rebalanced over time

7. Define modified (or adjusted) duration and calculate its value from the security's Macaulay duration

8. Estimate the horizon yield on a bond position from its yield to maturity and its modified duration

9. Define a bond's basis point value (BPV) and calculate it from the bond's modified duration

10. Explain the convexity behaviour of a fixed income security and give examples of instruments with:
    - Positive convexity
    - Negative convexity

11. For a straight bond, explain the relationship between convexity and:
    - Coupon rate
    - Maturity

12. Estimate a bond's percentage price change, given its duration, convexity and a specified change in yield

13. Using a financial calculator, compute a bond's BPV and its effective duration and explain why these differ from the analytical duration/convexity estimate

2. Fixed Income Laws

By now you should be aware of some fundamental laws of the fixed income markets:

Bond Market Law No 1

Bond price varies inversely with yield: it falls as market yield goes up, and vice-versa

The price of a bond is the sum of its discounted future cash flows: the higher the discount rate (or yield) applied, the more heavily are the future cash flows discounted.

Bond Market Law No 2

The longer the maturity of a fixed income security, the more sensitive is its price to a given change in yield.

An investor committing capital to the purchase of a fixed income security earns a fixed stream of cash flows over the term. If, during this time, interest rates were to rise the investor misses the opportunity to earn the higher rate. The fall in the price of the security reflects the present value of the interest rate difference.

- If you buy a security maturing a day later, and in the meantime interest rates rise, then you only lose one day's interest at the higher rate
- But if you invest in a 20-year bond the opportunity loss is correspondingly larger - and so is the change in the bond's price

Bond Market Law No 3

The lower the coupon rate on a bond, the more sensitive is its price to a given change in yield.

This law is not so intuitive. If interest rates rise, any coupon paid on the bond unlocks funds which can be reinvested at the higher rates. The higher reinvestment income offsets some of the opportunity loss on the principal tied up, so the higher the coupon rate on the bond the more stable is its price.

A zero-coupon bond pays no coupons which could be reinvested at the higher rates, so zeros are more price-sensitive to yield changes than equivalent coupon bonds.

3. Macaulay Duration

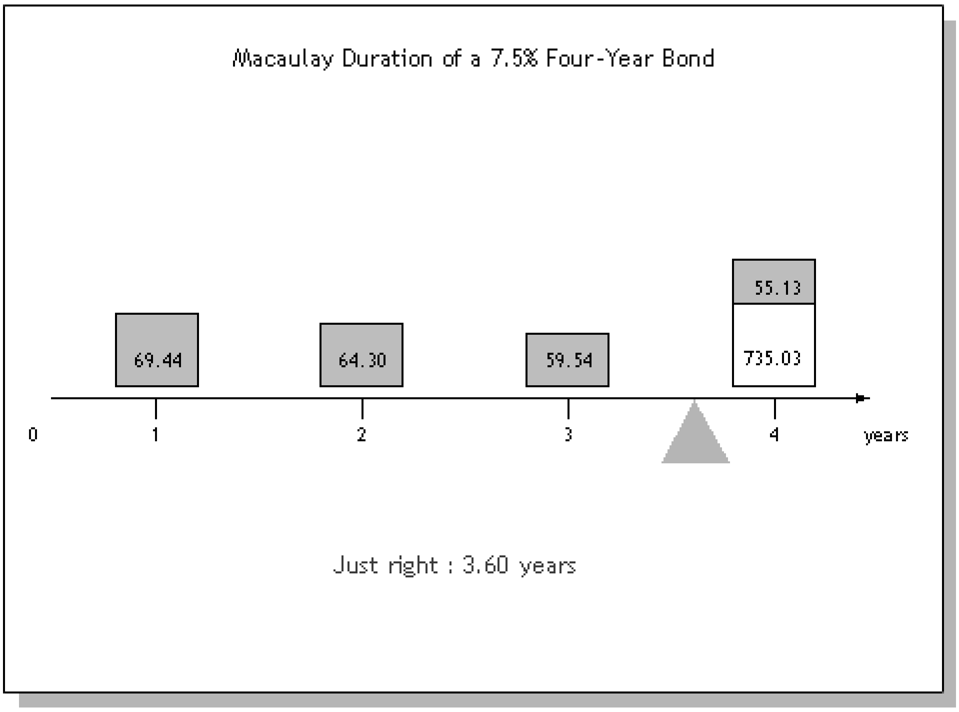
3.1. Definition

Macaulay duration is a weighted average of a bond's life, taking into account:

- The size of each cash flow
- Its timing.

Duration is a measure of market risk: the higher the bond's duration, the greater is its risk.

The concept was developed in the 1930s by the Scottish actuary Frederick R. Macaulay. The figure below shows duration as the 'centre of gravity' of a bond, taking into account the 'weight' and position of each cash flow along the timeline.



The weight of each coupon in the picture becomes progressively smaller as you look towards the right because the more distant cash flows are more heavily discounted.

Mathematically, duration is a weighted average of the time to each cash flow. The ‘weight’ given to each time period is the present value of the cash flow payable at that point in time, as a proportion of the present value of the whole bond (i.e. its dirty price).

3.2. Properties

Duration is useful because it brings all the factors that affect a bond's market risk 'under one roof'. Here are the bond market laws introduced earlier, restated in terms of duration, plus a new one:

- **Bond Market Law No 2** (restated): the longer the maturity of a bond, the higher is its duration - the pivot point in the see-saw above must be moved to the right.

- **Bond Market Law No 3** (restated): the lower the coupon rate on the bond the higher is its duration - again the pivot must be shifted to the right to preserve the balance. The duration of a zero coupon bond is equal to its maturity: a zero has only one cash flow, so the pivot point lies directly underneath it.

- **Bond Market Law No 4** (new): the duration of a bond increases as its yield falls, and vice versa. At higher yields the more distant cash flows are discounted proportionately more heavily, so the pivot point has to be moved to the left.

Properties of Macaulay Duration

Laws 2 and 4 indicate that duration is not a constant: it drifts over time as the bond approaches maturity and market yields change. Here are a couple of other interesting observations:

- The maximum duration of any coupon bond is about 14 years. Beyond a certain maturity, the present value of the more distant cash flows - even the principal - are so small as to be irrelevant! So the market risk on Walt Disney's 100-year Eurobond is virtually the same as the risk on British Telecom's 50-year bond. Only zero coupon bonds are able to breach this 14-year ‘sound barrier’.

- The duration of a floating rate note is equal to the length of its current coupon period. The coupons on a floater are reset periodically in line with current rates, so its market risk is limited to the time left to the next coupon fixing.

3.3. The General Formula

Macaulay Duration (D)

$$= \frac{1}{PV} \times \left[a \times \frac{C/t}{(1 + R/t)^{a}} + (a+1) \times \frac{C/t}{(1 + R/t)^{a+1}} + \ldots + (a+m) \times \frac{\text{Principal} + C/t}{(1 + R/t)^{a+m}} \right]$$

Where:
PV = Dirty price
C = Coupon rate (%)
R = Yield to maturity
t = Number of coupon payments per year (compounding period)
m = Number of complete coupon periods to maturity
a = (1 - Fractional coupon period)

$$\text{Fractional coupon period} = \frac{\text{Number of days since last coupon}}{\text{Number of days in current coupon period}}$$

Equivalent formulation:

$$D = \frac{C/t}{PV} \times \Sigma \frac{a + i}{(1 + R/t)^{a+i}} + \frac{(a + m)}{PV} \times \frac{\text{Principal}}{(1 + R/t)^{a+m}}$$

for i = 0 ... m

Example
Macaulay Duration calculation.

Security: 5% US Treasury note maturing 21 January 2005
Type: Semi-annual, actual/actual
Settlement date: 3 June 2003

Price: 95.48
Accrued: 1.84
Yield: 8.00%

? What is the bond's Macaulay duration?

We priced this bond in *Bond Pricing - Valuation Formula*!

Nr. days from settlement to next coupon	(3 June - 21 July)	=	48
Nr. days in current coupon period	(21 January - 21 July)	=	181
Fraction of period to next coupon	= 48/181	=	0.2652

In the table below we proceed in three stages:

1. Calculate the present value of each cash flow (column 2)
2. Multiply each PV by its corresponding time period (column 3)
3. Divide the sum of column 3 by the sum of column 2 (the bond's dirty price).

Coupon Date	**(1) Time Period**	**Cashflow**	**Discounted at 8%**	**(2) Present Value**	**(3) = (1) x (2)**
21 Jul 2003	0.2652	2.50	$\frac{2.50}{(1 + 0.08/2)^{0.2652}}$	2.47	0.655
21 Jan 2004	1.2652	2.50	$\frac{2.50}{(1 + 0.08/2)^{1.2652}}$	2.38	3.011
21 Jul 2004	2.2652	2.50	$\frac{2.50}{(1 + 0.08/2)^{2.2652}}$	2.29	5.187
21 Jan 2005	3.2652	102.50	$\frac{102.50}{(1 + 0.08/2)^{3.2652}}$	90.18	294.456
Sum				**97.32**	**303.309**

$$\text{Macaulay duration} = \frac{303.309}{97.32}$$

= 3.12 coupon periods
= **1.56** years

4. Using Duration

4.1. Example – Asset & Liability Management

An insurance company sold a single-premium savings plan giving the policy holder title to a fixed lump-sum cash payment when the plan matures in 9 years.

? How should the fund invest the premium proceeds to ensure it can meet this known future liability?

The Alternatives

- **Equities**: in the long run equities tend to outperform bonds but their return is uncertain. There is no guarantee that the value of the equities in 9 years will be sufficient to meet the company's obligations.

- **Money market investments**: although the market risk on short term paper may be small, the reinvestment risk is high: the interest and principal on the instruments would have to be rolled many times over the 9 years at varying yields, so the terminal value of the fund would be uncertain.

- **Coupon bonds**:
    - A bond with less than 9 years to maturity would expose the fund to reinvestment risk on the principal, as well as on the coupons.
    - One with more than 9 years to maturity would have to be liquidated before maturity, so it would expose the fund to market risk, as well as to reinvestment risk on the coupons.
    - One with exactly 9 years to maturity would have reinvestment risk on its coupons (though not on the principal), so its horizon yield would also be uncertain.

- **Zero coupon bonds**: only zeros repay a fixed cash flow at maturity with no reinvestment risk (there are no coupons to reinvest!). Ideally, the fund should purchase a zero maturing on the same date as the underlying savings plan.

    In practice, it is unlikely that there will be a zero that matches exactly each of the fund's future liabilities on thousands of different savings plans.

4.2. Duration-matching

The solution in this case is to apply a technique known as **duration-matching**. This involves building a portfolio of assets that has the same Macaulay duration as that of the fund's liabilities.

The duration of a fixed income asset portfolio is the weighted average of the durations of its constituent securities.

The weight of each security in the portfolio is the present value (PV) of that security relative to the PV of the total portfolio.

Duration of a Portfolio

$$D_{portfolio} = \frac{1}{PV} \times \Sigma (PV_i \times D_i)$$

for i = 1 ... n

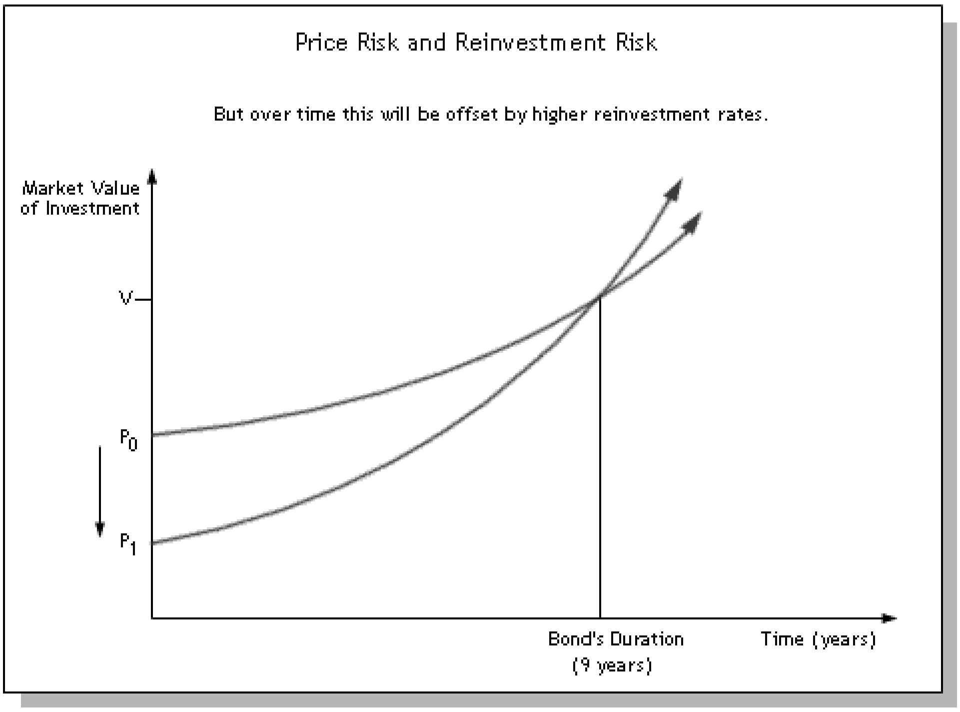
Where:

PV_i = Present value of asset i
D_i = Duration of asset i
n = Number of assets in the portfolio

Similarly, the duration of a liability portfolio is the weighted average of the durations of its constituent future liabilities.

Analysis

A fixed income portfolio that is duration-matched is said to be **immunised**: its future value will not be affected by interest rate movements. The figure below illustrates how this powerful technique works in the case of our 9-year forward liability.



Explanation

Suppose today the fund purchases a bond with a duration of 9 years at a price P_0. Over time the value of the fund will rise to V; the rise is exponential because interest compounds as the coupons are reinvested.

Now suppose that immediately after the bond is purchased market yields rise, resulting in a fall in the value of the fund to P1. Although on a mark to market the fund might now be technically insolvent, future coupons on this bond can be reinvested at higher rates. Other things being equal, by year 9 the higher reinvestment income should have fully offset the price loss today!

4.3. Managing Duration

A balance sheet is:

- **Short-funded** (or **net long duration**) if the duration of its assets exceeds that of its liabilities: a rise in interest rates will hurt its profit & loss, and vice versa

- **Long-funded** (or **net short duration**) if the duration of its assets is less than that of its liabilities: a rise in rates will improve its profit & loss, and vice versa

- **Match-funded** (or **duration matched**) if the duration of its assets is the same as that of its liabilities: a rise or a fall in rates will not impact on the profit & loss.

The technique of duration-matching is commonly used by banks, as well as fixed income funds, to manage the net interest rate risk on their balance sheets. The asset & liability managers calculate the net duration of the company as a whole and create interest rate hedging positions, using cash or derivative instruments, to ensure it is reasonably matched.

Of course, the duration of the assets (and the liabilities) will change daily, requiring the company to rebalance its hedging portfolio at regular intervals. As long as the company remains duration-matched, its balance sheet will be immune to fluctuations in market yields. This was Macaulay's contribution to interest rate risk management.

5. Modified Duration

Macaulay duration allows us to rank fixed income securities in terms of their market risk, but it does not tell us how risky the securities are in terms of profit or loss. However, Macaulay duration can be easily modified to quantify risk in those terms.

Modified duration (**IFID exam**)

$$= \frac{\text{Percentage change in a bond's dirty price}}{\text{1 percentage point change in yield}}$$

$$= \frac{\text{Macaulay duration (in years)}}{(1 + R/t)}$$

Where:
R = Yield to maturity, in decimal (i.e. 5% is 0.05)
t = Number of coupons per year (1= annual; 2= semi-annual, etc.)

Also known as: **Adjusted duration**.

We shall not prove the relationship between Macaulay and modified duration here, but we can illustrate its use with a simple example.

Example
Modified duration calculation

Security:	5% US Treasury note maturing 21 January 2005
Type:	Semi-annual, actual/actual
Settlement date:	3 June 2003
Price:	95.48
Accrued:	1.84
Yield:	8.00%
Macaulay duration:	1.56 years
Amount held:	USD 1 million.

? What is the potential loss on this investment if yields rose to 9%?

We calculated the Macaulay duration of this bond in section *Macaulay Duration*!

$$\text{Modified duration} = \frac{1.56}{(1 + 0.08/2)}$$

$$= \mathbf{1.50\%}$$

A rise in yield from 8% to 9% would result in a capital loss on this investment of approximately 1½%.

$$\text{Risk in cash terms} = \frac{1.50}{100} \times \frac{(95.48 + 1.84)}{100} \times 1{,}000{,}000$$

$$= \text{USD } \mathbf{14{,}598}$$

The capital loss is only approximate. As we said in section *Macaulay Duration*, Bond Market Law No. 4 means that Macaulay duration depends on the yield level at which we discount the bond's future cash flows. Therefore a yield change of this magnitude will also affect the numerator in the modified duration formula. We shall explore this phenomenon in section *Convexity.*

In any case, a 1% change in yield is a very large move in the US Treasury markets. Traders and risk managers typically estimate risk based on smaller yield changes, as we shall see in section *Basis Point Value*.

5.1. Application

Modified duration can provide you with a useful approximation to the horizon yield on a bond (see section Bond Pricing and Yield – Horizon Yield).

Provided the holding period on the bond is relatively short:

$$\textbf{Horizon yield} \approx \textbf{YTM} + \left(\textbf{MOD} \times \Delta\textbf{YTM} \times \frac{\textbf{Year basis}}{\textbf{Holding period}}\right)$$

Example

Settlement date: 12 March 2002
Security: 8% Eurobond maturing 12 March 2012
Type: annual, 30/360
Price: 90.00
Yield to maturity: 9.60%
Modified duration: 6.464%

? What is the horizon yield on this bond if it is held for 31 days, until 12 April 2002, when it is sold at 90.90 for a yield of 9.45%?

$$\text{Horizon yield} = 9.60 + \left[6.464 \times (9.60 - 9.45) \times \frac{360}{31}\right]$$

$$= \mathbf{20.86}\%$$

If you calculate the horizon yield on this position using the accompanying spreadsheet, the actual result is 20.89%. The approximation was very accurate here but can be less accurate, especially when the holding period is relatively long, because in that case the bond's duration will change (**duration drift**).

6. Basis Point Value

Basis point value (BPV): the change in the price of a bond, per 100 nominal, for a 1 basis point (0.01%) change in its yield.

$$BPV = \frac{\text{Modified duration(\%)}}{100} \times \frac{\text{Dirty Price}}{100}$$

Also known as: **Present value of a basis point** (PVBP), **Value of an 01** (Val-01), **Dollar modified duration**, **Risk factor**, **Delta**.

In the BPV formula we first divide modified duration by 100 to convert it from a percentage into decimal (i.e. 5% is 0.05). The second divisor of 100 reduces the scale of risk from a 100 basis point change in yield (modified duration) to just 1 basis point[1].

BPV is the basic unit of risk in the fixed income markets.

It is an essential building block in the design of a variety of trading strategies, as we shall see in module Outright and Spread Trading.

Example
Basis point value calculation

Security:	5% US Treasury note maturing 21 January 2005
Type:	Semi-annual, actual/actual
Settlement date:	3 June 2002
Price:	95.48
Accrued:	1.84
Yield:	8.00%
Macaulay duration:	1.56 years
Modified duration:	1.50%
Amount held:	USD 1 million.

We calculated the modified duration of this bond in section *Modified Duration*!

$$BPV = \frac{1.50}{100} \times \left(\frac{95.48 + 1.84}{100} \right)$$

$$= \mathbf{0.014598}$$

Thus, a 1 basis point rise in the bond's yield will result in:

- A fall in the price from 95.4800 to 95.4654
- A loss of 1.46 cents per USD 100 nominal (remember the bond price is expressed in percent)
- A loss of USD 145.98 on a USD 1 million position

BPV tends to come out as a very small figure with many decimal places. For convenience many bond analysis systems scale the BPV figure by a factor of 100, so in our example the **risk factor** would be 1.4598. Thus, a 100-point change in the bond's yield would result in:

- A fall in the price from 95.48 to 94.02 (minus 1.46)
- A loss of USD 1.46 per 100 nominal
- A loss of USD 14,598 on a USD 1 million position

[1] Risk factor and dollar modified duration are typically shown for 100 basis points – i.e. 100 x BPV. This is just to make the figures easier to read, as BPVs tend to be numerically small numbers.

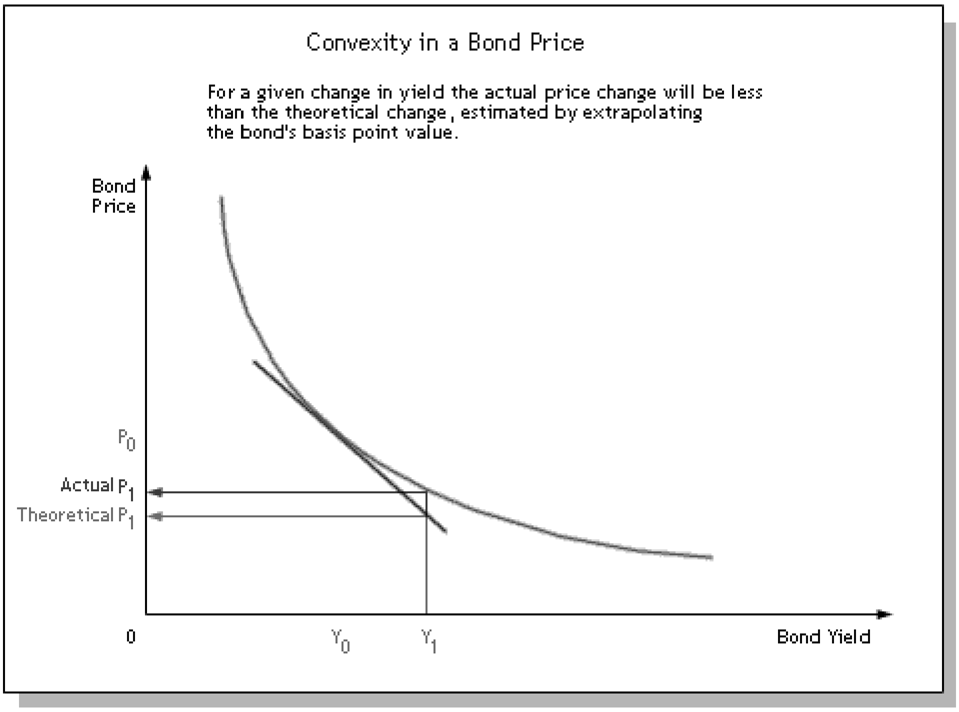
7. Convexity

7.1. Definition

The higher the yield on a straight bond the lower is its price risk. This is true whichever measure of bond price sensitivity you look at. As yield rises:

- The distant cash flows on the bond are discounted proportionately more heavily, so Macaulay duration falls
- The denominator in the modified duration formula is higher, so modified duration also falls
- The price of the bond falls, so BPV falls as well.

This phenomenon is depicted in the figure below which plots the relationship between bond price and yield. The shape of the curve indicates why the bond is said to have **positive convexity**.



7.2. Measurement

$$\text{Convexity (\%)} = \frac{\text{Change in modified duration}}{\text{100 basis points change in yield}}$$

Convexity may also be defined as the change in BPV for a given change in yield, but it is more frequently defined as above. Defining convexity in this way allows us to use a formula that gives a better approximation to what actually happens to a bond's price for a given change in yield (∆YTM).

Convexity-adjusted change in bond price
(IFID exam formula)

Percentage change in bond's dirty price:
$= -\text{Modified Duration} \times \Delta YTM \quad + \quad \text{Convexity} \times 1/2 \times \Delta YTM^2$

The proof of this approximation formula[2] lies outside the scope of this training programme, but the example following shows how it is used.

Implications

For a bond holder convexity is a Good Thing:

- As yields rise the loss on a bond portfolio becomes progressively less steep - as if the bond had a built-in 'parachute'

- As yields fall the gains on the bond portfolio become progressively larger - as if the bond had a turbo-booster!

All straight bonds have positive convexity but there are some bonds, for example callable bonds, where the issuer has the option to repay the principal ahead of its scheduled maturity, which display **negative convexity**. Callable bonds tend to trade with higher yields than equivalent straights, because for a bond holder negative convexity is a Bad Thing. The convexity behaviour of callable bonds is discussed in more detail in *Callable Bonds - Convexity.*

- Other things being equal, the longer the maturity of a bond and the higher its coupon rate, the greater is its convexity

- For straight bonds, convexity adjustments tend to be small and in most bond trading contexts they are usually ignored. However, for large institutional investors holding portfolios worth tens of billions of dollars, the convexity error becomes significant.

- Bonds with embedded options display much stronger convexity behaviour than straight bonds (for example, see Structured Securities – Callable Bonds). As we shall see in module Options Pricing and Risks – Gamma, this convexity is largely caused by the gamma of the options embedded in the bond.

Example
Convexity calculation

Security:	5% US Treasury note maturing 21 January 2005
Type:	Semi-annual, actual/actual
Settlement date:	3 June 2002
Price:	95.48
Accrued:	1.84
Yield:	8.00%
Modified duration:	1.50%
Convexity:	0.01%
Amount held:	USD 1 million.

? What is the potential loss on this investment if yields rose to 8.50%?

We calculated the modified duration of this bond in section *Modified Duration*! For a 50 basis point rise in yield, the percentage loss on the bond's dirty price will approximate:

$$= (-1.50 \times 0.5) + \left[\frac{0.01 \times (0.5)^2}{2} \right]$$

$$= -0.75 \quad + \; 0.00125$$

$$= \mathbf{-0.74875\%}$$

[2] The formula represents the first two terms of a **Taylor's expansion** of the bond pricing formula.

We can now use this convexity-adjusted figure in the same way as we used modified duration before:

$$\text{Risk in cash terms} = \frac{0.74875}{100} \times \frac{(95.48 + 1.84)}{100} \times 1{,}000{,}000$$

$$= \textbf{USD 7,287}$$

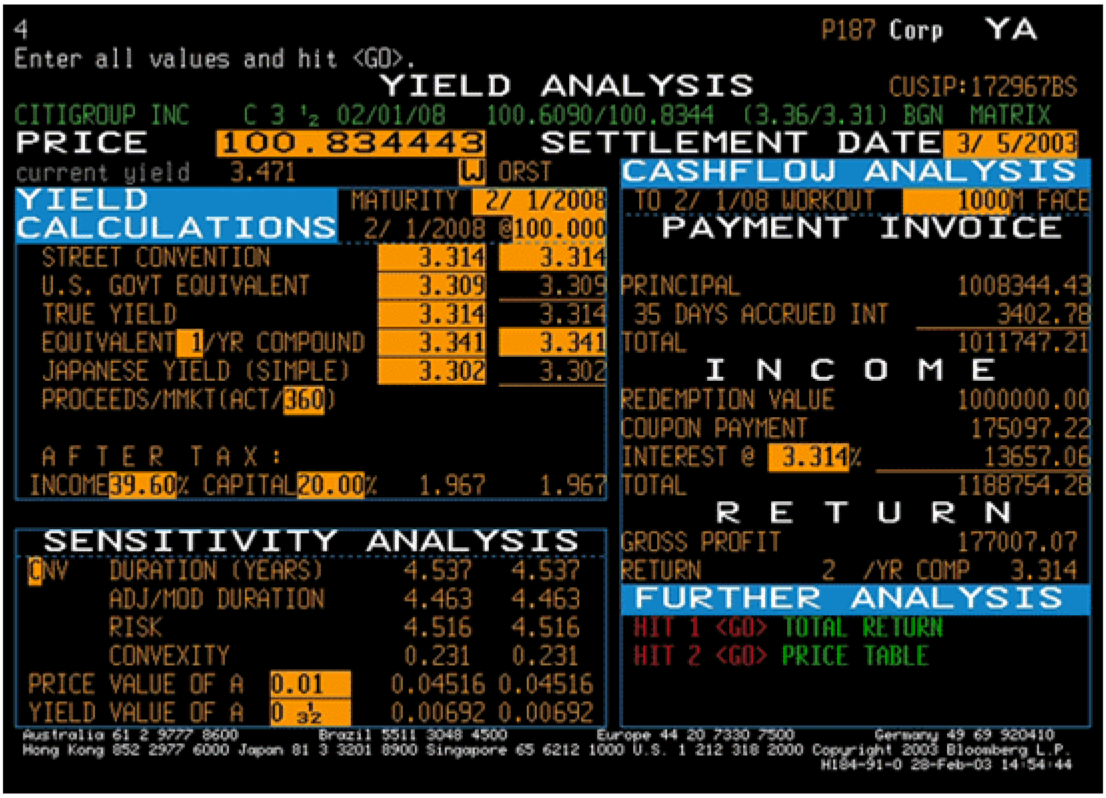
Analytic systems

Examples of Bloomberg and Reuters bond market risk analysis functions

Below are sample screens from two widely-used providers of market information and analytics.

These examples are for illustration purposes only and do not form part of the IFID Certificate syllabus.

Bloomberg yield analysis

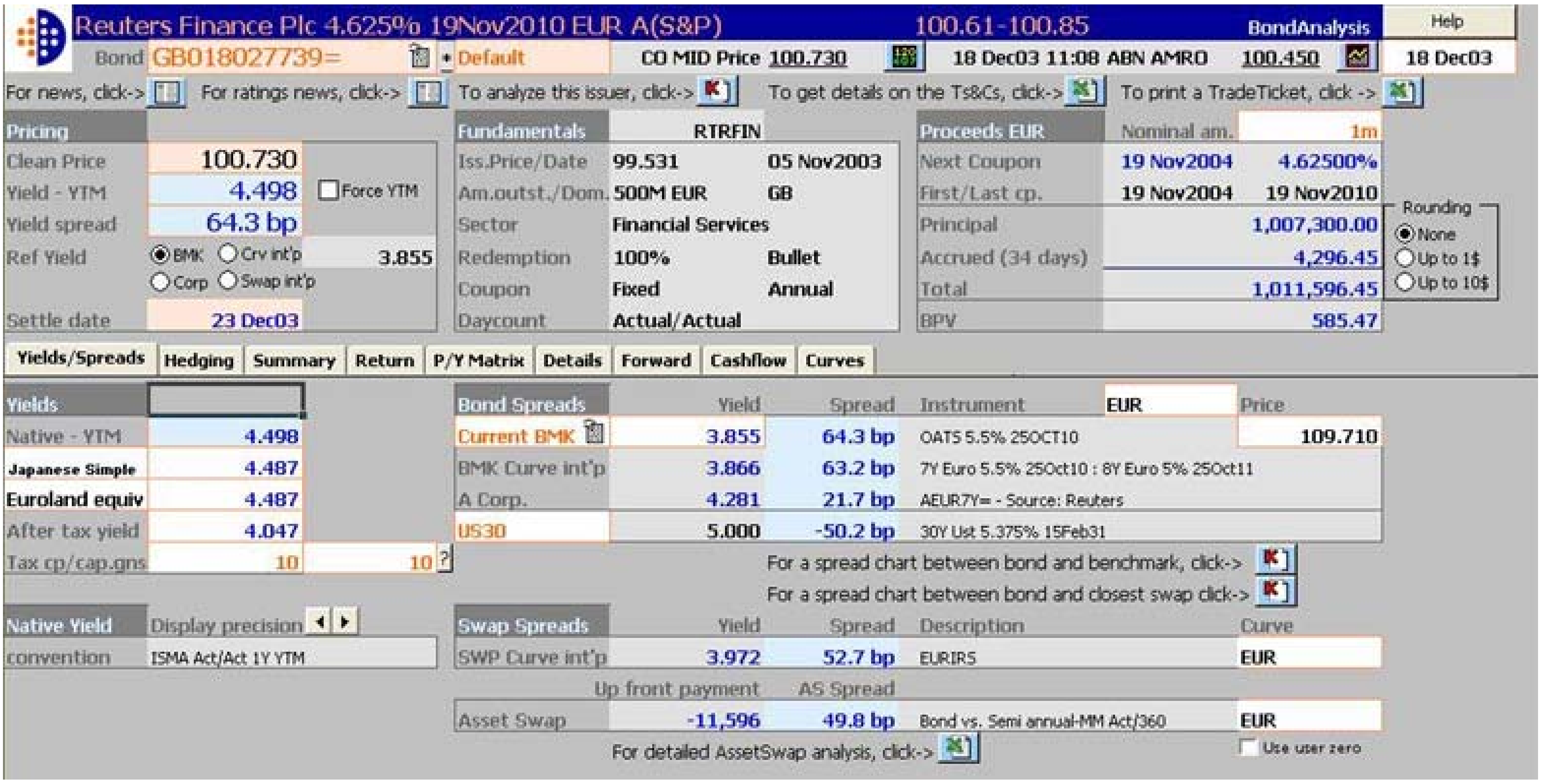


Notes

- **Risk** in the *Sensitivity Analysis* panel is 100 x BPV – in other words, the price value of 100 basis points. The reason for scaling up the BPV is simply to make the figure more easily comparable to the duration and the modified duration figures and also because BPV tends to be a very small figure, which makes it rather difficult to read!

- Notice how the system allows you to calculate not only the actual price value of a specified number of basis points (taking convexity into account) but also the yield value of a specified number of basis points on the price

Reuters bond analysis



Notes

- **BPV** in the *Proceeds* panel on this screen is measured on the value of the specified position, rather than on the bond price

- The *Hedging* tab is where you can specify what instruments you would use to hedge the market risk on this position and the system will calculate the amount of that instrument that you need to trade

8. Exercise

8.1. Question 1

Question 1

Try to estimate the Macaulay duration of each of the securities below just by visualising it on the Macaulay 'see-saw' (see section *Macaulay Duration*). Enter your answers in the boxes below, then validate.

a) Non-interest bearing demand deposit:

b) Six months Eurodeposit (in months):

c) Ten-year floating rate note where the coupon rate is reset semi-annually, in line with the 6 month LIBOR (in months):

d) Ten-year zero coupon bond (in years):

e) A 10-year 10% annual coupon bond (rounded to the whole number of years):

- 5 years
- 7 years
- 10 years

8.2. Question 2

Question 2

Settlement date:	16 April 2002
Security:	12¼% US T-Bond maturing on 4 January 2008
Type:	Semi-annual, Actl/Actl
Price:	124-11+[3]
Accrued:	3.4517%
Yield:	7.00%
Duration:	4.32 years

a) Using the bond's Macaulay duration, calculate its modified duration to 4 decimal places.

b) Using the result in (a), calculate the bond's BPV, per USD 100 nominal, to 5 decimal places.

c) Using the result calculated in (b), estimate the expected change in the bond's price if its yield were to rise by 10 basis points, from 7.00% to 7.10% (your answer to 2 decimal places).

[3] US Treasuries are quoted to the nearest 1/32% of par value, so 124-11 means 124 + 11/32%. Half a tick (i.e. 1/64%) is represented with a '+', so 124-11+ means 124 + 11.5/32%, or 124 + 23/64%.

8.3. Question 3

Question 3

Consider the following securities:

- 12 month T-bill
- 14 year 8% US Treasury bond yielding 12%
- 6 month CD
- 16 year 15% US Treasury bond yielding 12%

a) Rank these securities in order of price risk (basis point value) by entering the appropriate number (1 = lowest risk, 4 = highest risk) in each box below.

12 month T-bill	
14 year 8% US Treasury bond yielding 12%	
6 month CD	
16 year 15% US Treasury bond yielding 12%	

8.4. Question 4

Question 4

You currently hold the following:

Security:	10% bond maturing in 10 years
Type:	Annual, E30/360
Price:	103.00
Macaulay duration:	6.809 years

Portfolio held: USD 10 million.

a) If the yield on this bond were to rise by 100 basis points, what percentage of your capital would you lose? In other words, calculate the bond's modified duration.
Using a financial calculator or the bond pricing model supplied:

1. Calculate the yield on the bond at its current price
2. Add 1% to the calculated yield and re-price the bond[4]
3. Take the difference between the two prices
4. Divide this difference by the bond's original dirty price

Enter your answer in percent to 3 decimal places.

b) Using the analytical formula developed in section *Modified Duration*, calculate the bond's modified duration.

$$\textbf{Modified duration} = \frac{\textbf{Macaulay duration (in years)}}{\textbf{(1 + Yield/Nr Coupons per year)}}$$

Enter your answer in percent to 3 decimal places.

[4] Take care to work with your calculator's full precision – i.e. do not to round your result until you reach the final calculation.

c) Why do the results calculated in (a) and (b) differ?

- Because of rounding errors
- Because the bond has convexity
- Because the bond has credit risk, as well as market risk
- Because of specific supply and demand factors