ISMA CENTRE - THE BUSINESS SCHOOL
OF THE FINANCIAL MARKETS

UNIVERSITY OF READING
ENGLAND

IFID Certificate Programme

Fixed Income Analysis

Spot and Forward Yields

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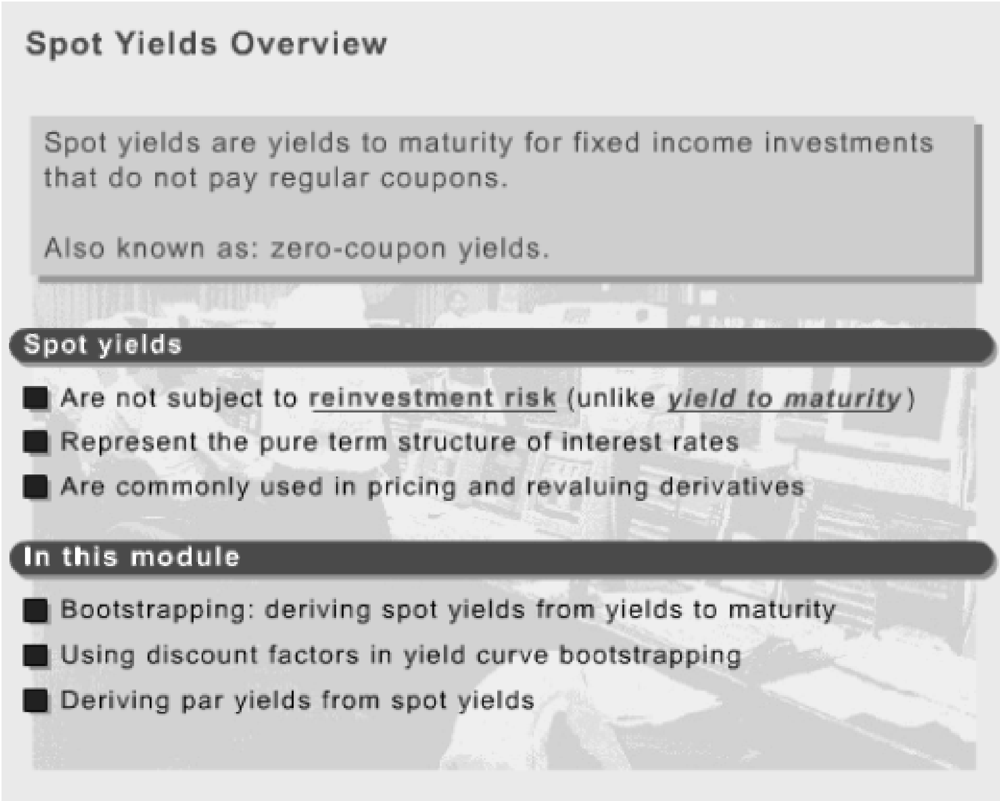
1. Overview



Learning Objectives

By the end of this module you will be able to:

1. Derive, using the method of bootstrapping, a theoretical discount function and its corresponding spot curve from a given coupon curve
2. Explain why a bond's coupon size may affect its yield to maturity
3. Explain why spot curves are commonly used to price bonds with irregular cash flows
4. Define a forward yield and derive a forward curve from the spot curve
5. Explain how the forward curve may be used to identify cheap maturities
6. Explain the relationship between forward yields, par yields and spot yields
7. Derive a par rate from a forward curve or a discount function

2. Coupon Stripping

2.1. Background

Money market instruments which mature in 12 months or less typically pay interest at maturity, so the yields on these instruments are already spot yields. For the longer maturities the spot yield curve is derived from the coupon yield curve using a technique known as **coupon stripping**.

Coupon stripping: selling the individual cash flows generated by a fixed coupon bond as a series (or **strip**) of separate zero-coupon securities.

The technique was pioneered by Merrill Lynch in 1982, when they ‘sliced’ USD 500 million of a 14% US Treasury of 2011 into a series of zero-coupon Treasury Income Growth Receipts (TIGRS).

Other investment banks soon followed suit, and even the US Treasury eventually joined in by allowing **Treasury strips** to be traded as government paper through the central book entry system. Today there are active markets in Treasury strips in the US, the UK and France.

Coupon stripping is not limited to government bonds: many investment banks have stripped good quality sovereign and supra-national Eurobond issues.

2.2. Example

Consider a dealer looking to strip the coupons on $100 million nominal of a 2-year bond with an annual coupon of 9%. At the time the (coupon) yield curve for that sector looks like this:

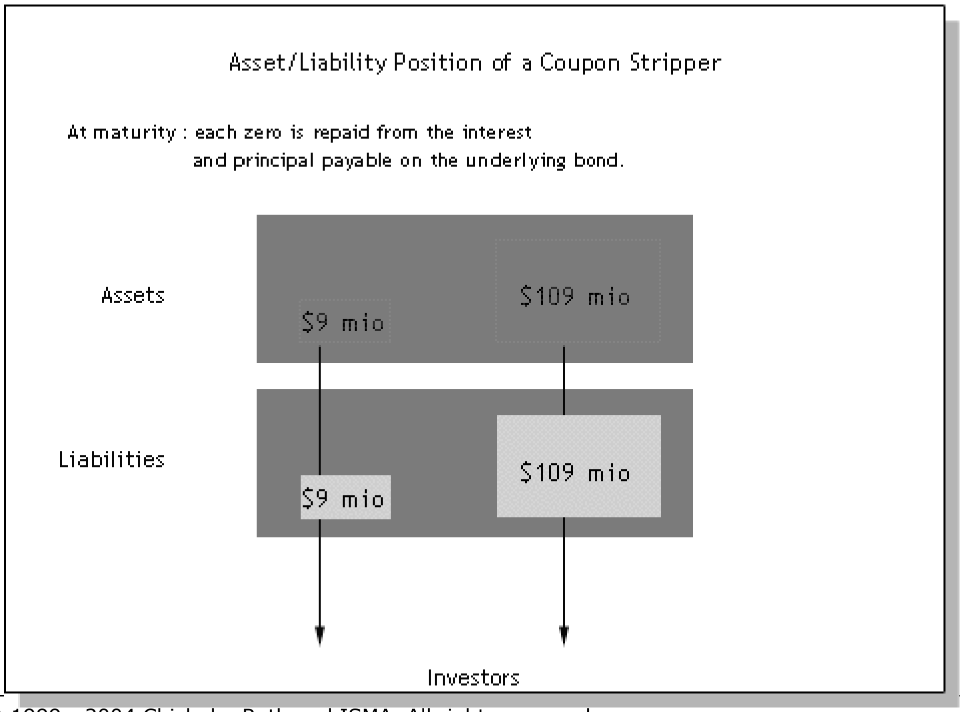
Maturity	YTM
1 year	8%
2 years	9%

The dealer creates a special purpose vehicle whose sole function is to purchase the 9% bond ('the underlying bond') and fund this purchase through the issue of two zero-coupon bonds:

- A 1 year zero with a face value of $9 million
- A 2 year zero with a face value of $109 million

In other words, each zero will be repaid out of the interest and principal receivable on the underlying bond. Since the zeros are fully backed by payments on the underlying bond their credit rating is the same as that on the underlying bond.

The figure below shows the assets & liabilities position of the special purpose vehicle.



? At what yields should the zeros be issued, so that the total issue proceeds exactly cover the cost of the underlying bond - i.e. so that there is no arbitrage profit in coupon stripping?

Analysis
Since they don't bear interest, both zeros will be priced at a discount to their face value.

Price of the 1-year zero $= \dfrac{9}{(1 + R_{01})}$

Price of the 2-year zero $= \dfrac{109}{(1 + R_{02})^2}$

Where R_{01} and R_{02} are, respectively, the yields on the 1- and 2-year zero-coupon yields - i.e. the **spot yields**.

The 1-year zero must be offered on a yield of 8%, otherwise the issue cannot compete with the YTM on ordinary 1-year bonds, which are in effect zeros. So R_{01} = 8%. R_{02} is solved from the following **no-arbitrage** condition:

Cost of 2-yr 9% bond = Proceeds from the sale of the two zeros

$$100.00 = \frac{9}{(1 + 0.08)} + \frac{109}{(1 + R_{02})^2}$$

$$100.00 = 8.33 + \frac{109}{(1 + R_{02})^2}$$

Rearranging this equation to solve for R_{02}:

$$91.67 = \frac{109}{(1 + R_{02})^2}$$

$$(1 + R_{02})^2 = 109 / 91.67 = 1.1890$$

$$R_{02} = \sqrt{(1.1890)} - 1$$

$$= 0.09045 \text{ or } \mathbf{9.05\%}$$

3. Bootstrapping

In section *Coupon Stripping*, we derived a 2-year spot yield (9.05%) from the par curve. Equipped with the 1- and 2-year spot yields, we could now go on to derive a 3-year spot yield following the same logic.

Maturity	**YTM**	**Spot Yield**
1 year	8%	8.00%
2 years	9%	9.05%
3 years	10%	

To derive the 3 year spot yield, we proceed as follows:

1. Purchase a 3-year 10% coupon bond, again at par ('the underlying bond')
2. Sell the first coupon for a yield of 8% and the second coupon for a yield of 9.05%
3. Calculate the breakeven 3-year spot yield (R_{03}) at which the final cash flow on the underlying bond must be sold in order to fully cover the cost of buying the underlying bond:

Cost of 3 year 10% bond = Proceeds from the sale of the three zeros

$$100 = \frac{10}{(1 + 0.0800)} + \frac{10}{(1 + 0.0905)^2} + \frac{110}{(1 + R_{03})^3}$$

$$100 = 9.26 + 8.41 + \frac{110}{(1 + R_{03})^3}$$

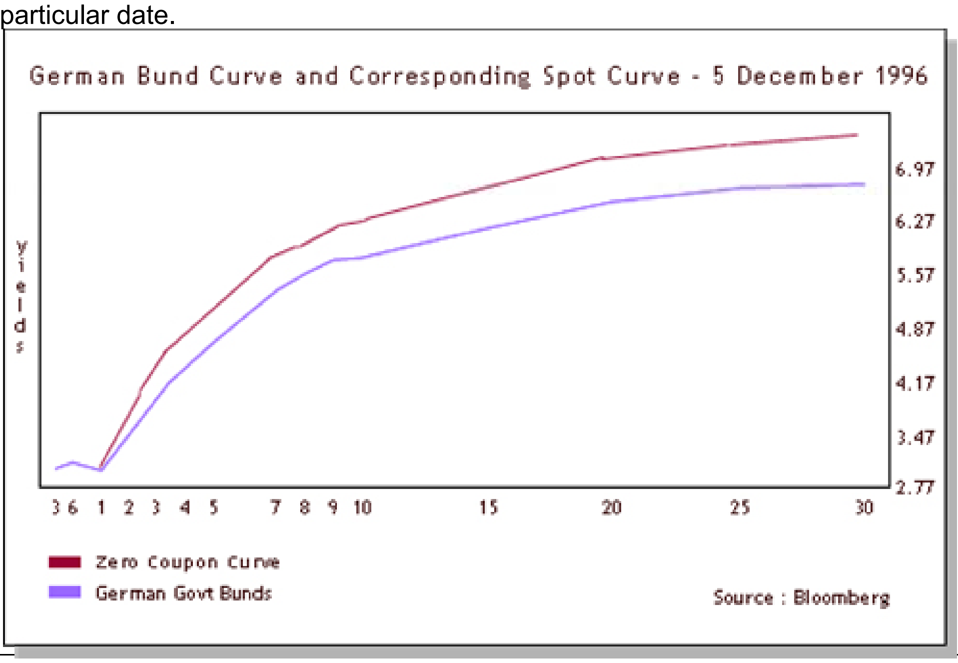
Rearranging this equation to solve for R_{03}:

$$82.33 = \frac{110}{(1 + R_{03})^3}$$

$$(1 + R_{03})^3 = 110 / 82.33 = 1.3361$$

$$R_{03} = (1.3361)^{(1/3)} - 1 = 0.101395 \text{ or } \mathbf{10.14\%}$$

By **bootstrapping** up the coupon curve in this way, one coupon period at a time, we can derive the entire spot curve. The figure below shows the spot curve derived from the German government bond curve on a particular date.



3.1. Spot Yields and Par Yields

This example shows that the yield to maturity on a coupon bond may not be the same as the yield to maturity on an equivalent zero-coupon. It follows that yields on low-coupon bonds may also be different from the yields on otherwise equivalent high-coupon bonds.

The yield on a bond depends on its coupon rate.

We now see that the concept of 'the yield curve' is ambiguous: there is a different curve for bonds with different coupon rates. When estimating 'the yield curve' from a set of observed bond prices (see *Curve Fitting*), analysts are therefore careful to include in the sample only those bonds which trade at or close to par. The resulting yield curve is referred to as the **par curve**, to distinguish it from the zero-coupon curve, or from any other curve.

Par curve:

- The yield curve estimated from coupon bonds trading at (or close to) par
- The benchmark curve against which new bond issues are priced

Also known as: **Coupon curve**

Spot curve: the yield curve for zero-coupon instruments that is consistent with the yields on the par curve.

Also known as: **Zero-coupon curve**

In this section we derived a spot curve from the **par curve** and therefore showed how the yield on a bond also depends on its coupon rate. Yields on low-coupon bonds are:

- Higher than par yields if the curve is positive
- Lower than par yields if the curve is inverted
- The same as par yields if the curve is flat.

The spot curve is the purest representation of the time value of money, as it does not depend on coupon size or indeed the structure of the bond's cash flows.

In government bond sectors that have active strips markets (e.g. US, France and UK) market participants actively look for bonds whose yields can be arbitraged against the spot curve. In most other markets, the effects of different coupon rates on bond yields with same maturities and credit ratings are often overshadowed by other factors such as income tax rates and market liquidity, so the spot curve is less important.

One area where the spot curve is important, however, is in the pricing of bonds with much less regular cash flows, such as sinking funds, callable bonds, and coupon-step-ups. In these cases, the valuation typically involves pricing each cash flow separately, using its corresponding spot rate.

4. Discount Factors

The mathematics of bootstrapping may be streamlined if we strip bonds with face values of $1, instead of $100, and work with discount factors.

Discount factor: the present value of $1 payable at a given future date, discounted at the corresponding spot rate.

Discount factor for period a + i:

$$D_{0,a+i} = \frac{1}{(1 + R_{0,a+i} / t)^{a+i}}$$

Where:

$R_{0,a+i}$ = Spot yield to period a+i
i = Number of whole coupon periods
a = 1 - Fractional coupon period
t = Compounding period (1 = annual, 2 = semi-annual, etc.)

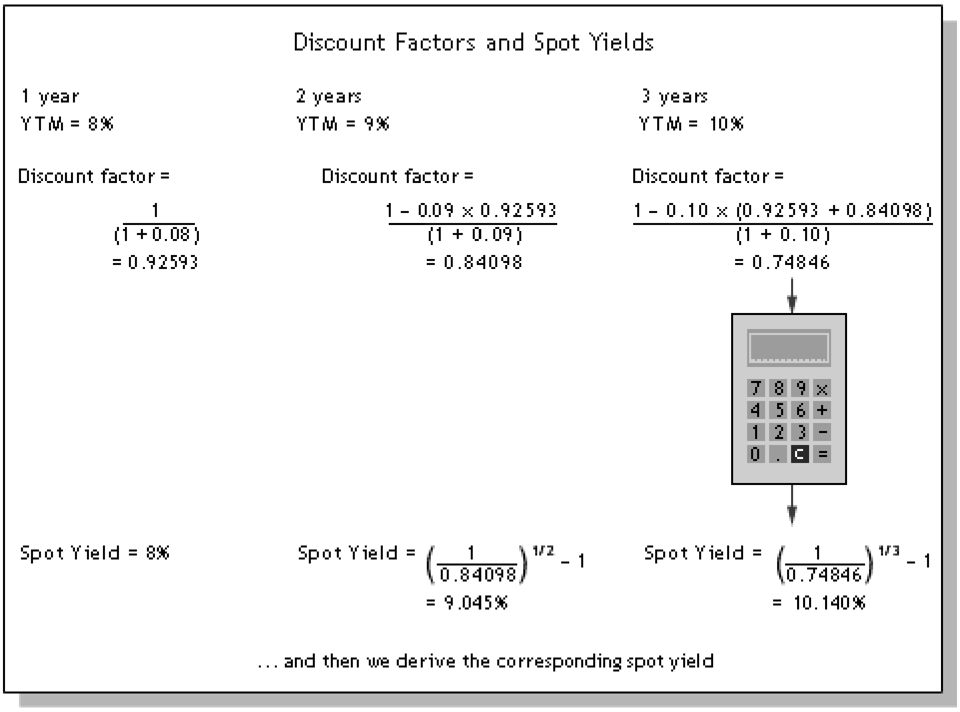
The bootstrapping procedure then consists in sequentially:

1. Calculating the discount factor for any given period, given the corresponding YTM and all the previous discount factors

2. Deriving the spot yield from the calculated discount factor.

4.1. Example

The figure below shows how you bootstrap the same yield curve used in the example in section *Bootstrapping*, this time using discount factors.



4.2. General Formulas

Discount factors from yields to maturity:

Discount factor for period a, $D_{0,a} = \dfrac{1}{(1 + Y_a / t)^a}$

Discount factor for period a+i, $D_{0,a+i} = \dfrac{1 - Y_{a+i} / t \times (\Sigma D_{0,a+k})}{(1 + Y_{a+i} / t)}$

for $k = 0 \ldots i - 1$

Where Y_j = Yield to maturity to period j
t = Compounding period (1 = annual, 2 = semi-annual, etc.)
a = 1 – Fractional coupon period

Spot yields from discount factors:

Spot yield to period a, $R_{0,a} = Y_a$

Spot yield to period a+i, $R_{0,a+i} = [(1 / D_{0,a+i})^{1/(a+i)} - 1] \times t$

Working with discount factors also simplifies the calculation of **forward yields**, as we shall see in section *Forward Yields*, below.

5. Par from Spot Yields

So far we have shown how to derive the spot curve from the par curve. It is also possible to go the other way - i.e. to derive par yields from spot yields. This technique is used in the interest rate swap market to price swaps off the spot curve. (see Interest Rate Swaps - Pricing).

Example

In the example in section *Bootstrapping*, we generated the following spot yields:

Maturity	**Spot Yield**
1 year	8.00%
2 years	9.05%
3 years	10.14%

A 3-year YTM (C%) may be derived from this set of spot yields using the following relationships:

- Price of a C% coupon bond to yield C% = Par (in this case $1)
- Price of par bond = PV of its future cash flows, each discounted at its own spot yield

$$1 = \frac{C}{(1 + 0.0800)} + \frac{C}{(1 + 0.0905)^2} + \frac{C}{(1 + 0.1014)^3} + \frac{1}{(1 + 0.1014)^3}$$

$$= C \times (0.92593 + 0.84098 + 0.74846) + 0.74846$$

$$= C \times 2.51537 + 0.74846$$

Therefore:

$$C = \frac{1 - 0.74846}{2.51537}$$

$= 0.10$ or **10.0%**

5.1. The General Formula

Par Yields from Spot Yields

$$1 = C/t \times \left[\Sigma \frac{1}{(1 + R_{0,i} / t)^i} \right] + \frac{1}{(1 + R_{0,L} / t)^L}$$

$$= C/t \times (\Sigma D_{0,i}) + D_{0,L}$$

Where:

C = Par yield to maturity
L = Number of whole coupon periods
t = Compounding period (1 = annual, 2 = semi-annual, etc.)
i = 1…L
$R_{0,i}$ = Spot yield to period i
$D_{0,i}$ = Discount factor for period i $= \frac{1}{(1 + R_{0,i})^i}$

Therefore,

$$\mathbf{C = \frac{(1 - D_{0,L}) \times t}{\Sigma D_{0,i}}}$$

6. Exercise 1

Question 1

The table below shows the first 3 points on an annual par curve. Calculate the corresponding spot yields and discount factors.

a) Enter your answers in each box below, then validate. Yields should be in percent, rounded to 2 decimal places, and discount factors should be rounded to 5 decimal places.

Year	**1**	**2**	**3**
Par Yield	6.00	6.70	6.85
Spot yield			
Discount factor			

b) In (a) spot yields are higher than par yields because:

- The yield curve is positive
- The yield curve is inverted
- Demand for zero coupon instruments is weaker
- Zero coupon instruments have higher credit risk

7. Forward Yields

Forward yields: yields that are effective (i.e. start accruing) at some specified future date.

Forward yields may be derived from the spot curve, as the following example illustrates.

Example - Par and Spot Yields (bond basis)

Maturity	YTM	Spot Yield
1 year	8%	8.00%
2 years	9%	9.05%
3 years	10%	10.14%

A dealer looking to lend $1 million for 1 year commencing in 1 year's time (a 1x2 year forward-forward loan) can fix his forward funding costs by performing the following operations:

- Raise $1 million by issuing a 2-year zero coupon bond yielding 9.05%
- Place the $1 million raised in the market for 1 year at 8%.

The dealer can now calculate his breakeven forward lending rate, R_{1x2}. This is the rate he must achieve on its forward loan such that:

Total cost of borrowing $1 for two years = Total return on investing $1 for two years

$$(1 + 0.09045)^2 = (1 + 0.0800) \times (1 + R_{1x2})$$

Therefore:

$$R_{1x2} = \frac{(1 + 0.09045)^2}{(1 + 0.08000)} - 1$$

$$= 0.10101, \text{ or } \mathbf{10.10\%}$$

R_{1x2} must be higher than the 2-year spot rate to compensate for the fact that the return on the first investment (8%) is lower than the underlying cost of funding (9.05%).

Using the same approach, the breakeven 2x3 year forward rate would be:

$$R_{2x3} = \frac{(1 + 0.1013952)^3}{(1 + 0.0904500)^2} - 1$$

$$= 0.123616 \text{ or } \mathbf{12.36\%}$$

7.1. Forward, Par and Spot Yields

The yield curve in the previous example, which is positive, was the same as the one used in section *Bootstrapping*, where we saw that spot yields were higher than par yields. Now we can see why:

- An investor purchasing the 10% 3-year coupon bond at par can lock in a reinvestment rate (using a forward LIBOR instruments such as short interest futures) of 10.11% for year 2 and 12.36% for year 3 on his future coupons. Since these reinvestment rates are higher than the bond's yield to maturity (YTM), the actual return on the bond will be higher than 10% (the reinvestment rate assumed in YTM).

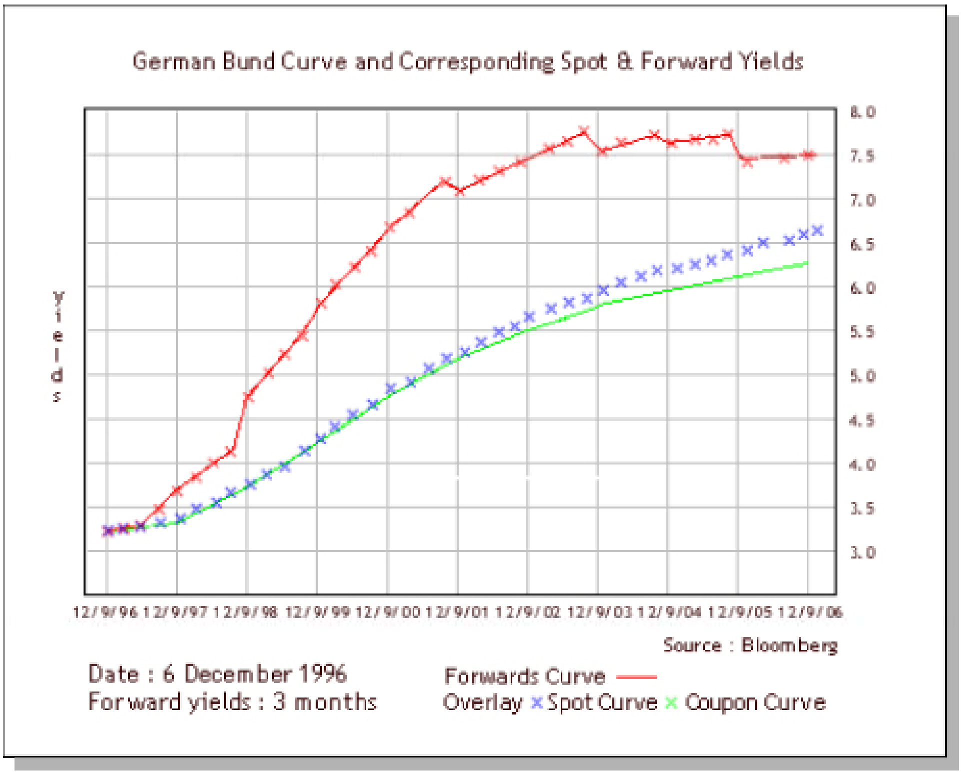
- In contrast, an investor in a 3-year zero cannot take advantage of these higher forward reinvestment rates, so by way of compensation this investor requires a higher YTM on his zero (10.14% instead of 10%).

Forward, par and spot yields are logically linked:

- **In a positive yield curve forward yields are higher than cash yields. Therefore spot yields must be higher than par yields.**

- **In an inverted yield curve forward yields are lower than cash yields. Therefore spot yields are lower than par yields.**

The figure below shows the spot and forward yield curves derived from the German government bond curve on a particular date.



For the bond analyst, a forward yield curve such as the one above is useful because:

- It maps the market's view about the future path of the short term interest rates (in this case the 3 month rate).

  Anyone taking a different view can take a position against it using a variety of interest rate derivatives of the types we discuss later in this programme (e.g. futures on LIBOR and interest rate swaps)

- It highlights which parts of the curve seem relatively cheap or rich to others at any one time.

  Any 'blip' on the shape of the par curve tends to be exaggerated on the forward curve, helping the trader to identify which issues may be trading rich relative to other issues and set up yield curve spread trades of the types that we discuss in module Outright and Spread Trading, in the expectation of a correction in the blip.

7.2. General Formula

In module Short Interest Futures – Forward Rates we develop the general arbitrage formula for forward yields money market style. Below is the corresponding formula for forward yields bond market style.

Forward Yields (zero coupon, bond basis)

Total cost for long period = Total return on the rollovers

$$(1 + R_{0,L} / t)^{L} = (1 + R_{0,S} / t)^{S} \times (1 + R_F / t)^{F}$$

$$R_F = \left[\left\{ \frac{(1 + R_{0,L} / t)^{L}}{(1 + R_{0,S} / t)^{S}} \right\}^{1/F} - 1 \right] \times t$$

$$= [(D_{0,S} / D_{0,L})^{1/F} - 1] \times t$$

Where:
$R_{0,L}$ = Spot yield for period 0 to L (the long rate)
$R_{0,S}$ = Spot yield for period 0 to S (the short rate)
$D_{0,L}$ = Discount factor for period 0 to L
$D_{0,S}$ = Discount factor for period 0 to S
S = Time to period S, in coupon periods and fractions
L = Time to period L, in coupon periods and fractions, L > S
F = L – S
R_F = Implied forward yield for period F
t = Compounding period (1 = annual, 2 = semi-annual, etc.)

This formula may be used to calculate forward yields for any term, as the example shows.

Example
Forward yield calculation

The table below summarises the forward yields for 1- and 2-year terms derived from the yield curve in the example above:

	YTM	Spot Yield	Discount Factor	1 year Term	2 year Term
1 year	8%	8.000%	**0.92593**	8.00%	9.045%
2 years	9%	9.045%	0.84098	10.10%	**11.23%**
3 years	10%	10.140%	**0.74846**	12.36%	

The 1x3 forward yield is calculated as follows:

$$R_{1x3} = \left[\frac{(1 + 0.1014)^{3}}{(1 + 0.08)} \right]^{1/2} - 1$$

$$= (0.92593 / 0.74846)^{1/2} - 1$$
$$= 0.11226 \text{ or } \mathbf{11.23\%}$$

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8. Spot from Forward Yields

8.1. Derivation Example

Since par yields, spot yields and forward yields are logically related to each other, any one of these may be derived from any other. Derivatives analysts seeking to generate a reliable spot curve often derive at least part of the curve from forward rates taken from the Eurocurrency futures market, rather than from the cash LIBOR market which is less liquid[1].

Example

Suppose we observe the following forward yields implied in the futures strips.

Period	Forward Yield
0x1 year	8.00%
1x2 years	10.10%
2x3 years	12.36%

We derive a 3-year spot yield from these forward yields as follows:

Total return on 3-year zero coupon investment = Total return on three annual rollovers

$$(1 + R_{03})^3 = (1 + 0.0800) \times (1 + 0.1010) \times (1 + 0.1236)$$

$R_{03} = 0.1014$ or **10.14%**

The technique (summarised next) may be used to derive any long period spot (or forward) zero coupon yield from a strip of single-period forward yields.

8.2. General Formulas

Spot Yields from Forward Strips

Total return on T-period zero coupon Investment, starting in S periods = Total return on T separate rollovers

$$(1 + R_T / t)^T = (1 + R_{S+1} / t) \times (1 + R_{S+2} / t) \times ... \times (1 + R_L / t)$$

$$= \Pi (1 + R_i / t)$$

Alternatively:

$$D_T = \Pi D_i$$
$$R_T = [(D_T)^{1/T} - 1] \times t$$

Where:
R_T = Zero coupon rate for total period T
i = Interest period; i = S … L
R_i = Forward rate for period i
D_i = Discount factor for period i
t = Compounding period (1 = annual, 2 = semi-annual, etc.)
T = Total number of interest periods = (L – S) x t

[1] The concept of the futures strip will be explained in module Short Interest Futures – The Futures Strip and in module Interest Rate Swaps – Pricing we'll show how to derive a par and a spot curve from the forward LIBORs implied in the futures strip.

9. Par from Forward Yields

9.1. Principles

In this section we show how you derive par yields (as opposed to spot yields) from the forward strip. The starting point is to establish an equivalence between the price of a straight par bond and the price of a floating rate note (FRN), both with a face value of $1:

Present value of par coupon bond = Present value of FRN

$$C/t \times \left[\Sigma \frac{1}{(1 + R_{0,i}/t)^i} \right] + \frac{1}{(1 + R_{0,L}/t)^L} = \Sigma \frac{R_{i-1,i}/t}{(1 + R_{0,i}/t)^i} + \frac{1}{(1 + R_{0,L}/t)^L}$$

for i = 1 ... L coupon periods. Using discount factors:

$$C/t \times (\Sigma D_{0,i}) + D_{0,L} = \Sigma (R_{i-1,i}/t \times D_{0,i}) + D_{0,L}$$

where $R_{0,i}$ is the spot rate from now to each period i and $R_{i-1,i}$ is the forward rate for each period i.

The present value of the principal repayment ($D_{0,L}$) is the same for both securities, so the expression reduces to an equality between the present value of the fixed coupons and the present value of the floating coupons, as shown below.

Par Yields from Forward Yields

$$C/t \times (\Sigma D_{0,i}) = \Sigma (R_i/t \times D_{0,i})$$

$$C = t \times \frac{\Sigma (R_i/t \times D_{0,i})}{\Sigma D_{0,i}}$$

The par yield is a weighted average of the corresponding forward rates, where the weight applied to each forward rate is $D_{0,i} / (\Sigma D_{0,i})$.

Applications

The relationship between the PV of a set of fixed cash flows and the PV of a set of floating (forward) cash flows is used in the interest swaps market to revalue swaps, or to arbitrage swaps against futures strips (see module Interest Rate Swaps - Pricing).

This relationship also establishes a link between the pricing of:

- **Swaptions**, which are options on par yields, and
- Interest rate **caps** and **floors**, which are options on forward rates

(see Interest Rate Options - Pricing).

9.2. Par Yields & Discount Factors

In section *Forward Yields*, above, we saw that a forward yield may be derived from the ratio of two discount factors:

$$\mathbf{R_i = (D_{0,i-1} / D_{0,i} - 1) \times t}$$

Substituting for R_i /t, it can be shown that the formula on the previous page reduces to:

$$C/t \times (\Sigma D_{0,i}) = \Sigma [(D_{0,i-1} / D_{0,i} - 1) \times D_{0,i}]$$

$$\mathbf{C = \frac{(D_{0,S} - D_{0,L})}{\Sigma D_{0,i}} \times t}$$

C = Coupon rate

i = Interest period, i = S+1...L

$D_{0,i}$ = Discount factor for period 0 to i

t = Payment frequency (1 = annual, 2 = semi-annual, etc.)

If the coupon rate is effective from time 0 then $D_{0,0}$ = 1 and the expression simplifies to:

$$C = \frac{(1 - D_{0,L})}{\Sigma D_{0,i}} \times t$$

This is the same expression that we derived in section *Par from Spot Yields*, above, starting with discount factors!

10. Exercise 2

10.1. Question 1

Question 2

The table below shows the first 3 points on an annual par curve, together with their corresponding spot yields and discount factors. Calculate the annual forward yields from this data.

a) Enter your answers in each box below, in percent, rounded to 2 decimal places, and validate.

Year	**1**	**2**	**3**
Par Yield	6.00000	6.70000	6.85000
Spot yield	6.00000	6.72361	6.87670
Discount factor	0.94340	0.87797	0.81913
Forward yield	---		

10.2. Question 2

Question 3

The table below shows one-year forward yields effective in years 1, 2 and 3. Calculate the 1, 2 and 3 year spot and par yields using the formulas developed in sections *Spot from Forward Yields* and *Par from Forward Yields*.

a) Yields should be in percent, rounded to 2 decimal places, and discount factors should be rounded to 5 decimal places.

Year	**1**	**2**	**3**
Forward yield	7.50	6.06	5.85
Spot yield	7.50		
Discount factor			
Par yield			

b) What is the rate for a loan commencing in 1 year's time and ending in 3 years (i.e. a 1x3 year forward yield), in percent to 3 decimal places:

(i) On a zero coupon basis?

(ii) On a coupon (i.e. par) basis?

c) Which (one or more) of the statements below explain(s) why in this example spot yields are lower than par yields?

- ☐ Zero coupon instruments have low credit risk
- ☐ Demand for zero coupon instruments is stronger
- ☐ The yield curve is inverted
- ☐ Forward yields are lower than par yields

