ISMA CENTRE - THE BUSINESS SCHOOL
OF THE FINANCIAL MARKETS

UNIVERSITY OF READING
ENGLAND

IFID Certificate Programme

Fixed Income Analysis

Time Value Of Money

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1. Overview

We prefer money today rather than in the future, because money today can be lent to earn interest. The **future value** (FV) of $1 today is therefore more than $1; the **present value** (PV) of $1, payable in the future, is therefore less than $1.

The price of any financial instrument is the discounted value of its expected future cash flows.

This fundamental principle applies to any financial instrument, whether it is a bond, an equity security or even a derivative such as an option. All financial instruments are priced using a **discounted cash flow** (DCF) model. DCF is a two stage methodology:

1. Identify the expected future cash flows (FVs) that are receivable or payable on the instrument and their timing[1]

2. PV those future cash flows using appropriate discount rates (or implied interest rates) for that category of instrument, taking into account the day-count[2] and the compounding conventions governing the rates used

In this module we distinguish between simple and compound interest and we introduce some general formulas for calculating FVs and PVs. We show how the formulas are used to price instruments such as annuities and perpetuities or to value irregular cash flow streams. We also explain how to calculate the investment return implied in a stream of cash flows.

Learning Objectives

By the end of this module, you will be able to:

1. List the key elements of a discounted cash flow model for valuing financial instruments

   Distinguish between:
2. (a) nominal rates and effective rates of interest
3. (b) discrete and continuously compounded rates of interest

4. Calculate an effective annual rate, given the nominal rate and the compounding period, and vice versa

5. Using different compounding periods, calculate the future value (FV) of:
   - A single cash flow
   - A stream of regular cash flows
   - A stream of irregular cash flows

6. Using different compounding periods, calculate the present value (PV) of:
   - A single cash flow
   - A stream of regular cash flows
   - A perpetuity
   - A stream of irregular cash flows

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[1] In the case of a straight bond this stage is very simple, as the future cash flows are stipulated in the certificate of debt. In the case of other instruments, for example an option, the future cash flow is uncertain and therefore has to be estimated either as a probability-weighted average or proxied by the forward price of the underlying instrument. We shall examine this in module Options Pricing and Volatility – Analytic Models.

[2] The day-count conventions specify, among other things, how many days are assumed to be in each compounding period and how many days in a given financial year. We shall examine the day-count conventions used in the different fixed income markets later on in this programme (see Money Market Instruments - Accrued Interest and Bond Pricing – Accrued Interest).

7. Calculate the sensitivity of the PV of a stream of regular cash flows to:
   - A change in the discount rate
   - The passage of time

8. Define the concept of internal rate of return (IRR) and explain how this is calculated

9. Explain the assumptions underlying IRR and the circumstances under which this may not be suitable as an investment or capital budgeting decision tool

2. Simple & Compound Interest

Compounding frequency
The number of times in a year that the interest on a loan is **capitalised** – i.e. added to the loan's principal - so that it can earn **interest on interest** (IOI).

Suppose we place $100 on a 12-month deposit with a bank and the bank quotes us a rate of 10% per annum. Suppose also that we a choice between:

1. Receiving all the interest and the principal in a single bullet at maturity;

   or

2. Receiving some of the interest every 6 months.

In the first case there is no compounding (**simple interest**) and in the second case the interest is **semi-annually compounded**. The two compounding methods affect the value of the total repayment at maturity. Let's see how.

2.1. Simple Interest

In the case of simple interest (case 1, on the previous page) the amount repaid at maturity would be just the $100 principal plus interest of $10:

Repayment amount = 100 + (100 x 0.10)
= 100 + 10
= **$110**

In this simple interest calculation, the interest amount paid is proportional to the term of the loan and is calculated by pro-rating the quoted annual rate by the term of the loan. Each day, the loan accrues the same amount of interest.

2.2. Compound Interest

Mankind's two most important inventions: the wheel and compound interest. Nathan M. Rothschild.

Semi-annual Compounding

In case 2 on the first page, above, the 10% annual rate on a 12 month loan for $100 is compounded every 6 months (semi-annually). Assuming that 6 months is exactly half a year, then after the first 6 months:

Principal amount = 100 + (100 x 0.10 x ½)

$= 100 \times (1 + 0.10/2)$
$=$ **\$105**

The first \$5 of interest is now paid or capitalised and the loan is **rolled over** for another 6 months at the annual rate of 10%. So at the end of the 12 months:

Repayment amount $= 105 \times (1 + 0.10/2)$
$= [\ 100 \times (1 + 0.10/2)\ ] \times (1 + 0.10/2)$
$= 100 \times (1 + 0.10/2)^2$
$=$ **\$110.25**

This is \$0.25 more than what would be earned on a simple interest basis, as in the example on the previous page. The extra \$0.25 represents interest on the \$5 interest paid at the six-month point: interest on interest. Thus, although the **nominal interest rate** on this deposit is 10%, its **effective rate** is 10.25%.

Quarterly Compounding

Had interest been compounded quarterly, then after 12 months:

Repayment amount $= 100 \times (1 + 0.10/4)^4$
$=$ **\$110.38!**

Quarterly compounded, a nominal rate of 10% represents an effective rate of 10.38%. The more frequent the compounding period, the higher is the effective rate and in theory, compounding can be as frequent as you like - quarterly, monthly, weekly, daily, hourly and even **continuously**, as we shall see in a later section.

2.3. Nominal & Effective Rates

What you see is not what you get!

The examples on the previous pages illustrate that, with compounding, the interest rate that you see is not necessarily the interest that, effectively, you earn or you pay. If the compounding frequency is other than annual, it becomes necessary to distinguish between a quoted nominal rate and its effective annual rate and.

The general formula below establishes the relationship between the nominal compounded rate and the effective rate and can be used to convert from one type of rate to the other.

Annually compounded return = Equivalent return, compounded t times

$(1 + \text{Effective rate}) = (1 + \text{Nominal rate}/t)^t$

Therefore:

Effective rate[3] $= \mathbf{(1 + \text{Nominal rate}/t)^t - 1}$

Nominal rate $= \mathbf{[\,(1 + \text{Effective rate})^{1/t} - 1\,] \times t}$

Where t = Compounding frequency
(1 = annual, 2 = semi-annual, 4 = quarterly, etc.)

[3] Also known (in the UK) as the **annual percentage rate** (APR).

3. Periodic & Continuous Compounding

Periodic (or **discrete**) **compounding** is when there is a finite number of compounding periods in a year – e.g.:

- 4 as in quarterly compounding
- 12 as in monthly compounding
- 52 as in weekly compounding
- 365 as in daily compounding
- 8,750 as in hourly compounding
- Etc!

Continuous compounding is the limiting case, when the compounding period is infinitesimally small.

In the limit, as the number of compounding period's approaches infinity:

$$X_T = 1 \times e^{R.T}$$

Where:

X_T = Value of $1 in T years from today

T = Number of years (or fraction of a year)

R = Nominal interest rate (in decimal - i.e. 10% is 0.10)

e = 2.71828182846... (the exponential coefficient[4])

In other words, with continuous compounding, the value of your dollar grows exponentially! The general formula below establishes the relationship between the nominal continuously compounded rate and the effective annual rate and can be used to convert from one type of rate to the other.

Annually compounded return	=	Equivalent return, continuously compounded
(1 + Effective rate)	=	e^R

Therefore:

Effective rate	**=**	$e^R - 1$
Nominal rate (R)	**=**	**ln(1 + Effective rate)**
Where ln()	=	Natural logarithm[5] (i.e. logarithm to base e)

Who uses continuous compounding?

Continuous compounding implies that interest on a loan is capitalised as soon as it accrues and in practice nobody can compound interest that fast!

[4] For a refresher of this concept, see the special note on the exponential coefficient and logarithms.
[5] It is not too difficult to derive this closed-form solution from the arithmetic series for the FV, but its proof lies outside the scope of this programme.

However, continuous compounding is an important concept because this interest formula (known as an **exponential function**) is algebraically much easier to manipulate than the discrete compounding formula and is therefore often used in mathematical models for pricing options and other derivatives (see for example Options Pricing and Volatility – Analytic Models).[6]

More on exponentials and logarithms
More information about the exponential coefficient, the continuous compounding formula introduced here and natural logarithms. This will help you gain a deeper understanding of why exponential functions and natural logarithms are so widely used in financial modelling, but you are *not* required to know this for the exam!

The exponential coefficient (e) is a constant that is the limiting value of the expression:

$(1 + 1/m)^m$, as m approaches infinity.

Formally:

$$\lim_{m \to \infty} (1 + 1/m)^m = e$$

e = 2.71828182846...

Given this definition, it is easy to derive the continuous compounding **exponential function** introduced in this section.

$$X_T = (1 + R/t)^{t.T}$$
$$= [(1 + R/t)^{t/R}]^{R.T}$$

Where:

X_T = Value of $1 today in year T

R = Nominal interest rate (in decimal - i.e. 10% is 0.10)

T = Number of years (or fraction of a year)

t = Number of compounding periods per year
(t x T = N, total number of compounding periods)

Let m = t/R, so:

$$X_T = [(1 + 1/m)^m]^{R.T}$$

As n approaches infinity, so does m and *by definition* $(1 + 1/m)^m$ approaches the value of the exponential coefficient.

Therefore:

$$\mathbf{X_T = e^{R.T}}$$

[6] Of course, any such model has to convert the given market interest rate, which is quoted on a discrete basis, into a continuously compounded equivalent before pricing the derivative, otherwise the effective rate used by the model will not be the same as the market's!

Natural logarithms

The natural logarithm (X) of a number Y, ln(Y), is the **index** of an exponential function to the **base e** (the exponential coefficient).

$X = \ln(Y)$
$Y = e^{X}$

In this definition X is the **logarithm** (or log) of Y and Y is the **anti-logarithm** (or anti-log) of X.

Logarithms (especially natural logs) are often used to simplify financial calculations. For example, if two values of a variable X at different points in time (X_{t-1} and X_t) are numerically close, then $\ln(X_t / X_{t-1})$ gives a good approximation of the percentage change between them. Formally:

$$\begin{aligned}\text{Percentage change} &= (X_t - X_{t-1}) / X_{t-1} \times 100 \\ &= (X_t / X_{t-1} - 1) \times 100 \\ &= \ln(X_t / X_{t-1}) \times 100 \qquad \text{if } X_t \approx X_{t-1}\end{aligned}$$

Thus, if you have an investment that accrues interest at the rate of R% per annum, then:

$X_t = X_{t-1} \times e^{R.\Delta t}$

Where Δt = [t – (t-1)], in years or fraction of a year

Therefore:

$X_t / X_{t-1} = e^{R.\Delta t}$

$\ln(X_t / X_{t-1}) = R.\Delta t$

The *percentage growth* of an investment earning continuously compounded interest is directly proportional to the rate of interest applied and the length of time of the investment.

In section *Simple & Compound Interest* we saw that the absolute growth of an investment earning simple interest is directly proportional to the rate of interest applied and the length of time of the investment. Now we have the equivalent rule for a continuously compounded investment!

Common logarithms

The common logarithm (X) of a number Y, log(Y), is the index of an exponential function to the base 10:

$X = \log(Y)$
$Y = 10^{X}$

Common logarithms share the same algebraic properties as natural logarithms but not the continuous compounding growth rule described above, so they are not as useful in finance as are natural logarithms.

4. Exercise 1

4.1. Question 1

Question 1

In the table below, please enter your answers in percent to 3 decimal places.

a) What is the effective interest rate payable on a 1 year loan at 5% per annum nominal, when the rate is compounded:

Compounding period	Effective Rate
Annually	
Semi-anually	
Quarterly	

4.2. Question 2

Question 2

Consider the following loan:

Principal:	$100
Nominal rate:	10%
Maturity:	2 years
Compounding period	Daily (1/365 of a year)

a) What is the effective annual rate on this loan? Enter your answer in percent per annum, rounded to 3 decimal places.

Effective rate

b) Another bank quotes an effective rate of 10.49%, monthly compounded, for the same loan. Which of the two is cheaper?

- The daily compounded one
- The monthly compounded one

c) What would be the nominal rate on a loan that pays an effective rate of 10.49%, monthly compounded? Enter your answer in percent per annum, rounded to 3 decimal places.

Nominal rate

4.3. Question 3

Question 3

What is the annually compounded yield on a bond yielding 4%, semi-annually?

a) Enter your answer in percent, rounded to 3 decimal places.

4.4. Question 4

Question 4

a) Consider the following loan:

Principal:	$100
Nominal rate:	10%
Maturity:	1 year
Compounding period:	Continuous

What is the effective annual rate on this loan? Enter your answer in percent per annum, rounded to 3 decimal places.

Effective rate

b) A derivatives system developer models the forward price of an underlying asset based on the following continuously compounding formula:

$$\mathbf{X_T = e^{R.T}}$$

Where:

X_T = Value of $1 today in year T

T = Number of years (or fraction of a year)

R = Nominal interest rate (in decimal - i.e. 10% is 0.10)

e = 2.71828182846... (the exponential coefficient)

If the developer takes the simple interest rate of 10% quoted by a broker to calculate the 1 year forward price for the asset, what should be the continuously compounded equivalent nominal rate that he should enter into his model? Enter your answer in percent per annum, rounded to 3 decimal places.

Continuously compounded equivalent rate

5. Future Value

5.1. Single Cash Flow

The future value (FV) of an amount of money that we borrow (or lend) today is the value that you will have to pay (or have earned) at some time in the future. The process of computing the future value is also referred to as **compounding** and is grounded on the concepts of simple and compounded interest presented in the previous section.

Future value of a single cash flow A:

$FV(A) = A \times (1 + R \times T)$ simple interest (no compounding)

$FV(A) = A \times (1 + R/t)^{t.T}$ discrete compounding

$FV(A) = A \times e^{R.T}$ continuous compounding

Where:

T = Number of years (or fraction of a year)

t = Number of compounding periods per year

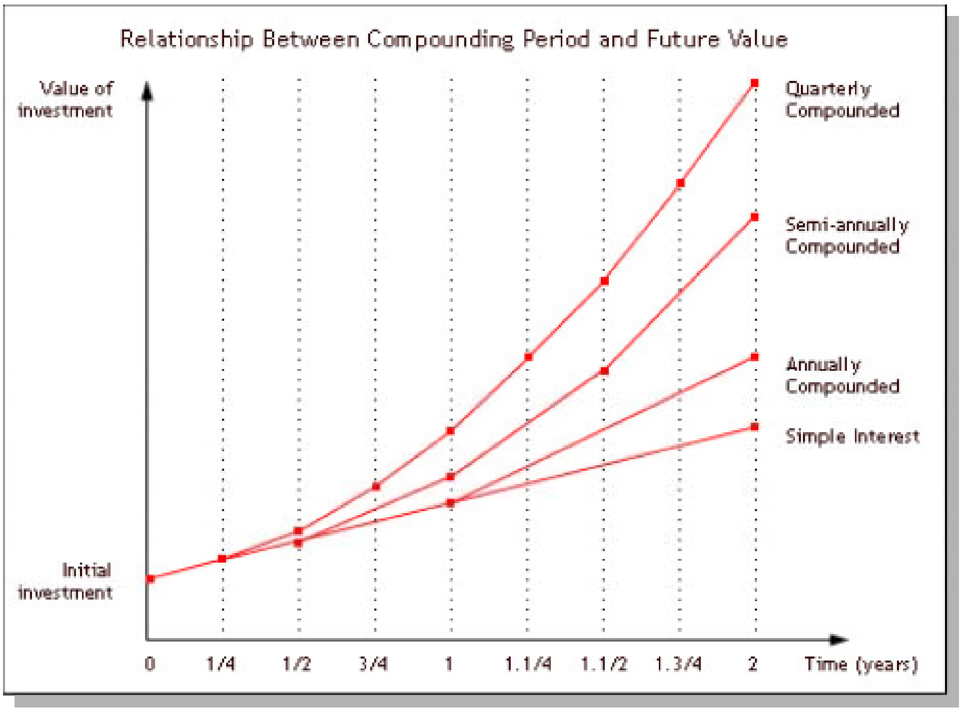
R = Nominal interest rate (in decimal - i.e. 10% is 0.10)

e = 2.71828182846... (the exponential coefficient)

The expression by which A is multiplied in each case, for example $(1 + r/t)^{t \times T}$ in the case of discrete compounding, is the future value of $1 at time T and is known as the **compounding factor**.

Example

If you deposit $100 today in a bank account for 10 years to earn an annual interest rate of 10%, this investment will grow to a different future amount depending on the compounding frequency: the more frequent the compounding the faster the money grows.



5.2. Stream of Regular Cash Flows

Annuity is a regular stream of equal cash flows payable at regular intervals over a fixed period of time.

The future value of an annuity is the sum of the future values of each of the individual cash flows:

$$FV = C \times (1 + R/t)^{t.T-1} + C \times (1 + R/t)^{t.T-2} + \ldots + C \times (1 + R/t)^{1} + C$$

$$= C \times \Sigma\, (1 + R/t)^{t.T-i} \qquad \text{where } i = 1 \ldots t.T$$

$$\mathbf{= C \times \left[\frac{(1 + R/t)^{t.T} - 1}{R/t} \right]} \qquad \text{(closed-form solution[7])}$$

Where:
C = Regular cash flow
R = Interest rate (the required return on the annuity)
t = Number of payments per year (= compounding period)
T = Number of years

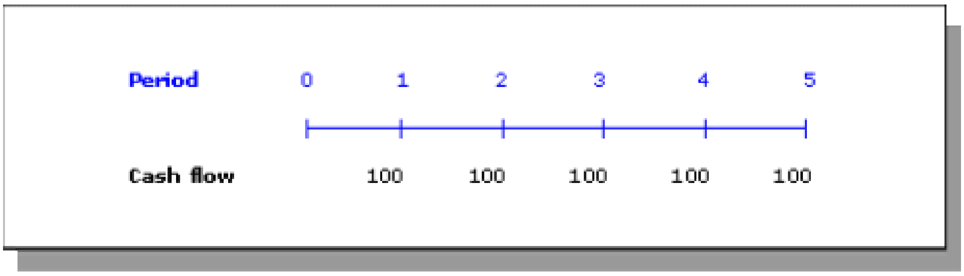
Annuities are found in contracts such as rents, leases, amortised loans and pension plans. They also exist as components of derivative instruments such as the fixed leg of an interest swap (see Interest Rate Swaps – Generic Structure).

The term in the square brackets above is the future value of an annuity of $1 per period. Multiplying this by the amount of the annuity (C) gives the future value of that annuity amount.

The future value of an annuity is calculated at the end of the period in which the final payment occurs. Thus, the FV of a 5-year annuity is computed at the end of year 5.

Example

? What is the future value of an annuity of $100 per year for 5 years, if the rate of rate is 10%?



$$FV = 100 \times \left[\frac{(1 + 0.10)^{5} - 1}{0.10} \right]$$

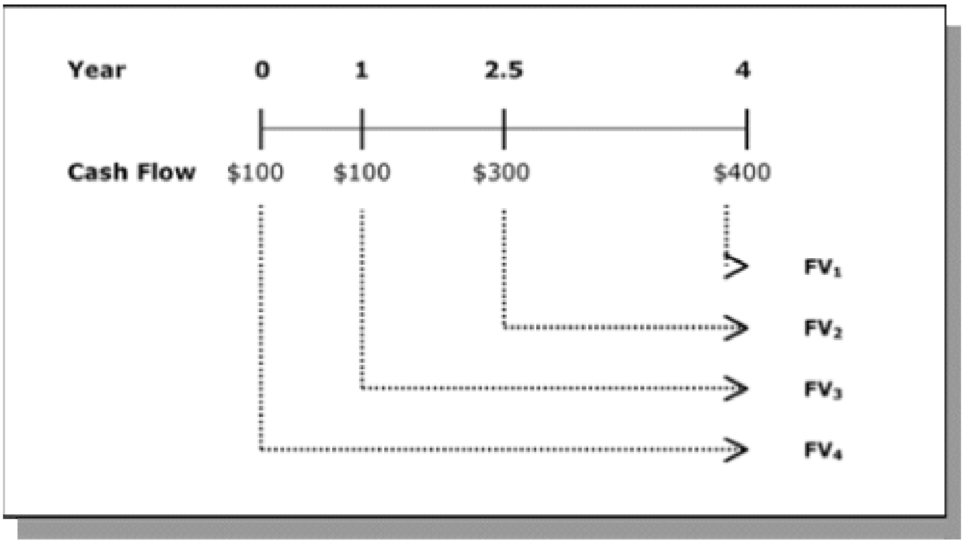
$$= \mathbf{\$610.51}$$

[7] It is not too difficult to derive this closed-form solution from the arithmetic series for the FV, but its proof lies outside the scope of this programme.

5.3. Stream of Irregular Cash Flows

When the pattern of the cash flow stream is not regular, no simple formula exists to compute the future value in one single step; the future value must be calculated as the sum of the future values of each individual cash flow.

Example



FV_1	$= \$400 \times (1 + 0.10)^{0}$	=	$400.00
FV_2	$= \$300 \times (1 + 0.10)^{1.5}$	=	$346.10
FV_3	$= \$100 \times (1 + 0.10)^{3}$	=	$133.10
FV_4	$= \$100 \times (1 + 0.10)^{4}$	=	$146.40
Total FV			**$1,025.60**

Reinvestment Risk

Notice how, when we apply the compounding formula to a stream of cash flows, we are implicitly assuming that all the cash flows paid will be reinvested at the same rate, which of course in reality may not be possible.

In bullet instruments, the interest due at the end of each compounding period is capitalised - in other words, it is added to the balance of the principal outstanding - and therefore continues to accrue at the same rate as the original loan.

In other contracts, such as bonds, the interest due is actually paid out in instalments (known as **coupons**) and there is no guarantee that the lender may be able to reinvest the coupons received at the same rate as the original loan. In such cases the total return on the investment depends on the reinvestment rate achieved. **Reinvestment risk** is discussed in more detail in the context of bond yields.

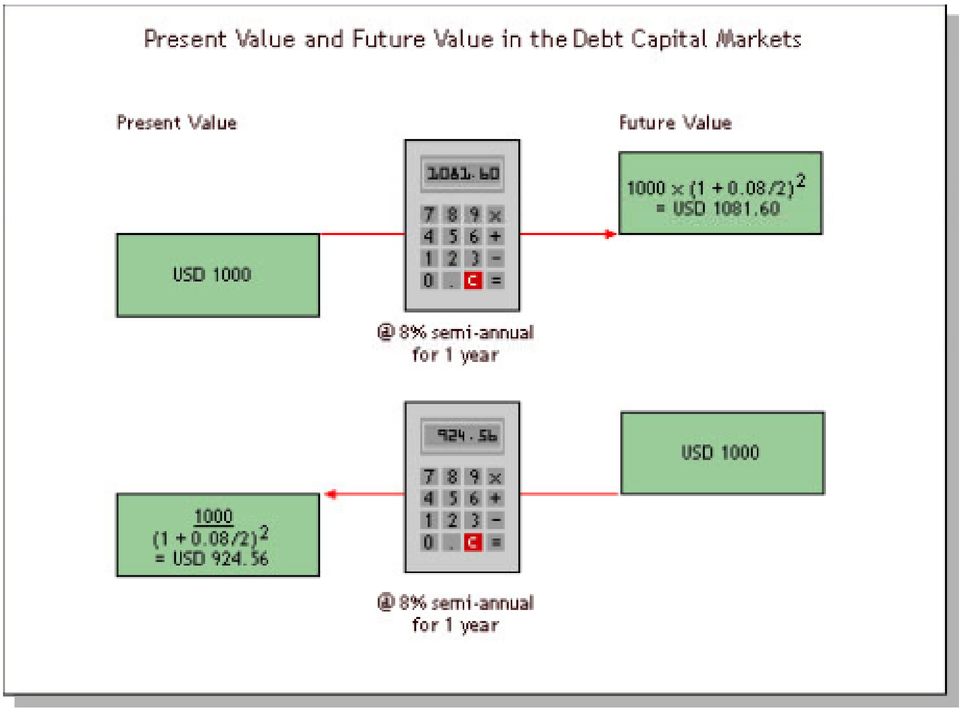
6. Present Value

6.1. Single Cash Flow

Present valuing is the process of determining what a cash flow to be received in the future is worth in dollars today – i.e. what is the cash price of that future cash flow.

Computing the PV of a future cash flow is referred to as **discounting** and the interest rate used to calculate the present value is called the **discount rate**.

In section *Future Value* we showed how to calculate the FV of a sum borrowed or invested today, given rate of interest, the maturity and the compounding period. The figure below illustrates how the same methodology is used to calculate the PV of a future cash flow.



In the example, $924.56 is the PV of $1,000 payable in 1 year: the sum of capital which, when invested at 8% per annum, semi-annually compounded, accumulates to exactly $1,000 after 1 year.

Notice the logic between PV and FV:

- When we go forward in time we compound a present value by multiplying it by its compounding factor (1+ ...)
- When we go back, we discount a future value by multiplying it by the reciprocal of the same factor, called the **discount factor**

A discount factor is the present value of $1 payable (or receivable) at a given point in the future.

For example, a sum of money receivable today will have a discount factor of 1, making its PV and FV the same. The further out in time the cash flow is received, the smaller will be its discount factor and hence it's PV.

From the formulas for FV introduced in section *Future Value* above, we can easily derive corresponding ones for calculating PV.

$PV(A) = FV(A) \times (1 + R \times T)^{-1}$ simple interest (no compounding)

$PV(A) = FV(A) \times (1 + R/t)^{-t.T}$ discrete compounding

$PV(A) = FV(A) \times e^{-R.T}$ continuous compounding

Where:

T = Number of years (or fraction of a year)

t = Number of compounding periods per year

R = Nominal interest rate (in decimal - i.e. 10% is 0.10)

e = 2.71828182846... (the exponential coefficient)

As we shall see in later sections of this programme, different sectors of the fixed income market use one or the other version of these PV formulas to price the instruments. For example:

- The simple interest version is used in the money markets
  (see Money Market Instruments – Present & Future Value)

- The discrete compounding version is used in the bond markets
  (see Bond Pricing – Valuation Formula)

- The continuous compounding version is used in the options markets
  (see Option Pricing & Hedging – Analytic Models)

6.2. Stream of Regular Cash Flows

The present value of a stream of cash flows is equal to the sum of the present values of the individual cash flows. In the special case of an annuity, the calculation of the PV of the cash flow stream can be greatly simplified[8]:

$PV = C \times (1 + R/t)^{-1} + C \times (1 + R/t)^{-2} + \ldots + C \times (1 + R/t)^{-t.T}$

$= C \times \Sigma\, (1 + R/t)^{-i}$ where $i = 1 \ldots t.T$

$\mathbf{= C \times \Sigma D_i}$

$\mathbf{= C \times (D_1 - D_{t.T+1}) / (1 - D_1)}$ (closed form solution)

Where:

D_i = i^{th} discount factor = $1/(1 + R/t)^i$

C = Regular cash flow

R = Discount rate (the required return on the annuity)

t = Number of payments per year (= compounding period)

T = Number of years

Example

? How much could you borrow, at 5% per annum, if you intended to repay the loan in 4 equal annual instalments of $5,000?

[8] It is not too difficult to derive this closed-form formula shown below from the arithmetic series for the PV, but its proof lies outside the scope of this programme.

Analysis
Using the formula for the annuity presented above and making
C = 5,000, R = 0.05, T = 4 and t = 1:

$$\text{PV of annuity} = \frac{5{,}000}{1.05} + \frac{5{,}000}{1.05^2} + \frac{5{,}000}{1.05^3} + \frac{5{,}000}{1.05^4}$$

$$= 4{,}761.90 + 4{,}535.15 + 4{,}319.19 + 4{,}113.51$$

$$= \mathbf{\$17{,}729.75}$$

For long-dated annuities, it is certainly easier to use the closed-form equation presented above, which always has the same size regardless of the number of cash flows.

6.3. Perpetuities

Perpetuity: a regular stream of equal cash flows payable at the same time intervals forever.

Perpetuities are rarer than annuities but by no means uncommon: some governments and corporations have issued fixed-coupon perpetual bonds.

When valuing perpetuities, clearly we have to use a closed-form equation rather than an arithmetic series! In fact, the formula for pricing a perpetuity is very simple indeed. If we take the closed form solution for the PV of an annuity, on the previous page, and we increase T to infinity we find that the discount factor $D_{t.T+1}$ = 0 and the formula reduces to:

$$\text{PV of perpetuity} = C \times (D_1 - 0) / (1 - D_1)$$

$$= \mathbf{\frac{C \times t}{R}}$$

Example

The UK Government 2½% Consols are perpetual bonds that were issued after World War I.

? What is the price of this bond to yield 6.70%?

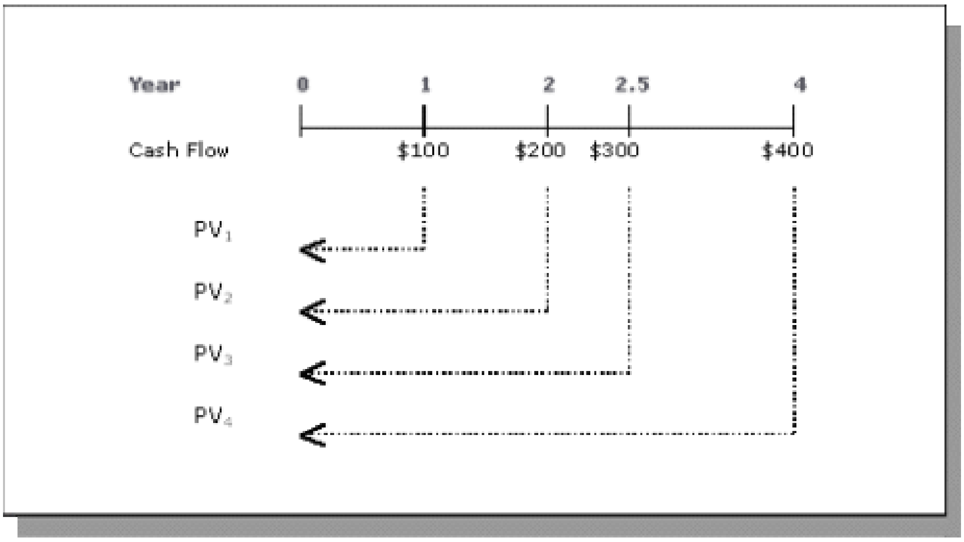
Analysis
In the perpetuity formula, C is the regular cash flow and t is the number of payments per year, so (Cxt) is the bond's annual coupon rate and therefore:

Price = 2.50 / 0.067
= 37.31% - simple as that!

6.4. Stream of Irregular Cash Flows

When the cash flow amounts are not identical, or when they occur at irregular intervals, there is no simple formula and the PV must be calculated as the sum of the PVs of each individual cash flow.

Example



PV_1	$= 100 \times 1/(1 + 0.1)^1$	=	\$90.90
PV_2	$= 200 \times 1/(1 + 0.1)^2$	=	\$165.30
PV_3	$= 300 \times 1/(1 + 0.1)^{2.5}$	=	\$236.40
PV_2	$= 400 \times 1/(1 + 0.1)^4$	=	<u>\$273.20</u>
Total PV			**\$765.80**

7. Exercise 2

7.1. Question 1

Question 5

Consider the following loan:

Principal:	\$100
Nominal rate:	10%
Maturity:	2 years
Compounding period:	Daily (1/365 of a year)

a) What is its repayment amount at maturity? Enter your answer rounded to the nearest cent.

Repayment amount

7.2. Question 2

Question 6

What is the future value of USD 100 deposited at 10% interest, annually compounded, for 7 years?

a) Type your answer rounded to 2 decimal places and validate.

USD

7.3. Question 3

Question 7

In this question, we show how to manipulate the terms of the annuity formula to calculate the amount of an annuity, given its PV and the rate of interest.

a) What is the annual amortisation amount on a \$100 loan repayable over 3 years in fixed instalments (principal plus interest), at a rate of interest of 10%?

Enter your answer to 2 decimal places.

USD

8. PV Sensitivities

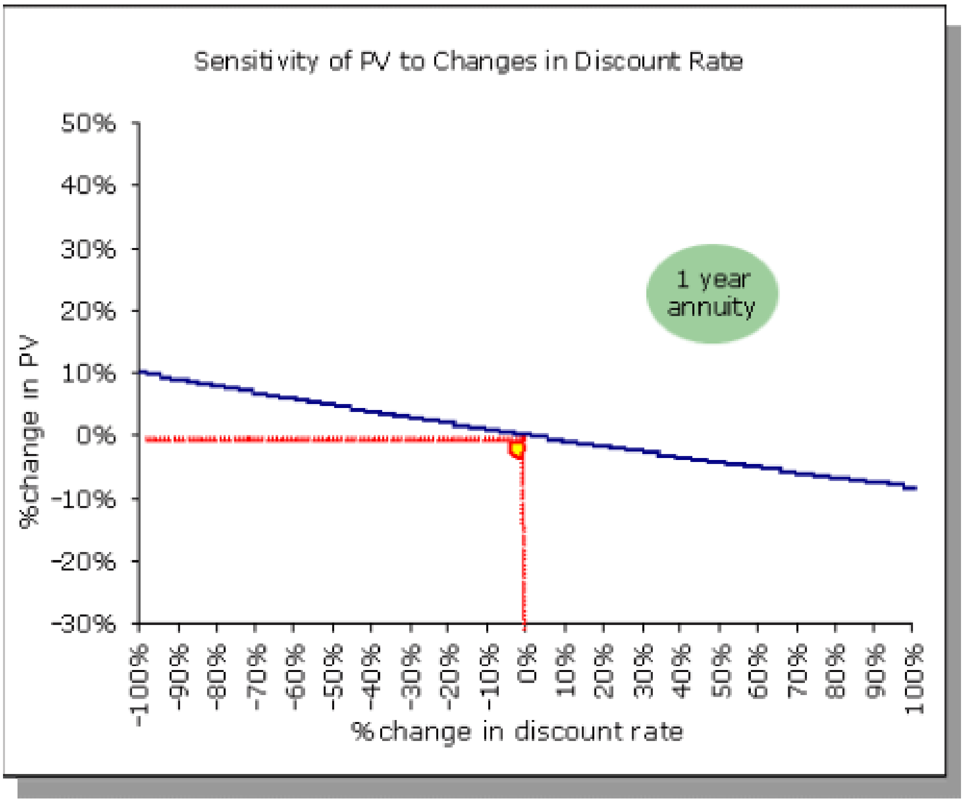
8.1. Sensitivity to the Discount Rate

? What happens to the PV of a stream of cash flows if the discount rate decreases or increases?

The higher the discount rate, the more heavily are the future cash flows discounted, so the PV of the future cash flow stream obviously decreases. But there is one law about the relationship between PV and the discount rate that you should note at this stage:

Other things being equal, the longer the maturity of the future cash flows the greater is the sensitivity of the PV to a change in the discount rate.

The figure below plots the percentage change in the PV of annuities with different maturities for different percentage changes in the discount rate, starting with a 1 year annuity.



The figure shows clearly that, other things being equal, the longer the maturity of the instrument the greater its market risk, but notice that the inverse relationship between PV and discount rates is not linear:

- The percentage falls in PV, as the discount rate is raised, tends to be smaller and smaller
- The percentage rise in PV, as the discount rate is lowered, tends to be larger and larger

This pricing behaviour is known as convexity and is common to nearly all fixed income instruments[9]. Notice one other phenomenon:

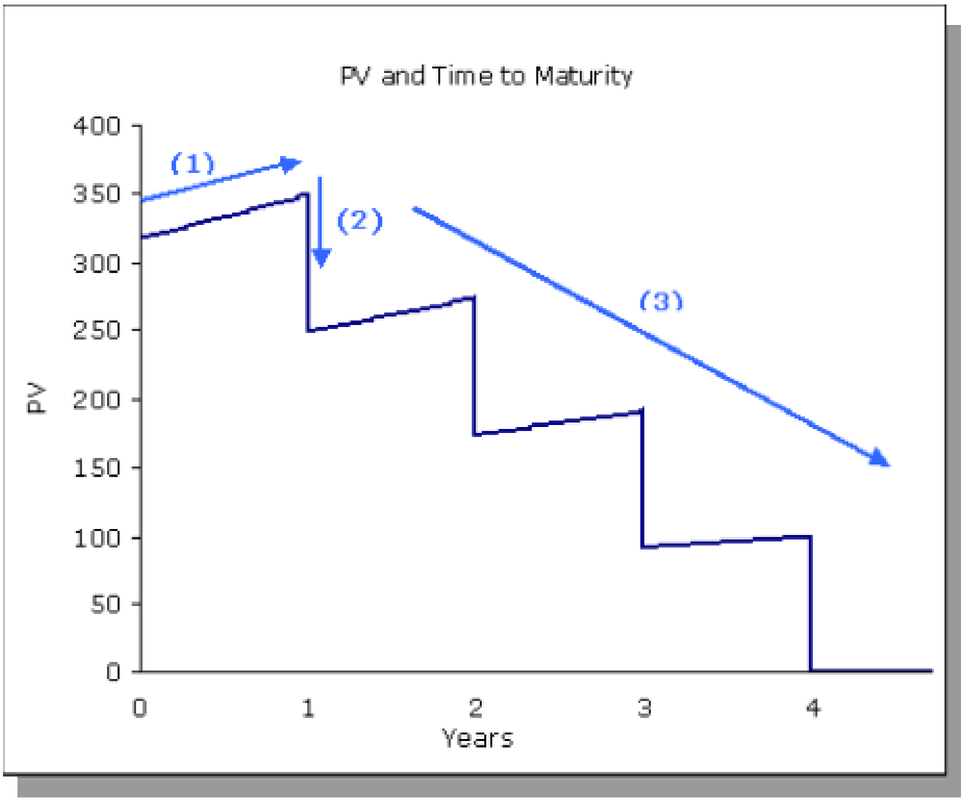
[9] See Measuring Interest Rate Risk – Convexity for a discussion of the convexity behaviour of bonds and Interest Rate Swaps – Market Risk for a discussion of the convexity behaviour of swaps).

Other things being equal, the longer the maturity of the future cash flows the greater is the convexity of the fixed income instrument.

8.2. Sensitivity to Time

Even when the discount rate does not change, the PV of any stream of cash flows (regular or irregular) will change over time.

Example
The figure below shows the evolution of an annuity paying $100 each year for 4 years. At a discount rate of 10%, we can calculate the PV of this annuity at $316.95 using the formula introduced in section *Present Value*. The figure below shows how, other things being equal, the PV of this annuity changes over time.



- Initially, as time approaches the date of the first cash flow the PV of this annuity stream increases, as indicated by arrow (1). This is because as the future cash flow stream approaches us, it must be worth more[10].

[10] How do we PV the annuity part-way into the current compounding period? The way this is typically done is by pro-rating the powers in the PV formula, as shown below:

$$PV = C \times (1 + R/t)^{-a} + C \times (1 + R/t)^{-(a+1)} + \ldots + C \times (1 + R/t)^{-[a+(t-1).T]}$$

Where:
a = Days to next cash flow / Days in current compounding period
D_i = i^{th} discount factor = $1/(1 + R/t)^i$
C = Regular cash flow
R = Discount rate (the required return on the annuity)
T = Number of payments per year (= compounding period)
T = Number of years

- But then at the first anniversary the first \$100 cash flow is paid and from now on this will be excluded from future PV calculations. As a result we have a sudden and sharp drop in the PV, as indicated by arrow (2)[11].

- As more and more payments are made, there are less remaining cash flows and so the PV of the remaining annuity continues to fall, as indicated by arrow (3).

Not only does the PV of the annuity converge to zero but also its PV sensitivity to changes in the discount rate and its price convexity will decline, as explained earlier.

9. Internal Rate of Return

We can also use the formulas linking PV with FV developed earlier in a slightly different way, to derive the implied return on an investment or a loan when we know its PV and its FVs.

The internal rate of return (IRR) is:

- The rate of interest implied in a given PV and stream of FVs
- The discount rate that makes the PV of an investment or loan equal to its cash market price

For example, in the case of a stream of irregular cash flows which are all one year apart, calculating the IRR involves finding the rate R which makes the right hand side of the equation below (the PV of the instrument's future cash flows) equal to its left hand side (its market price).

$$P = C_1 \times (1 + R/t)^{-1} + C_2 \times (1 + R/t)^{-2} + \ldots + C_{t.T} \times (1 + R/t)^{-t.T}$$

Where:
P = Cash market price of investment or loan (given)
C_i = Cash flow at period i (which could be positive or negative)
R = IRR (the implied return)
t = Number of payments per year (= compounding period)
T = Number of years

(PV – P) is the net present value of the financial instrument, so IRR is also the discount rate that makes the NPV of an investment equal to zero.

Since we now know P and each C_i, the only unknown in this equation is R. The calculation is difficult to perform because the formula cannot be re-arranged into an expression for R. So R has to be found by trial-and-error (or **iteratively**) by computing the right hand side of this formula using different discount rates until we find the one that makes it equal to the left hand side (i.e. the given cash price). Fortunately, financial calculators can do this in an instant!

The only case where there is a formula for IRR is when the instrument generates only one future cash flow. In that case:

$$P = C_{t.T} \times (1 + R/t)^{-t.T}$$

Therefore:

$$\mathbf{R = [(C_{t.T} / P)^{1/(t.T)} - 1] \times t}$$

Please note that the IRR found when solving the equations introduced here is a **notional rate**. To find the **annual effective** IRR, we must apply the equation introduced in section *Simple and Compound Interest*, above.

[11] In section Bond Pricing – Clean and Dirty Prices we will see that there is a convenient way to smooth the PV of the cash flows over time, enabling us to isolate the change in value due to the passage of time and/or in the discount rate, from the change in value simply due to the payment of a cash flow.

9.1. Advantages and Limitations

In most cases, whenever fixed income market investors talk about the yield on a bond or money market security, they mean its IRR[12].

IRR is used extensively in the fixed income markets as a **capital budgeting** (CB) decision rule for screening investments. Other things being equal:

- Instruments that yield the highest IRR are preferable to those that yield less
- Any instrument whose yield is higher than the cost of funding it has a positive NPV and therefore is worth investing in

Advantages

Reasons for its popularity are all linked to its property of being a percentage rate rather than an absolute dollar amount. So IRR can be useful:

- In comparing the return available from different types of investment, generating different types of cash flow (e.g. comparing bonds or equities vs. bonds)
- Making return comparisons over different periods of time and for different sized investments

Limitations

1. IRR does not keep track of the sign or the volatility of the future cash flows: you could have two financial instruments with very different cash flow streams; their IRR may be the same but their liquidity implications may be very different
2. In the case mutually exclusive non-scaleable investments (i.e. you cannot invest in more of the same) the IRR rule may reject the investment with lower IRR but higher NPV in dollar terms.
3. In some cases the IRR formula has no solution, for example:
   - When all the cash flows, including the cash price, are either all positive or all negative (in which case the NPV can never be zero)
   - In the case of complex cash flows
4. In other cases the IRR formula may have more than one solution. In particular, there can be as many IRR solutions as there are changes in the sign of the future cash flows (for example -10,+20,-5 involves two sign changes, so there are two possible IRRs in this case)
5. IRR implicitly assumes that all the cash flows generated are reinvested at the same IRR, which of course may not be true in reality[13]

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[12] See module Bond Yield – Yield to Maturity and also Money Market Instruments – Pricing CDs.

[13] This becomes clear if we multiply both the left and the right hand side of the equation by the same compounding factor:

$$P \times (1 + R/t)^{t.T} = C_1 \times (1 + R/t)^{t.T-1} + C_2 \times (1 + R/t)^{t.T-2} + \ldots + C_{t.T}$$

Now the equation is saying that the IRR is the rate of interest which, when you reinvest all the future cash flows at that rate, gives you a future value equal to investing the cash sum (P) at the same rate.

10. Exercise 3

10.1. Question 1

Question 8

In the table below, express the bond prices in percentages of face value, rounded to 2 decimal places.

a) What is the PV of a 2-year zero coupon bond to yield 10%, when the compounding period is:

Compounding period	PV
1. Annual	
2. Semi-annual	

b) Price of bond 2 above is lower than that of bond 1 because:

- Bond 1 earns a higher effective yield
- Bond 2 is in higher demand
- Bond 1 earns more coupon income
- Bond 2 earns a higher effective yield

10.2. Question 2

Question 9

What is the PV of a 3 year annuity paying USD 100 annually when discounted at 10%?

a) Enter your answer to 2 decimal places.

USD

10.3. Question 3

Question 10

In this question we compare the PV of a long-dated annuity with the PV of a perpetuity for the same annual cash flow.

a) What is the PV of a 99-year rental lease with an annual payment of $1,000, when discounted at 10%? Round your answers to 2 decimal places.

USD

b) What is the present value of a straight perpetuity paying $1,000 annually, when discounted at 10%?

USD

10.4. Question 4

Question 11

A bank lends $1,400 to a client and ask the client to repay the entire debt, including interest, over 2 years in equal instalments of $375 every 6 months. What is the internal rate of return on this loan? Enter your answer in percentage per annum, rounded to the nearest 3 decimal places.

a) Nominal IRR

Effective IRR