

## Pricing Derivatives on Financial Securities Subject to Credit Risk

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### ABSTRACT

This article provides a new methodology for pricing and hedging derivative securities involving credit risk. Two types of credit risks are considered. The first is where the asset underlying the derivative security may default. The second is where the writer of the derivative security may default. We apply the foreign currency analogy of Jarrow and Turnbull (1991) to decompose the dollar payoff from a risky security into a certain payoff and a "spot exchange rate." Arbitrage-free valuation techniques are then employed. This methodology can be applied to corporate debt and over the counter derivatives, such as swaps and caps.

THE PURPOSE OF THIS article is to provide a new theory for pricing and hedging derivative securities involving credit risk. Two sources of credit risk are identified and analyzed. The first is where the asset underlying the derivative security may default, paying off less than promised. This is the case, for example, with imbedded options on corporate debt. The second is the credit risk introduced by the writer of the derivative security, who may also default. Examples include over-the-counter writers of options on Eurodollar futures, swaps, and swaptions.

For pricing derivative securities involving credit risk, there are currently two approaches. The first views these derivatives as contingent claims not on the financial securities themselves, but as "compound options" on the assets underlying the financial securities. This is the case, for example, with the pricing of imbedded options on corporate debt (see Merton (1974, 1977), Black and Cox (1976), Ho and Singer (1982), Chance (1990), and Kim, Ramaswamy, and Sundaresan (1993)) or the pricing of vulnerable options (see Johnson and Stulz (1987)). In practice, however, this valuation methodology is difficult to use. First, the assets underlying the financial security are often not tradeable and therefore their values are not observable. This makes application of the theory and estimation of the relevant parameters problematic. Second, as in the case of corporate debt, all of the other liabilities of the firm senior to the corporate debt must first (and simultaneously) be valued. This generates significant computational difficulties. As a result, this approach has not

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proven very effective in practice for pricing corporate liabilities (see Jones, Mason, and Rosenfeld (1984)). Unfortunately, these same complications arise when this technology is applied to pricing swaps (see Cooper and Mello (1990, 1991)). It has shown more promise, however, for valuing commercial mortgages where these problems are present to a lesser extent (see Titman and Torous (1989)).

As a pragmatic alternative, the second approach to the pricing of derivative securities involving credit risk is to ignore the credit risk and price the imbedded options as default-free interest rate options (see Ho and Singer (1984), and Ramaswamy and Sundaresan (1986)). This, however, is inconsistent with the absence of arbitrage and the existence of spreads between the yields on corporate debt and Treasuries.

The purpose of this article is to present a third approach for pricing derivative securities involving credit risk. This approach uses the foreign currency analogy of Jarrow and Turnbull (1991) which takes as given a stochastic term structure of default-free interest rates *and* a stochastic maturity specific credit-risk spread.<sup>1</sup> Given these two term structures, option type features can then be priced in an arbitrage free manner using the martingale measure technology.<sup>2</sup>

Three alternative approaches have recently and independently been developed for pricing derivative securities involving credit risk. These are: (i) Hull and White (1991), (ii) Litterman and Iben (1991), and (iii) Longstaff and Schwartz (1992), and Nielsen, Saá-Requejo and Santa-Clara (1993). Hull and White only study the pricing of options whose writer may default, called vulnerable options.<sup>3</sup> They do not price options on assets with credit risks nor do they analyze the hedging of vulnerable options. Litterman and Iben (1991) is a limiting case of the discrete-time model presented below where there is zero payoff in default. Litterman and Iben do not study vulnerable options. The third approach is a modification of the compound options approach previously discussed. In this approach, capital structure is assumed to be irrelevant. Bankruptcy can occur at any time and is modeled by assuming that when the identical but unlevered firm's value hits some exogenous

<sup>1</sup> This is similar in spirit to the stated purpose of Ramaswamy and Sundaresan (1986; Section 4). Unfortunately, their valuation equation (11) combined with their terminal condition (p. 269) implies that their debt is *default free*. The imposition of the local expectation hypothesis (9), p. 268 (versus (5) p. 260) uniquely identifies their premium  $p(t)$  as a stochastic market price for risk. Thus, they are still valuing riskless debt, albeit with an equilibrium model in which the market price of risk is prespecified by their expression (10).

<sup>2</sup> The existence of the credit spread is taken as exogenous. Anderson and Sundaresan (1994) allow for strategic interaction between the debtholders and equityholders, and value the claims of the firm in a general dynamic setting, so that the credit spread is endogenous. They do not consider counterparty risk.

<sup>3</sup> Hull and White (1991) obtain a similar result to our expression (26); however, there is a technical problem with the Hull-White article. Their model is in continuous time where the event of default causes a discontinuity in the option's value. The existence and uniqueness of the equivalent probability measure is simply assumed, without comment about the significance of such discontinuities.

boundary, default occurs in the levered firm and the firm's debt pays off a fixed fractional amount. The valuation of options, which would require the explicit determination of the exogenous boundary, is not addressed.

An outline of this article follows. Section I provides the notation and formalizes the foreign currency analogy. Section II provides the economic intuition underlying our methodology via a discrete-time model in a two-period economy. This setup allows us to demonstrate how to price and hedge derivatives written on credit risky assets and vulnerable derivatives using standard and non-technical arguments. This model readily generalizes to an arbitrary number of trading dates. A simple continuous-time model is presented in Section III. The analysis here replicates the results from the discrete-time setting. Necessary and sufficient conditions are provided for the existence and uniqueness of the equivalent martingale measure. By imposing various restrictions on the processes for the different term structures, we obtain a number of closed-form results. This section includes Johnson and Stulz (1987) as a special case. We also price equity options when there is a positive probability of default, generalizing Merton's (1976) result. Additional generalizations and extensions are discussed. A summary is given in Section IV.

## I. The Economy

We consider a frictionless economy with a trading horizon  $[0, \tau]$ . The set of trading dates can be either discrete or continuous. If discrete, the set of trading dates is  $\{0, 1, 2, \dots, \tau\}$ . If continuous, the set of trading dates is the entire time interval  $[0, \tau]$ . Two classes of zero-coupon bonds trade.

The first class is *default-free*, zero-coupon bonds of all maturities. Let  $p_0(t, T)$  be the time  $t$  dollar value of the default-free zero-coupon bond paying a certain dollar at time  $T \geq t$ . We assume that the zero-coupon bond prices are strictly positive, i.e.,  $p_0(t, T) > 0$ , and default-free, i.e.,  $p_0(t, t) \equiv 1$ .

A money market account can be constructed from this term structure by investing a dollar in the shortest maturity default-free zero-coupon bond, and rolling it over at each future date. Let  $B(t)$  denote the time  $t$  value of this money market account initialized with a dollar at time 0.

The second class is XYZ zero-coupon bonds of all maturities. These XYZ zero-coupon bonds are risky and subject to default.<sup>4</sup> Let  $v_1(t, T)$  denote the time  $t$  value of the XYZ zero-coupon bond *promising* a dollar at date  $T \geq t$ . We assume that the XYZ zeros have strictly positive prices, i.e.,  $v_1(t, T) > 0$ . This restriction is imposed for analytic convenience (to avoid dividing by zeros) and is easily relaxed.

Next, to aid our intuition, to help in modeling, and to facilitate estimation, we decompose these XYZ zero-coupon bonds into the product of two hypothet-

<sup>4</sup> Strictly speaking, the entire term structure of zero-coupon XYZ debt need not trade directly. Enough XYZ coupon-bearing debt must trade, however, so that the prices of all the zeros can be recovered.

ical quantities: a zero-coupon bond denominated in a hypothetical currency, a promised XYZ dollar called an XYZ, and a price in dollars of XYZs.

First, we define

$$e_1(t) \equiv v_1(t, t). \quad (1)$$

The quantity  $e_1(t)$  represents the time  $t$  dollar value of one promised XYZ dollar delivered immediately (at time  $t$ ) and is analogous to a spot exchange rate. If XYZ is not in default, the exchange rate will be unity as each promised XYZ dollar is actually worth a dollar. However, if XYZ is in default, then each promised XYZ dollar may be worth less than a dollar. The exact specification of this exchange rate process is a crucial step in our model and it is given in subsequent sections. Note that the spot exchange rate can alternatively be interpreted as a payoff ratio in default.

Next, we construct a hypothetical, XYZ paying zero-coupon bond. Define

$$p_1(t, T) \equiv v_1(t, T)/e_1(t). \quad (2)$$

This quantity is the time  $t$  value in units of XYZs, of one XYZ delivered at time  $T$ . Note that by expression (1), these zero-coupon bonds are default-free in XYZs, i.e.,

$$p_1(T, T) \equiv 1. \quad (3)$$

Rearranging expression (2) gives a tautological decomposition of the XYZ zero-coupon bond

$$v_1(t, T) = p_1(t, T)e_1(t). \quad (4)$$

This decomposition is useful for modeling purposes as we can use it to separately characterize the term structure of XYZs in terms of  $p_1(t, T)$  and the payoff ratio  $e_1(t)$ . It also aids our intuition by revealing the *foreign currency analogy*, i.e., the dollar value of an XYZ bond is the XYZ value of the bond times the spot exchange rate of dollars per XYZ. The foreign currency analogy is useful because foreign currency option pricing techniques are well-understood (see Amin and Jarrow (1991)), and these same techniques can now be applied in a modified form to price derivatives involving credit risk.

## II. The Two-Period Discrete Trading Economy

To illustrate the foreign currency analogy as applied to credit risky options, we first study a two-period economy. The discrete-time example selected for analysis is analytically simple, yet realistic enough that its multiperiod generalization should prove useful in actual practice. This example is generalized to the continuous-time setting in a subsequent section.

### A. The Setup

There are two time periods with trading dates  $t \in \{0, 1, 2\}$ . We first describe the term structure for the default-free zero-coupon bonds and then the term structure for the XYZ zero-coupon bonds.

#### A.1. The Default-Free Term Structure

The bond price process for default-free debt is assumed to depend only on the spot interest rate. The stochastic evolution of the default-free spot interest rate is shown in Figure 1. The current ( $t = 0$ ) one period spot rate of interest is defined by

$$r(0) = 1/p_0(0, 1). \quad (5a)$$

In the “up-state,” the one period spot interest rate is

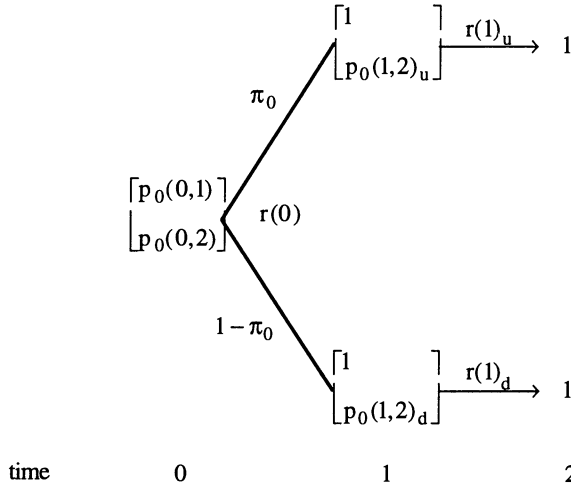
$$r(1)_u \equiv 1/p_0(1, 2)_u, \quad (5b)$$

and in the “down-state”

$$r(1)_d \equiv 1/p_0(1, 2)_d. \quad (5c)$$

The pseudo- or risk-neutral probability of state  $u$  occurring is denoted by  $\pi_0$ . Without loss of generality we assume that  $p_0(1, 2)_u < p_0(1, 2)_d$ .

The money market account's values are given by  $B(0) \equiv 1$ ,  $B(1) \equiv r(0)$ ,  $B(2)_u \equiv r(0)r(1)_u$  and  $B(2)_d \equiv r(0)r(1)_d$ . Note that  $B(t + 1)$  is known at time  $t$ .



**Figure 1. The default-free zero-coupon bond price process for the two-period economy.** This binomial tree describes the evolution of the spot interest rate process and the zero-coupon bond prices over the time periods 0, 1, and 2 where  $p_0(t, T)_\omega$  is the time  $t$  price of a zero-coupon bond paying a sure dollar at time  $T$  given state  $\omega \in \{u, d\}$ ,  $r(t)_\omega$  is the spot interest rate at time  $t$  given state  $\omega \in \{u, d\}$ , and  $\pi_0$  is the pseudoprobability.

## A.2. The XYZ Term Structure

We consider zero-coupon bonds belonging to a particular risk class, which we refer to as XYZ. For the one-period zero-coupon bond, two states are possible at maturity. If default has not occurred, the payoff is the face value of the bond. If default has occurred, the payoff is less than the face value of the bond.

Modeling the actual payoff in default is a complex problem. First, the absolute priority rule is often violated.<sup>5</sup> Second, numerous other factors affect the payoff, such as the relative bargaining power of the different stake holders, and the percentage of managerial ownership.<sup>6</sup> Consequently, as a first approximation, we take the payoff to the bond holder in the event of default as an exogenously given constant. This payoff per unit of face value is denoted by  $\delta$ , and it is assumed to be the same for all instruments in a given credit risk class.<sup>7</sup>

In terms of the foreign currency analogy, the spot exchange rate at time 0 is unity,  $e(0) = 1$ , and at times 1 and 2 the spot exchange rate  $e(t)$  takes on the two values shown in Figure 2. This discrete-time binomial process was selected to approximate a continuous-time Poisson bankruptcy process. At time 1, with pseudoprobability  $\lambda\mu_0$ , default occurs. If in default at time 1, XYZ remains in default at time 2 and the payoff ratio remains fixed at  $\delta$  per unit of face value. At time 1, with pseudoprobability  $(1 - \lambda\mu_0)$ , default does not occur. Conditional upon this state, the pseudoprobability that default occurs at  $t = 2$  is denoted by  $\lambda\mu_1$ .

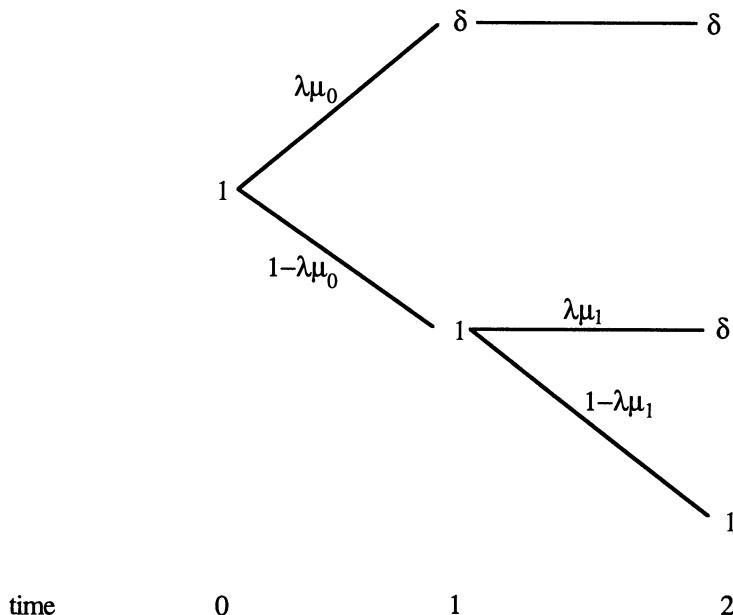
The stochastic evolution of the XYZ zero-coupon bonds in the hypothetical XYZ currency is depicted in Figure 3. This figure is similar to Figure 1 for the default-free bonds except that there are now more states. The states correspond to all possible combinations of spot interest rate movements and bankruptcy. Figure 4 depicts the stochastic evolution of the XYZ zero-coupon bonds in dollars. This process is the multiplicative product of the processes in Figures 2 and 3.

It is assumed that *the spot interest rate process in Figure 1 and the bankruptcy process in Figure 2 are independent under the pseudoprobabilities*. This is reflected by the fact that the pseudoprobabilities shown in Figure 4 are the product of the separate pseudoprobabilities in Figures 1 and 2. If the market prices of risk are nonrandom in the economy, then this assumption is equivalent to independence under the true (empirical) process. This assumption is imposed because it simplifies the analysis and facilitates the deriva-

<sup>5</sup> Eberhart, Moore, and Roenfeldt (1990) report that in a sample of 24 firms, the absolute priority rule was violated in 23 cases. Weiss (1990) examines 37 firms and reports violations of the absolute priority rule in 27 cases.

<sup>6</sup> These issues are discussed in Weiss (1990) and Schwartz (1993).

<sup>7</sup> Bankruptcy often involves the acceleration of claims in that all debt becomes immediately payable. This acceleration of debt can be partially incorporated into our model by recognizing that different risk classes of debt within the same firm can have different recovery rates at different times ( $\delta_t$ 's).  $\delta_t$  can be greater for those classes of debt that would receive larger payoffs at different times due to acceleration.



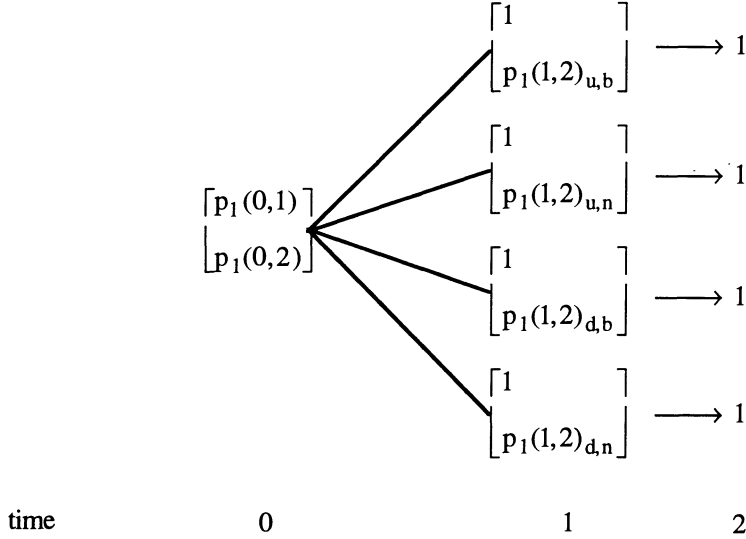
**Figure 2. The payoff ratio process for XYZ debt in the two-period economy.** This tree describes the evolution of the payoff ratio for XYZ debt over the time periods 0, 1, and 2 where  $\lambda\mu_t$  represents the pseudoprobability at date  $t$ , and  $\delta$  represents the payoff per promised dollar in default.

tion of numerous results. Its usefulness in practice still needs to be evaluated via empirical investigation.

### B. Arbitrage-Free Restrictions

In a discrete-time, discrete-state space economy (as analyzed above), Harrison and Pliska (1981) show that the nonexistence of *arbitrage opportunities* is equivalent to the existence of pseudoprobabilities  $\pi_0$ ,  $\lambda\mu_0$ ,  $\lambda\mu_1$  such that  $p_0(t, 1)/B(t)$ ,  $p_0(t, 2)/B(t)$ ,  $v_1(t, 1)/B(t)$ , and  $v_1(t, 2)/B(t)$  are martingales; and that market *completeness* is equivalent to uniqueness of these pseudoprobabilities. Relative prices being martingales implies that trading in these securities is a fair-game, i.e., expected values equal current values. Completeness implies that any contingent claim written against these securities can be constructed synthetically via trading in the primary securities. This section characterizes the necessary and sufficient conditions for the existence of these unique pseudoprobabilities. Thus, it characterizes the necessary and sufficient conditions for arbitrage-free prices and complete markets.

We can obtain these conditions via an investigation of each separate market, the default-free bond market and the risky debt market. The pseudoprobability  $\pi_0$  is determined in the default-free bond market. From Figure 1



**Figure 3. The XYZ zero-coupon bond price process for the two-period economy in XYZs.** This tree describes the evolution of the XYZ zero-coupon bond prices in XYZs over the time periods 0, 1, and 2 where  $p_1(t, T)_\omega$  represents the time  $t$  price in units of XYZs of one XYZ paid at time  $T$  given state  $\omega \in \{ub, un, db, dn\}$ .

we get

$$p_0(0, 2) = [\pi_0 p_0(1, 2)_u + (1 - \pi_0) p_0(1, 2)_d] / r(0). \quad (6)$$

This condition states that the time 0 long-term zero-coupon bond price is its time 1 discounted expected value, using the pseudoprobabilities. Using expression (6),  $\pi_0$  is given by

$$\pi_0 = [p_0(1, 2)_d - r(0) p_0(0, 2)] / [p_0(1, 2)_d - p_0(1, 2)_u]. \quad (7)$$

Thus,  $\pi_0$  exists, is unique, and satisfies  $0 < \pi_0 < 1$  if and only if

$$p_0(1, 2)_u < r(0) p_0(0, 2) < p_0(1, 2)_d. \quad (8)$$

These are the standard conditions. They state that the long-term zero-coupon bond should not be dominated by the short-term zero-coupon bond. It earns more return in one state ( $d$ ), and less in the other ( $u$ ).

As the time 0 prices for risky debt depend on their time 1 prices, we must first analyze the time 1 risky debt market. This market determines the time 1 pseudoprobability ( $\lambda\mu_1$ ). From Figure 4 we get

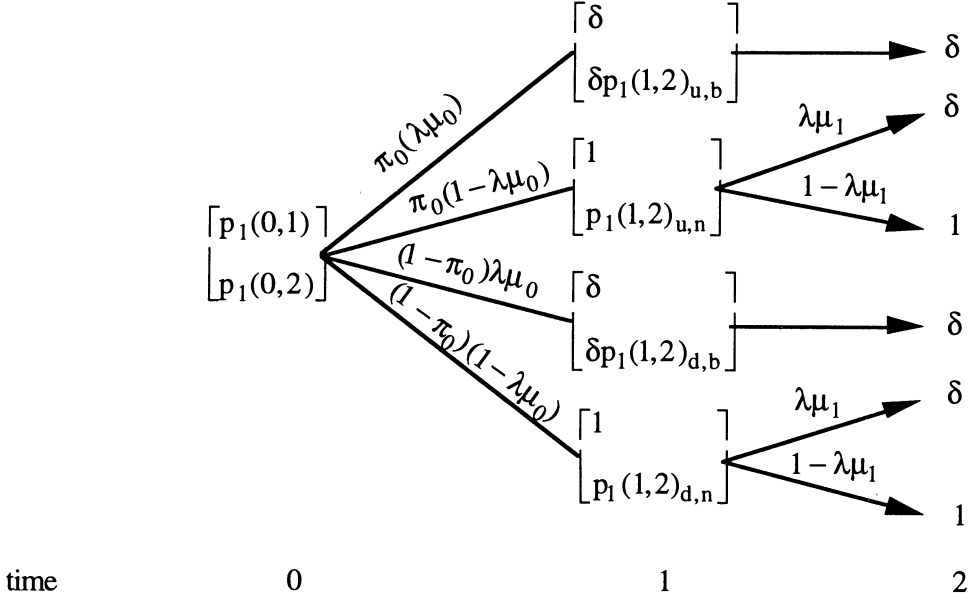
$$v_1(1, 2)_{u,b} = \delta p_1(1, 2)_{u,b} = \delta / r(1)_u \quad (9a)$$

$$v_1(1, 2)_{u,n} = p_1(1, 2)_{u,n} = [\lambda\mu_1 \delta + (1 - \lambda\mu_1)] / r(1)_u \quad (9b)$$

$$v_1(1, 2)_{d,b} = \delta p_1(1, 2)_{d,b} = \delta / r(1)_d \quad (9c)$$

$$v_1(1, 2)_{d,n} = p_1(1, 2)_{d,n} = [\lambda\mu_1 \delta + (1 - \lambda\mu_1)] / r(1)_d. \quad (9d)$$





**Figure 4. The XYZ zero-coupon bond price process for the two-period economy in dollars.** This tree describes the evolution of the XYZ zero-coupon bond prices in dollars over the time periods 0, 1, and 2. This tree is the multiplicative product of the trees in Figures 3 and 4.  $\pi_0$  is the pseudoprobability for the spot interest rate process,  $\lambda\mu_0$  and  $\lambda\mu_1$  are the default process pseudoprobabilities,  $\delta$  is the payoff ratio in default, and  $p_1(t, T)_\omega$  is the time  $t$  price in units of XYZs of one XYZ paid at time  $T$  given state  $\omega \in \{ub, un, db, dn\}$  where  $0 \leq t < T \leq 2$ .

Here again, the time 1 long-term XYZ bond price is its time 2 discounted expected value, using the pseudoprobabilities.

Using expression (9),  $\lambda\mu_1$  is given by

$$\lambda\mu_1 = [1 - p_1(1, 2)_{u,n}r(1)_u]/[1 - \delta] = [1 - p_1(1, 2)_{d,n}r(1)_d]/[1 - \delta]. \quad (10)$$

Thus,  $\lambda\mu_1$  exists, is unique, and satisfies  $0 < \lambda\mu_1 < 1$  if and only if

$$p_1(1, 2)_{u,b} = 1/r(1)_u \quad (11a)$$

$$p_1(1, 2)_{d,b} = 1/r(1)_d \quad (11b)$$

$$\delta/r(1)_u < p_1(1, 2)_{u,n} < 1/r(1)_u \quad (11c)$$

$$\delta/r(1)_d < p_1(1, 2)_{d,n} < 1/r(1)_d \quad (11d)$$

$$r(1)_u p_1(1, 2)_{u,n} = r(1)_d p_1(1, 2)_{d,n}. \quad (11e)$$

Conditions (11a) and (11b) show that in the state of bankruptcy, the default-free bonds in units of dollars and the XYZ denominated XYZ bonds are equal. This is because there is no remaining bankruptcy risk, and the only uncertainty left is due to default-free spot interest rates.

Conditions (11c) and (11d) state that given no bankruptcy at time 1, the dollar value for the XYZ zero bond must be less than the dollar value of a claim paying one dollar for sure and greater than a claim paying  $\delta$  dollars for sure.

Condition (11e) guarantees the independence of the pseudoprobability ( $\lambda\mu_1$ ) from the spot interest rate process. It adds additional structure to the model, and, therefore, restricts the possible term structures ( $p_0(t, T)$ ,  $v_1(t, T)$ ) that will be consistent with this specification of the model. This restriction is testable, and if rejected, it could be easily removed.

Finally, the time 0 pseudoprobability ( $\lambda\mu_0$ ) is determined in the time 0 risky debt market. From Figure 4 we get

$$v_1(0, 1) = p_1(0, 1) = [\lambda\mu_0\delta + (1 - \lambda\mu_0)]/r(0) \quad (12a)$$

$$\begin{aligned} v_1(0, 2) = p_1(0, 2) = & [\pi_0(\lambda\mu_0)\delta p_1(1, 2)_{u,b} + \pi_0(1 - \lambda\mu_0)p_1(1, 2)_{u,n} \\ & + (1 - \pi_0)\lambda\mu_0\delta p_1(1, 2)_{d,b} \\ & + (1 - \pi_0)(1 - \lambda\mu_0)p_1(1, 2)_{d,n}]/r(0) \end{aligned} \quad (12b)$$

These conditions guarantee that time 0 prices are their time 1 discounted expected values, using pseudoprobabilities. Substituting conditions (7) and (10) into (12b) and simplifying yields

$$v_1(0, 2) = p_1(0, 2) = p_0(0, 2)[\lambda\mu_0\delta + (1 - \lambda\mu_0)r(1)_d p_1(1, 2)_{d,n}]. \quad (13)$$

Using expressions (12) and (13),  $\lambda\mu_0$  is given by

$$\begin{aligned} \lambda\mu_0 = & [1 - r(0)p_1(0, 1)]/[1 - \delta] \\ = & [r(1)_d p_1(1, 2)_{d,n} - p_1(0, 2)/p_0(0, 2)]/[r(1)_d p_1(1, 2)_{d,n} - \delta] \end{aligned} \quad (14)$$

Thus,  $\lambda\mu_0$  exists, is unique, and satisfies  $0 < \lambda\mu_0 < 1$  if and only if

$$\delta/r(0) < p_1(0, 1) < 1/r(0) \quad (15a)$$

$$\delta p_0(0, 2) < p_1(0, 2) < p_0(0, 2)r(1)_d p_1(1, 2)_{d,n} \quad (15b)$$

$$\frac{[r(1)_d p_1(1, 2)_{d,n} - p_1(0, 2)/p_0(0, 2)]}{[r(1)_d p_1(1, 2)_{d,n} - \delta]} = \frac{[1 - r(0)p_1(0, 1)]}{[1 - \delta]}. \quad (15c)$$

Condition (15a) states that the dollar value of the XYZ zero-coupon bond maturing at time 1 must be worth less than receiving a dollar for sure and greater than receiving  $\delta$  dollars for sure. Condition (15b) states that the XYZ zero coupon bond maturing at time 2 must be worth more than receiving  $\delta$  dollars for sure and less than receiving  $r(1)_d p_1(1, 2)_{d,n}$  dollars for sure at time 2. Finally, condition (15c) guarantees that under the pseudoprobabilities, the bankruptcy process is independent of the default-free spot interest rate process. This restriction is imposed for analytic convenience and is easily relaxed.

For the remainder of this section, we assume conditions (8), (11), and (15) are satisfied. As argued above, this is equivalent to assuming that there are no arbitrage opportunities in this economy and that the market is complete.

### C. XYZ Zero-Coupon Bonds

Under the above structure, the XYZ zero-coupon bond prices can be rewritten in an equivalent form. First, note that under the pseudoprobabilities the expected payoff ratios at future dates can be calculated. These are

$$\tilde{E}_1(e_1(2)) = \begin{cases} \delta & \text{if bankrupt at time 1} \\ \lambda\mu_1\delta + (1 - \lambda\mu_1) & \text{if not bankrupt at time 1.} \end{cases} \quad (16a)$$

$$\tilde{E}_0(e_1(2)) = \lambda\mu_0\delta + (1 - \lambda\mu_0)[\lambda\mu_1\delta + (1 - \lambda\mu_1)], \text{ and} \quad (16b)$$

$$\tilde{E}_0(e_1(1)) = \lambda\mu_0\delta + (1 - \lambda\mu_0), \quad (16c)$$

where  $\tilde{E}_t(\cdot)$  is the time  $t$  conditional expected value under the pseudoprobabilities.

Expression (16a) states that at time 1, the expected payoff ratio at time 2 is either  $\delta$ , if the firm is bankrupt at time 1, or  $(\lambda\mu_1)\delta + (1 - \lambda\mu_1)$ , if the firm is not bankrupt at time 1. The expected payoff ratio at time 2, as viewed from time 0, is given in expression (16b). It is a weighted average of the payment,  $\delta$ , from going bankrupt at time 1, plus not being bankrupt at time 1 and the expected payoff at time 2,  $[\lambda\mu_1\delta + (1 - \lambda\mu_1)]$ . Expression (16c) provides the expected payoff ratio at time 1 as seen at time 0. Note that all of expressions (16a–c) are strictly less than 1.

Expressions (13), (11), and (9) in conjunction with expression (16) imply

$$v_1(t, T) = p_0(t, T)\tilde{E}_t(e_1(T)). \quad (17)$$

The XYZ zero-coupon bond price is its discounted expected payoff at time  $T$  (using the pseudoprobabilities). The discount factor is the default-free zero-coupon bond price.

This decomposition allows one to implicitly estimate the expected time  $T$  payoff  $\tilde{E}_t(e_1(T))$  given observations of both bond prices ( $v_1(t, T)$  and  $p_0(t, T)$ ). Alternatively, given an estimate of the payoff ratio in default ( $\delta$ ), expression (17) provides us with a recursive estimation procedure for the pseudoprobabilities  $\lambda\mu_0$  and  $\lambda\mu_1$ . This recursive estimation procedure is explained as follows. Given  $p_0(0, 1)$  and  $v_1(0, 1)$ , estimate  $\lambda\mu_0$  by expression (17) and (16c). Second, given these,  $p_2(0, 2)$  and  $v_1(0, 2)$ , estimate  $\lambda\mu_1$  by expressions (17) and (16b). This recursive procedure is easily generalizable to multiple periods.

Expression (17) is useful in clarifying the restrictions upon  $v_1(t, T)$  (and therefore  $p_1(t, T)$ ) imposed by the no arbitrage conditions contained in expressions (9) and (12). As the expected payoff ratio is strictly less than one ( $\tilde{E}_1(e_1(T)) < 1$ ), we see that the XYZ zero-coupon bond is strictly less valuable than a default-free zero-coupon bond of equal maturity (i.e.,  $v_1(t, T) < p_0(t, T)$ ). In other words, in the presence of bankruptcy, a strictly positive credit risk spread is a necessary condition for an arbitrage-free price system. This insight generalizes and obtains even with the relaxation of the statistical independence assumption in the pseudoprobabilities imposed upon  $e_1(t)$

and  $p_0(t, T)$ . This insight is the reason for rejecting the models of Ho and Singer (1984) and Ramaswamy and Sundaresan (1986) as inconsistent (see the introduction).

The statistical independence condition under the pseudoprobabilities ((11e) and (15c)) is the reason that expression (17) admits such a simple decomposition. This decomposition is an additional restriction imposed upon the economy, and as such it restricts the possible term structures for XYZ zero-coupon debt ( $v_1(t, T)$ ). This is significant as it implies that, in special cases, not all of the XYZ maturity debt must trade to apply the model. For example, consider a multiperiod extension of the above model where the pseudodefault probability  $\lambda\mu_t$  is constant over time. In this case, given any two XYZ zero-coupon bonds, one can deduce  $\delta$  and  $\lambda\mu$ . With these parameters, the prices for all the remaining XYZ zero-coupon bonds can be computed. Thus, in this case, only two XYZ traded zero-coupon bonds are needed. This observation is significant in applications where there is a sparsity of XYZ zero-coupon bonds trading. In this situation, coupon-bearing XYZ debt may also be useful. This is discussed in the next section.

#### D. XYZ Coupon Bonds

Consider an XYZ coupon-bearing bond with promised dollar coupons of  $k_1$  at time 1 and  $k_2$  at time 2, where  $k_2$  includes the principal repayment. Let  $D(t)$  represent the time  $t$  dollar value of this XYZ coupon-bond. Using the risk-neutral valuation procedure, we know that the XYZ coupon-bond's price equals its discounted expected payoff (using the pseudoprobabilities), i.e.,

$$D(0) = \tilde{E}_0(k_1 e_1(1)/B(1) + k_2 e_1(2)/B(2)). \quad (18)$$

Simple algebra, along with expression (17), yields

$$D(0) = k_1 v_1(0, 1) + k_2 v_1(0, 2). \quad (19)$$

The coupon-bearing bond is equivalent to a portfolio consisting of  $k_1$  zero-coupon bonds of maturity 1 and  $k_2$  zero-coupon bonds of maturity 2. This is analogous to an identical result which obtains for default-free coupon bonds. This result is valid even without the statistical independence assumptions on the pseudoprobabilities contained in expressions (11e) and (15c).

Expression (19) allows one to deduce the term structure of XYZ zero-coupon debt from XYZ coupon-bearing debt prices, in the same manner that it is currently done for default-free debt. In multiperiod generalizations of this model and in conjunction with additional restrictions upon the pseudo-default probabilities (e.g.,  $\lambda\mu_t$  is constant over time), this insight allows one to deduce the prices of the XYZ zero-coupon bonds ( $v_1(t, T)$ ) from the traded prices of only a few issues of XYZ coupon-bearing debt.

#### E. Options on XYZ Debt

Options written against the XYZ zero-coupon term structure can now be analyzed using the standard procedures. We illustrate this approach with an

example. Let  $C(t)$  be the time  $t$  price of a European call option on the two-period XYZ zero-coupon bond. Let the option's exercise price be  $K$  and its exercise date be time 1. Its value at expiration is

$$C(1) = \max[v_1(1, 2) - K, 0]. \quad (20)$$

The risk-neutral valuation procedure allows us to compute this option's value at time 0 as the discounted expectation of its time 1 payoff (using the pseudoprobabilities), i.e.,

$$\begin{aligned} C(0) = & (1 - \lambda\mu_0)[\pi_0 C(1)_{u,n} + (1 - \pi_0)C(1)_{d,n}]/r(0) \\ & + (\lambda\mu_0)[\pi_0 C(1)_{u,b} + (1 - \pi_0)C(1)_{d,b}]/r(0). \end{aligned} \quad (21)$$

The first term on the right hand side in square brackets is the value of the option given default does not occur. The second term in square brackets is the value of the option given that default has occurred.

Because the tree for XYZ zero-coupon debt has four branches, three traded assets and the money market account are needed to hedge the call option. The hedge consists of  $\alpha$  shares of the two-period XYZ zero-coupon bond,  $\beta$  shares of the one-period XYZ zero-coupon bond,  $\gamma$  shares of the two-period default-free zero-coupon bond, and  $\varepsilon$  shares of the money market account such that

$$\alpha v_1(1, 2)_{u,b} + \beta v_1(1, 1)_{u,b} + \gamma p_0(1, 2)_{u,b} + \varepsilon r(0) = C(1)_{u,b} \quad (22a)$$

$$\alpha v_1(1, 2)_{u,n} + \beta v_1(1, 1)_{u,n} + \gamma p_0(1, 2)_{u,n} + \varepsilon r(0) = C(1)_{u,n} \quad (22b)$$

$$\alpha v_1(1, 2)_{d,b} + \beta v_1(1, 1)_{d,b} + \gamma p_0(1, 2)_{d,b} + \varepsilon r(0) = C(1)_{d,b} \quad (22c)$$

$$\alpha v_1(1, 2)_{d,n} + \beta v_1(1, 1)_{d,n} + \gamma p_0(1, 2)_{d,n} + \varepsilon r(0) = C(1)_{d,n}. \quad (22d)$$

A unique solution can be shown to exist because the market is complete, i.e., conditions (8), (11), and (15) hold. In fact, it can also be shown that the initial cost of constructing this portfolio equals the call's price, i.e.,

$$C(0) = \alpha v_1(0, 2) + \beta v_1(0, 1) + \gamma p_0(0, 2) + \varepsilon. \quad (23)$$

This is a well-known implication of arbitrage-free price systems.

### F. Vulnerable Options

Consider options on the XYZ zero-coupon bonds, but written by a third party, who could also default. These options have been labelled *vulnerable options* by Johnson and Stulz (1987). This section studies the valuation and hedging of vulnerable options.

We assume that this third party has zero-coupon bonds issued against its assets, with prices denoted by  $v_2(t, T)$  for  $t \leq T$ . For simplicity, we suppose these dollar denominated bonds satisfy the same processes as given in Figure 4, but with the "1" subscript replaced by a "2." Expanding the economy appropriately, we assume that there exists unique pseudoprobabilities such that the relative prices of all bonds, i.e.,  $p_0(t, 1)/B(t)$ ,  $p_0(t, 2)/B(t)$ ,

$v_1(t, 1)/B(t)$ ,  $v_1(t, 2)/B(t)$ ,  $v_2(t, 1)/B(t)$ , and  $v_2(t, 2)/B(t)$  are martingales. This is the no arbitrage and complete markets condition. Denote these pseudoprobabilities by the expectations operator  $\tilde{E}(\cdot)$ . Furthermore, we assume that *the bankruptcy process for the payoff ratio,  $e_2(t)$ , is independent of the default-free spot interest rate process and independent of the payoff ratio of XYZ, both under the pseudoprobabilities.*

Consider the European call option valued in expression (21), but this time, let it be written by the third party. At maturity, the cash flow to the buyer represents a promise by firm 2 to make the payment  $C(1)$ , i.e.,

$$e_2(1)C(1). \quad (24)$$

The value to the buyer at time 0,  $C_2(0)$ , is therefore the discounted expected value of this payment (under the pseudoprobabilities).

$$\begin{aligned} C_2(0) &= \tilde{E}_0(e_2(1)C(1)/B(1)) \\ &= \tilde{E}_0(e_2(1))C(0). \end{aligned} \quad (25)$$

The statistical independence of the bankruptcy process for the option writer from both the bankruptcy process for XYZ and the spot interest rate process under the pseudoprobabilities implies the second equality. The price of a call option written by a risky firm is equal to the price of a call written by a default-free writer discounted by the expected payoff from the risky writer. Given expression (17), we can rewrite this as

$$C_2(0) = \frac{v_2(0, 1)}{p_0(0, 1)} C(0). \quad (26)$$

As  $v_2(0, 1)/p_0(0, 1) < 1$ , we see that a vulnerable option is always less valuable than a non-vulnerable option (i.e.,  $C_2(0) < C(0)$ ).

### G. A Numerical Example

In Table I we are given two sets of prices of zero-coupon bonds. The first column is for default-free bonds and the second column is for XYZ zero-coupon bonds. These prices are taken as exogenous. For this credit class, if default occurs the bond holder receives  $\delta = 0.32$  dollars per promised dollar. First we need to determine the pseudoprobabilities of default.

For the one period bond using equations (17) and (16c) we have

$$v_1(0, 1) = p_0(0, 1)[\lambda\mu_0\delta + (1 - \lambda\mu_0)]. \quad (27)$$

Using the prices in Table 1, this implies that  $\lambda\mu_0 = 0.01$ . For the two period bond, using equations (17) and (16.b), we have that

$$v_1(0, 2) = p_0(0, 2)\{\lambda\mu_0\delta + (1 - \lambda\mu_0)[\lambda\mu_1\delta + (1 - \lambda\mu_1)]\}, \quad (28)$$

implying  $\lambda\mu_1 = 0.03$ .

The process for the default-free spot interest rate is shown in Figure 5, Panel A. This process is consistent with Black, Derman, and Toy (1990).

**Table I**  
**The Initial Term Structures of Default-Free Debt and XYZ Debt**

This table gives the initial prices, in dollars, for default-free and XYZ zeros with maturities 1 and 2. The symbol  $T$  represents the maturities of the zeros,  $p_0(0, T)$  represents the time 0 price of the default-free zero with maturity date  $T$ , and  $v_1(0, T)$  represents the time 0 price of the XYZ zero with maturity  $T$ .

Maturity	Prices of Default-Free Zero Coupon Bonds	Prices of XYZ Zero Coupon Bonds
$T$	$p_0(0, T)$	$v_1(0, T)$
1	94.8627	94.2176
2	89.5343	87.1168

*G.1. Option on a Credit Risky Bond*

Consider a European put option with maturity one year on an XYZ zero-coupon bond with maturity date time 2. At the maturity of the option, the option holder can sell the XYZ zero-coupon bond for the strike price of 92. Let the face value of the XYZ zero-coupon bond be 100. The option's value at maturity is shown in Figure 5, Panel B. Using equation (21), the time 0 value of this option is

$$C(0) = (1 - 0.01)[0.5(0.07) + 0.5(0)]0.9486 + 0.01[0.5(61.97) + 0.5(61.62)]0.9486 = 0.62 \tag{29}$$

*G.2. Vulnerable Options*

Consider a European put option with maturity one year and a strike price of 95 on a default-free zero-coupon bond with maturity date time 2. Let the face value of the zero-coupon bond be 100. If there is no risk of the *writer* defaulting, the value of the option at maturity is

$$C(1) = \begin{cases} 95 - 93.84 = 1.16 \\ 95 - 94.93 = 0.07 \end{cases} \tag{30}$$

and the value of the option today is

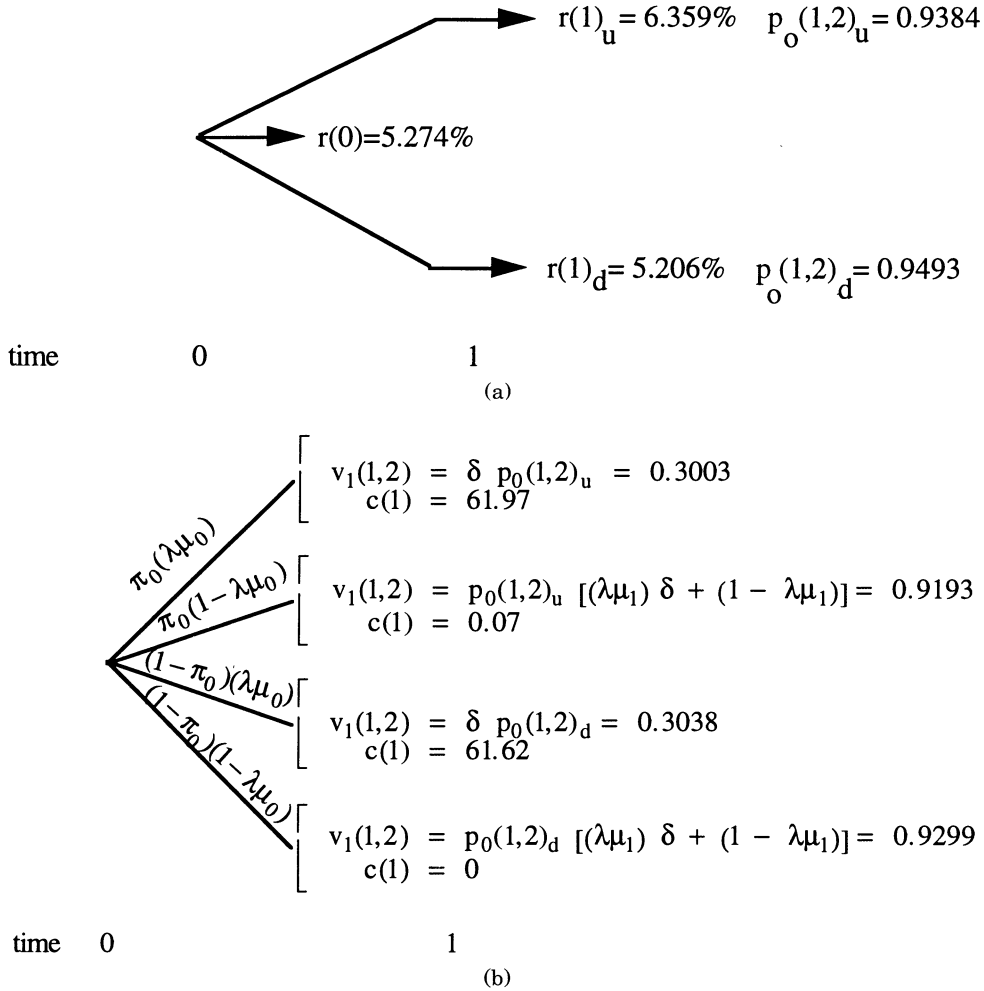
$$C(0) = (0.5(1.16) + 0.5(0.07))0.9486 = 0.5834 \tag{31}$$

Let the institution writing this option belong to the XYZ credit class. Using equation (26), the value of the vulnerable option is

$$C(0) = [94.2176/94.8627]0.5834 = 0.5794 \tag{32}$$

*G.3. Swaps*

Consider an existing off-market interest rate swap with two periods remaining where we are receiving fixed payments of 6 percent from a counter-



**Figure 5. A spot-interest rate for the default-free term structure, and B zero-coupon bond prices and option values.** Panel A describes the evolution of the spot interest rate and bond price process over the time periods 0 and 1 where  $\pi_0 = 0.5$ .  $r(t)_\omega$  is the spot interest rate at time  $t$  given state  $\omega \in \{u, d\}$  and  $p_0(t, T)_\omega$  is the time  $t$  price of a default-free bond paying a dollar at time  $T$  given state  $\omega \in \{u, d\}$ . Panel B describes the evolution of the dollar values of XYZ zero-coupon bonds and the call option's values over the time periods 0 and 1 where  $\pi_0 = 0.5$ ,  $\lambda \mu_0 = 0.01$ ,  $\lambda \mu_1 = 0.03$ , and  $\delta = 0.32$ .  $\pi_0$  is the pseudoprobability for the spot interest rate process,  $\lambda \mu_0$  and  $\lambda \mu_1$  are the default process pseudoprobabilities,  $\delta$  is the payoff ratio in default, and  $p_0(1, 2)_\omega$  is the time 1 value of a default-free dollar paid at time 2 given state  $\omega \in \{u, d\}$ ,  $v_1(1, 2)$  is the time 1 dollar value of an XYZ paid at time  $T$ , and  $C(1)$  is the value of a call option at time 1.

party belonging to the credit class XYZ, and we are making floating rate payments. For simplicity, we assume that there is no chance of default on the floating rate payments. The time 0 value of the first floating rate payment is

$$\text{Float}(0, 1) \equiv 1 - p_0(0, 1), \quad (33a)$$



and the second payment is

$$\text{Float}(0, 2) \equiv p_0(0, 1) - p_0(0, 2). \quad (33b)$$

In a standard swap, the bankruptcy rules are such that if default occurs, all future payments are null and void.<sup>8</sup> We can incorporate this provision into our model by defining a new payoff ratio conditional upon no default at time  $t - 1$ , i.e.

$$\bar{e}(t) = \begin{cases} 1; & \text{probability } 1 - \lambda\mu_{t-1} \\ 0; & \text{probability } \lambda\mu_{t-1} \end{cases} \quad (34a)$$

If default has occurred at  $t - 1$ , then

$$\bar{e}(t) = 0 \text{ with probability } 1, \quad (34b)$$

implying that the swap is null and void. The value of the swap today at  $t = 0$  is

$$\begin{aligned} v_s(0) &\equiv [\bar{R}p_0(0, 1) - \text{Float}(0, 1)]E_0[\bar{e}(1)] \\ &\quad + [\bar{R}p_0(0, 2) - \text{Float}(0, 2)]E_0[\bar{e}(2)] \end{aligned} \quad (35)$$

where  $\bar{R}$ , the fixed payment, equals 6 percent.<sup>9</sup> We can compute these terms, given that we have determined  $\lambda\mu_0$  and  $\lambda\mu_1$  from the term structure. From Table 1, the value of the swap, assuming a notational principal of \$100 million is

$$\begin{aligned} v_s(0) &= \{[0.06(0.9486) - (1 - 0.9486)](1 - .01) \\ &\quad + [0.06(0.8953) - (0.9486 - 0.8953)] + [(1 - .03)(1 - .01)]\}100m \\ &= 55,160(1 - .01) + 4,180(1 - .03)(1 - .01) \\ &= 58,622 \end{aligned} \quad (36)$$

If we ignore the credit risk, the value of the swap is 59,340, a difference of 718. In comparison to the notational principal, this difference is insignificant, though in comparison to the value of the swap (calculated ignoring default), it represents approximately 1 percent of the total value.

## H. Generalizations

As discussed earlier, it is a straightforward exercise to generalize this two-period example. Four extensions will prove useful in applications. The first is the multiperiod generalization that follows by augmenting the number of periods in the previous model. Retaining the statistical independence assumption of the bankruptcy processes from the spot rate process under the pseudoprobabilities implies that the separations exhibited in the expressions for the XYZ zero-coupon bond price (expression (17)), the XYZ coupon bond

<sup>8</sup> For a more complete description of different possible contingencies, see Cooper and Mello (1991).

<sup>9</sup> When the swap is first entered into,  $\bar{R}$  is determined such that  $v_s(0) = 0$ .  $\bar{R}$  is then called the swap rate.

price (expression (19)), the option value on XYZ debt (expression (21)), and the vulnerable options (expression (26)) still obtain.

The second extension (in the multiperiod setting) is to introduce a correlation (in the pseudoprobabilities) between the bankruptcy process on XYZ debt and the default-free term structure. This is equivalent to changing the probabilities in Figure 4 to a different distribution. In the discrete-time, discrete-state space setting, this generalization imposes no additional complication, except that the separation exhibited in the previous sections will not obtain. All the previous results generalize in a straightforward fashion.

The third extension is to introduce traded common equity on XYZ into the above economy. This entails the introduction of additional randomness (additional branches on the tree). This extension, albeit messy, is straightforward and illustrated in the next section.

The fourth extension is to analyze the continuous-time limit of the discrete-time model. The continuous-time limit is useful for estimation and computation. It aids estimation because the pseudoprobabilities for the default-free term structure ( $\pi_0$ ) can be reparameterized in terms of instantaneous volatilities. Volatilities can be easily estimated and interpreted. This same statement is not true for the pseudoprobabilities themselves.

### III. The Continuous Trading Economy

This section extends the previous two-period example to its multiperiod, continuous-time limit. For pedagogical reasons, we retain the statistical independence assumption under the pseudoprobabilities between the bankruptcy process for XYZ debt and the default-free term structure. This allows a direct comparison of the results across the two models. Generalizations to more complicated economies will be subsequently discussed. To be consistent with Heath, Jarrow, and Morton (1992), we specify the exogenous stochastic processes followed by the forward rates and the payoff ratio. Although various parameterizations are possible, we choose one convenient for application of the techniques available in Jarrow and Madan (1995).

#### A. The Setup

We consider continuous trading over the time interval  $[0, \tau]$ . Let  $\tau_1^*$  denote the time of bankruptcy of firm XYZ. We assume that  $\tau_1^*$  is exponentially distributed over  $[0, \infty)$  with parameter  $\lambda_1$ . Alternative distributions for the bankruptcy time could have been utilized (see Longstaff and Schwartz (1992) for an endogenously derived process based on a lognormally distributed random variable for the firm's value).

The default-free forward rate is defined by

$$f_0(t, T) \equiv -\partial \log p_0(t, T) / \partial T. \quad (37a)$$

The default-free spot interest rate is defined by

$$r_0(t) \equiv f_0(t, t), \quad (37b)$$

and the default-free money market account is defined by

$$B(t) = \exp \left\{ \int_0^t r_0(s) ds \right\}. \quad (37c)$$

Analogously, we define for XYZ debt

$$f_1(t, T) \equiv -\partial \log p_1(t, T) / \partial T, \quad (37d)$$

$$r_1(t) \equiv f_1(t, t), \text{ and} \quad (37e)$$

$$B_1(t) \equiv \exp \left\{ \int_0^t r_1(s) ds \right\}. \quad (37f)$$

We now impose the exogenous stochastic structure directly on the forward rates  $f_0(t, T)$ ,  $f_1(t, T)$ , and payoff ratio  $e_1(t)$ .

ASSUMPTION 1: *Default-free forward rates*

$$df_0(t, T) = \alpha_0(t, T) dt + \sigma(t, T) dW_1(t), \quad (38)$$

where  $W_1(t)$  is a Brownian motion,  $(\alpha_0(t, T), \sigma(t, T))$  satisfy some smoothness and boundedness conditions,<sup>10</sup> and  $\sigma(t, T)$  is deterministic (non-random).

The default-free forward rate's change over a small instant in time is seen to be equal to a drift  $(\alpha_0(t, T))$  plus a random shock with volatility  $(\sigma(t, T))$ . The assumption of a deterministic volatility function is made only to facilitate the derivation of closed-form solutions. This assumption can easily be relaxed. The deterministic volatility  $\sigma(t, T)$  makes this a Gaussian economy. As such, it is a continuous limit of the process given in Figure 1. If  $\sigma(t, T) \equiv \sigma > 0$  is a positive constant, one gets the continuous-time analogue of Ho and Lee (1986). If  $\sigma(t, T) = \sigma e^{-\xi(T-t)}$  for  $\sigma, \xi$  constants, one gets the model of Hull and White (1990) and Musiela, Turnbull, and Wakeman (1993). Assumption 1 is easily generalized to multiple, independent Brownian motions.

ASSUMPTION 2: *XYZ forward rates*

$$df_1(t, T) = \begin{cases} [\alpha_1(t, T) - \theta_1(t, T)\lambda_1]dt + \sigma(t, T)dW_1(t) & \text{if } t < \tau_1^* \\ [\alpha_1(t, T) - \theta_1(t, T)\lambda_1]dt \\ \quad + \sigma(t, T)dW_1(t) + \theta_1(t, T) & \text{if } t = \tau_1^* \\ \alpha_1(t, T)dt + \sigma(t, T)dW_1(t) & \text{if } t > \tau_1^*, \end{cases} \quad (39)$$

where  $\alpha_1(t, T)$ ,  $\theta_1(t, T)$  satisfy some smoothness and boundedness conditions.<sup>11</sup>

<sup>10</sup> These measurability and boundedness conditions can be found in Jarrow and Madan (1995).

<sup>11</sup> See Jarrow and Madan (1995) for these measurability and integrability conditions.

The process for XYZ forward rates mimics the stochastic process for default-free forward rates. Indeed, XYZ forward rates change over a small instant in time by a drift plus a random shock. Prior to bankruptcy ( $t < \tau_1^*$ ), the drift is adjusted downward to reflect the expected change  $\theta_1(\tau_1^*, T)\lambda_1$  that occurs at the bankruptcy time. After bankruptcy ( $t > \tau_1^*$ ), the forward rate process is identical to expression (38), except for subscripts. Without loss of generality, the coefficient preceding the Brownian motion component equals that in the default-free forward rate process.<sup>12</sup>

ASSUMPTION 3: *The XYZ payoff ratio*

$$e_1(t) = \begin{cases} 1 & \text{if } t < \tau_1^* \\ \delta_1 & \text{if } t \geq \tau_1^* \end{cases} \quad (40)$$

where  $0 < \delta_1 < 1$ .

The payoff ratio is unity until bankruptcy, at which time it is equal to  $\delta_1 < 1$ . This is a continuous-time limit of the bankruptcy process in Figure 2. As in the discrete-time setting, the payoff ratio ( $\delta_1$ ) can differ depending on the seniority of the debt. Although the payoff ratio is constant, this restriction is imposed for simplicity. It is a straightforward mathematical exercise to make  $e_1(t)$  random and dependent on an additional Brownian motion representing the randomness generating the value of the firm. Given trading in a sufficient number of XYZ zeros, the market for XYZ debt will still be complete, and our methodology still applies. This generalization would include Merton's (1974) model as a special case. The difficulty with this generalization is that the valuation formulae become more complex, and estimation/computation becomes more involved. Empirical validation of the simpler model is needed to determine whether this additional complexity is warranted.

One can derive the following stochastic processes for  $p_0(t, T)$ ,  $v_1(t, T)$ , and  $B_1(t)e_1(t)$ .<sup>13</sup> These are the continuous-time analogues of Figures 1 and 4. For default-free zeros

$$dp_0(t, T)/p_0(t, T) = [r_0(t) + \beta_0(t, T)]dt + a(t, T)dW_1(t) \quad (41a)$$

where

$$\beta_0(t, T) \equiv - \int_t^T \alpha_0(t, u) du + (1/2)a(t, T)^2 \quad (41b)$$

and

$$a(t, T) \equiv - \int_t^T \sigma(t, u) du. \quad (41c)$$

<sup>12</sup> See footnote 14. This is a no arbitrage restriction.

<sup>13</sup> For the derivation, see Jarrow and Madan (1995). Note that  $t - \equiv \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \geq 0}} (t - \varepsilon)$ .

Expression (41) gives the return process followed by the default-free zeros. The random return equals the spot interest rate  $r_0(t)$ , an adjustment for risk ( $\beta_0(t, T)$ ), plus a random shock with volatility  $a(t, T)$ . By construction, the volatility  $a(t, T)$  approaches 0 as the bond matures.

For the XYZ zeros

$$\begin{aligned} dv_1(t, T)/v_1(t-, T) \\ = \begin{cases} [r_1(t) + \beta_1(t, T) - \Theta_1(t, T)\lambda_1]dt + a(t, T)dW_1(t) & \text{if } t < \tau_1^* \\ [r_1(t) + \beta_1(t, T) - \Theta_1(t, T)\lambda_1]dt \\ \quad + a(t, T)dW_1(t) + (\delta_1 e^{\Theta_1(t, T)} - 1) & \text{if } t = \tau_1^* \\ [r_1(t) + \beta_1(t, T)]dt + a(t, T)dW_1(t) & \text{if } t > \tau_1^* \end{cases} \end{aligned} \quad (42a)$$

where

$$\beta_1(t, T) \equiv - \int_t^T \alpha_1(t, u) du + (1/2)a(t, T)^2, \quad (42b)$$

and

$$\Theta_1(t, T) \equiv - \int_t^T \theta_1(t, u) du. \quad (42c)$$

Expression (42) gives the return process followed by the XYZ zeros. The random return mimics the default-free return process. Indeed, prior to bankruptcy ( $t < \tau_1^*$ ), the random return consists of a drift, adjusted for the change at the time of bankruptcy, and a random shock with volatility  $a(t, T)$ . At bankruptcy ( $t = \tau_1^*$ ), the return changes discretely by  $(\delta_1 e^{\Theta_1(t, T)} - 1)$ . Subsequently to bankruptcy ( $t > \tau_1^*$ ), the return process is that given by expression (41) with only the subscripts changed.

$$d[B_1(t)e_1(t)]/B_1(t-)e_1(t-) = \begin{cases} r_1(t)dt & \text{if } t < \tau_1^* \\ [r_1(t)dt + (\delta_1 - 1)] & \text{if } t = \tau_1^* \\ r_1(t)dt & \text{if } t > \tau_1^* \end{cases} \quad (43)$$

Expression (43) gives the return process (in dollars) followed by the XYZ money market account. It returns the XYZ spot interest rate, except at bankruptcy ( $t = \tau_1^*$ ), where it drops  $(\delta_1 - 1) < 0$  percent.

### B. Arbitrage-Free Restrictions

To ensure that the economy has no arbitrage opportunities and that the market is complete, using Harrison and Pliska (1981) we need to provide conditions that guarantee the existence of a unique equivalent probability making the relative prices  $v_1(t, T)/B(t)$ ,  $B_1(t)e_1(t)/B(t)$ , and  $p_0(t, T)/B(t)$

martingales. This is analogous to conditions (8), (11), and (15) derived in the discrete-time setting. To obtain these conditions, we impose<sup>14</sup>

ASSUMPTION 4: *The existence of unique equivalent martingale probabilities*

$$(\delta_1 e^{\Theta_1(t,T)} - 1) \neq 0 \text{ for all } t \leq \tau_1^* \text{ and } T \in [0, \tau]. \quad (44)$$

This assumption can be understood by referring to the XYZ bond process in expression (42). The bankruptcy process' impact on the XYZ bond's return is the quantity  $(\delta_1 e^{\Theta_1(t,T)} - 1)$ . For this risk to be relevant (and hedgeable), this coefficient must be nonzero. This condition is satisfied as long as XYZ bond prices change at the time of bankruptcy.

Under this assumption, the following system of equations can be shown to have a unique solution  $(\gamma_1(t), \mu_1(t))$ .

$$\beta_0(t, T) + \gamma_1(t)a(t, T) = 0 \quad (45a)$$

$$\begin{aligned} r_1(t) - r_0(t) + \beta_1(t, T) + \gamma_1(t)a(t, T) - \Theta_1(t, T)\lambda_1 \\ + (\delta_1 e^{\Theta_1(t,T)} - 1)\lambda_1 \mu_1(t) = 0 \text{ if } t < \tau_1^* \end{aligned} \quad (45b)$$

$$r_1(t) - r_0(t) + \beta_1(t, T) + \gamma_1(t)a(t, T) = 0 \text{ if } t \geq \tau_1^*. \quad (45c)$$

$$r_1(t) = r_0(t) + (1 - \delta_1)\lambda_1 \mu_1(t) \text{ if } t < \tau_1^* \quad (45d)$$

$$r_1(t) = r_0(t) \text{ if } t \geq \tau_1^* \quad (45e)$$

The quantities  $(\gamma_1(t), \mu_1(t))$  have the interpretation of being market prices of risk. This is most easily seen via expressions (45a) and (45b). Expression (45a) shows that the excess expected return on the  $T$ -maturity default-free zero  $(\beta_0(t, T))$  is proportional to its volatility  $(a(t, T))$ . The proportionality factor is the risk premium  $\gamma_1(t)$ , which is independent of the  $T$ -maturity bond selected. This is the standard no arbitrage condition for the default-free debt market as given in Heath, Jarrow, and Morton (1992). It implies their forward rate drift restriction.

Expressions (45b) and (45c) are the analogous restrictions for the XYZ zero-coupon bond market. To interpret these conditions, we combine them with (45a) and (45e).

$$\beta_1(t, T) - \Theta_1(t, T)\lambda_1 = \beta_0(t, T) - \delta_1(e^{\Theta_1(t,T)} - 1)\lambda_1 \mu_1(t) \text{ if } t < \tau_1^* \quad (46a)$$

$$\beta_1(t, T) = \beta_0(t, T) \text{ if } t \geq \tau_1^*. \quad (46b)$$

Expression (46a) is (45b) rewritten. We see that prior to bankruptcy ( $t < \tau_1^*$ ), the excess expected return on the XYZ zero  $(\beta_1(t, T) - \Theta_1(t, T)\lambda_1)$  is

<sup>14</sup> See Jarrow and Madan (1995). Expressions (45a)–(45e) make the stochastic processes in expressions (41)–(43) martingales under the transformation  $d\tilde{W}_1 = dW_1 - \gamma_1(s)ds$  and with  $\tau_1^*$  distributed exponentially with parameter  $\lambda_1 \mu_1(t)dt$ .

equal to the excess expected return on the default-free zero  $\beta_0(t, T)$  plus an adjustment for default risk. The adjustment is proportional to the bankruptcy shock  $\delta_1(e^{\Theta_1(t, T)} - 1)$ . The proportionality factor is the risk premium  $(\lambda_1 \mu_1(t))$ , which is independent of the  $T$ -maturity bond selected. Subsequent to bankruptcy ( $t \geq \tau_1^*$ ), as there is no more bankruptcy risk, the excess expected return on both XYZ zeros and default-free zeros is identical (see (46b)).

Expressions (45e) and (46b) imply that the return processes for the XYZ zeros and the default-free zeros are identical after bankruptcy ( $t \geq \tau_1^*$ ), which implies that  $v_1(t, T) = \delta_1 p_0(t, T)$ . That is, after bankruptcy, the Treasury and XYZ term structures are identical, i.e.,  $p_0(t, T) = p_1(t, T)$ . This result was seen in the discrete time setting as expressions (11a) and (11b).

An additional implication of Assumption 4 is that the market is complete (see Harrison and Pliska (1981)). Define a *contingent claim*  $X$  as a suitably bounded random cash flow at time  $T < \tau$ .<sup>15</sup> Then, the time  $t$  “arbitrage-free” price of this contingent claim is its expected discounted value under the martingale probabilities, i.e.,

$$\tilde{E}_t(X/B(T))B(t), \text{ where} \quad (47)$$

$\tilde{E}_t(\cdot)$  is the time  $t$  expectation under the martingale probabilities. This expression provides the method for pricing derivative securities involving credit risk.

Next, for simplicity, we add

**ASSUMPTION 5:** *The Poisson bankruptcy process under the martingale probabilities*

$$\mu_1(t) \equiv \mu_1 > 0 \text{ a positive constant.} \quad (48)$$

The previous structure implies that the bankruptcy process is independent of the spot interest rate process under the true (empirical) probabilities. This additional Assumption 5 implies the statistical independence of the bankruptcy process from the default-free interest rate process *under the martingale probabilities*. It does so because it also makes the time of bankruptcy process an exponential distribution under the martingale probabilities with parameter  $\lambda_1 \mu_1$ , which is independent of the spot interest rate process. It is imposed to simplify the subsequent analysis. This is a subtle condition as it imposes implicit structure on the risk premia in the economy. This assumption can easily be relaxed, with correspondingly more complex valuation formulae. We retain Assumption 5 to facilitate the understanding of the subsequent material. Empirical validation of the model under Assumption 5 is needed to determine whether this additional complexity is warranted.

<sup>15</sup> That is, letting  $\tilde{Q}$  be the martingale probability with expectation operator  $\tilde{E}(\cdot)$ ,  $\tilde{E}([X/B(T)]^2) < +\infty$ .

## C. XYZ Bonds

Under Assumptions 1 to 5, we can simplify  $v_1(t, T)$  further

$$\begin{aligned} v_1(t, T) &= \tilde{E}_t(e_1(T)/B(T))B(t) = \tilde{E}_1(e_1(T))p_0(t, T) \\ &= \begin{cases} (e^{-\lambda_1 \mu_1(T-t)} + \delta_1(1 - e^{-\lambda_1 \mu_1(T-t)}))p_0(t, T) & \text{if } t < \tau_1^* \\ \delta_1 p_0(t, T) & \text{if } t \geq \tau_1^* \end{cases} \quad (49) \end{aligned}$$

This decomposition is the continuous-time analogue of expressions (16) and (17). In this form, we see that to compute the stochastic process for  $v_1(t, T)$ , we only need the parameters  $(\lambda_1 \mu_1)$  and  $\delta_1$ . As in the discrete-time model, these can be obtained via a recursive estimation procedure. In bankruptcy,  $v_1(t, T) = \delta_1 p_0(t, T)$ .

Merton (1974) also derives an expression for the value of a zero-coupon corporate bond. In Merton's model, default occurs if the value of the firm's assets are less than the amount owed to bondholders at maturity. If default occurs, it is assumed that bondholders costlessly take over the firm. In the simplest version of Merton's model, to compute the value of the zero-coupon bond, it is necessary to know both the current value of the firm's total assets and the total asset's volatility. At this point, the two models appear quite similar, each involving two unknowns. The relevant differences in the models appear in the application. Typically a firm has many different forms of liabilities outstanding. To use Merton's model, it is necessary to simultaneously solve for the value of all of these claims, which is a nontrivial exercise (see Jones, Mason, and Rosenfeld (1984)) and necessitates strong assumptions about the relevance of capital structure and the treatment of claims in the event of bankruptcy. Secondly, to use Merton's model, one must also be able to measure the current value of the firm's assets. This is a difficult task.

In contrast, our model circumvents these difficulties by taking as given the term structure of interest rates for the relevant credit risk class. This, however, introduces its own set of problems. The bonds used to construct this term structure must have the same probability of default, and if default occurs, the payment rule must be known. In default, we assume that the claim holders receive some fixed amount per promised dollar. This assumption can, in fact, be relaxed to include Merton's (1974) bankruptcy condition as a special case.

In a generalized version of Merton's model, the value of the firm and the term structure of interest rates can be correlated. In our model, we assume independence because it facilitates the derivation of closed form solutions. This assumption can also be relaxed.

An examination of expression (49) reveals that nowhere does the parameter  $\Theta_1(t, T)$  appear. This implies that the martingale restrictions under Assumption 5 completely specify  $\Theta_1(t, T)$  in terms of the parameters of the bankruptcy process under the martingale probabilities. These restrictions could prove useful for empirical estimation and for testing this particular form of the model. They are provided in the following lemma.



LEMMA 1: *Martingale restrictions under Assumption 5*<sup>16</sup>

$$(\delta_1 e^{\Theta_1(t, T)} - 1) = \frac{e^{-\lambda_1 \mu_1(T-t)}(\delta_1 - 1)}{e^{-\lambda_1 \mu_1(T-t)} + \delta_1(1 - e^{-\lambda_1 \mu_1(T-t)})} \quad \text{for } t < \tau_1^*. \quad (50)$$

*Proof:* In the Appendix.

Lemma 1 is the additional restriction imposed on the bond's volatilities by the statistical independence assumption under the martingale probabilities, Assumption 5. It is the analogous restriction to that given in the discrete-time setting via expressions (11e) and (15c).

#### D. Options on XYZ Debt

This section provides a closed-form solution for a European type call option with exercise price  $K$  and maturity  $m$  on an XYZ zero-coupon bond with maturity  $M \geq m$ . Let  $C_1(t, K)$  denote the call's time  $t$  value with exercise price  $K$ . Using risk-neutral valuation, we have that

$$C_1(t, K) = \tilde{E}_t(\max[v_1(m, M) - K, 0]/B(m))B(t). \quad (51)$$

Expression (49) implies that this can be written as

$$C_1(t, K) = \tilde{E}_t(\max[p_0(m, M)\tilde{E}_m(e(M)) - K, 0]/B(m))B(t). \quad (52)$$

Using the fact that

$$\tilde{E}_m(e_1(M)) = \begin{cases} [e^{-\lambda_1 \mu_1(M-m)} + \delta_1(1 - e^{-\lambda_1 \mu_1(M-m)})] & \text{if } m < \tau_1^* \\ \delta_1 & \text{if } m \geq \tau_1^* \end{cases}, \quad (53)$$

and the specifics of Assumptions 1 to 5, we can rewrite this as

$$\begin{aligned} C_1(t, K) &= \delta_1(1 - e^{-\lambda_1 \mu_1(M-t)})C_0(t, K') \\ &\quad + (e^{-\lambda_1 \mu_1(M-m)} + \delta_1(1 - e^{-\lambda_1 \mu_1(M-m)}))e^{-\lambda_1 \mu_1(m-t)}C_0(t, K'') \text{ if } t < \tau_1^* \end{aligned} \quad (54)$$

where

$$\begin{aligned} C_0(t, L) &\equiv \tilde{E}_t(\max[p_0(m, M) - L, 0]/B(m))B(t) \\ &= p_0(t, M)\Phi(h(L)) - Lp_0(t, m)\Phi(h(L) - q) \end{aligned} \quad (55a)$$

and

$$h(L) \equiv [\log(p_0(t, M)/p_0(t, m)L) + (1/2)q^2]/q \quad (55b)$$

$$q^2 \equiv \int_t^m [a(u, M) - a(u, m)]^2 ds. \quad (55c)$$

$$K' \equiv K/\delta_1 \quad (55d)$$

$$K'' \equiv K/[e^{-\lambda_1 \mu_1(M-m)} + \delta_1(1 - e^{-\lambda_1 \mu_1(M-m)})]. \quad (55e)$$

<sup>16</sup> If we had allowed  $\sigma(t, T)$  in Assumption 1 to differ from  $\sigma(t, T)$  in Assumption 2, then this lemma would have implied their equality as well.

Expression (54) gives a closed-form solution for the value of a European option on XYZ risky debt, and it is the continuous time analogue of expression (21). It is a linear combination of the value of two distinct European options on otherwise identical *default-free debt*. This result is important because it allows one to compute option values on risky debt using software developed for riskless debt. Indeed, the first term is equal to the risk-neutral probability that default occurs prior to time  $m$  ( $1 - e^{-\lambda_1 \mu_1(m-t)}$ ) times the value of XYZ's option in that case ( $\delta_1 C_0(t, K')$ ). After bankruptcy, recall that XYZ debt is riskless. The second term is equal to the risk-neutral probability that default occurs after time  $m$  ( $e^{-\lambda_1 \mu_1(m-t)}$ ) times the value of the option in that case ( $(e^{-\lambda_1 \mu_1(M-m)} + \delta_1(1 - e^{-\lambda_1 \mu_1(M-m)}))C_0(t, K'')$ ). If bankruptcy occurs after the option matures ( $m$ ), then XYZ debt is again riskless at the option's expiration date, but for a different reason. Note that  $C_0(t, K') < C_0(t, K'')$  as  $K' > K''$ . The option on the default-free debt is valued under Assumption 1, and the formula (55a-e) is obtained from Heath, Jarrow, and Morton (1992). When  $\sigma(t, T) \equiv \sigma > 0$ , we get Ho and Lee's (1986) model, or when  $\sigma(t, T) \equiv \sigma e^{-\xi(T-t)}$ , we get the model of Hull and White (1990) and Musiela, Turnbull, and Wakeman (1993). For  $\mu_1 \lambda_1 \equiv 0$ , no default risk, expression (54) reduces to  $C_1(t, K) = C_0(t, K)$ , which is the standard no default interest rate option pricing formula in a Gaussian economy.

Expression (54) is easily computed, and easily extended to multiple factors for the Brownian motion risk.<sup>17</sup>

Equation (54) is for a time prior to bankruptcy. After bankruptcy, the call's value is

$$C_1(t, K) = \delta_1 C_0(t, K') \text{ for } t \geq \tau_1^*. \quad (56)$$

Hence, at bankruptcy, there is a discrete and negative drop in the call's value. This is most easily seen by comparing expressions (54) and (56).

### E. Vulnerable Options

The previous sections value options on financial securities subject to default. Implicit in this procedure is that the secondary market buyer/seller of these options are default free. This would happen, for example, when the option transaction is guaranteed by a regulated and organized exchange. In the absence of such a guarantee, the writer of the option contract can also default.<sup>18</sup> Such options are called vulnerable options (see Johnson and Stulz

<sup>17</sup> If there are  $b$  independent Brownian motions,  $W_i(t)$  for  $i = 1, \dots, b$  with volatilities  $a_i(t, T)$  for  $i = 1, \dots, b$ , then  $q^2$  in (55) becomes  $\sum_{i=1}^b \int_t^M [a_i(u, M) - a_i(u, m)]^2 ds$ . Otherwise, the formula remains unchanged.

<sup>18</sup> The position of an option writer is of unlimited liability and, therefore, default risk to the purchaser is relevant. In contrast, the purchaser of the option contract has limited liability, so the writer does not face a symmetric default risk from the purchaser. In addition, when creating a synthetic long position in an option, the borrowed funds are default-free as their value is always covered by the long position in the underlying security. Recall that the (synthetic call) portfolio's value is nonnegative with probability one.

(1987)).<sup>19</sup> This section extends the pricing methodology of the previous sections to price vulnerable options.

To price these vulnerable options, the economy in the previous section needs to be extended. Let the writer of the option be another firm, whose risky zero-coupon bonds ( $v_2(t, T)$ ) are also traded. Using the foreign currency analogy, we can decompose these zero-coupon bonds into

$$v_2(t, T) = e_2(t)p_2(t, T) \quad \text{where} \quad p_2(T, T) = 1 \text{ for all } T. \quad (57)$$

It is assumed that the forward rates  $f_2(t, T)$  and the payoff ratio  $e_2(t)$  satisfy Assumptions 2 and 3 with the index “2” replacing the index “1.” Following the same analysis as before, a sufficient condition for the existence of unique equivalent martingale probabilities is Assumption 4 applied to  $(v_2(t, T))$ . Adding Assumption 5, all the preceding results apply in an identical manner.

Next, consider the option writer, writing a European call option with exercise price  $K$  and maturity  $m$  on the XYZ zero-coupon bond with maturity  $M \geq m$ . This is the option valued in the last section. The option’s time  $t$  price will be denoted  $C_2(t, K)$ , the “2” subscript indicating the fact that the option writer is involved.

The option writer *promises* to pay  $C_1(m, K)$  dollars at time  $m$ . However, the option writer may default. Thus, this option contract has a time  $m$  value equal to

$$C_2(m, K) = e_2(m)C_1(m, K). \quad (58)$$

Using the risk-neutral valuation procedure, we get that

$$C_2(t, K) = \tilde{E}_t(e_2(m)C_1(m, K)/B(m))B(t). \quad (59)$$

Using the statistical independence of the bankruptcy processes from each other and the spot rate process under the martingale probabilities, expression (59) simplifies to:

$$\begin{aligned} C_2(t, K) &= \tilde{E}_t(e_2(m))\tilde{E}_t(C_1(m, K)/B(m))B(t) \\ &= \tilde{E}_t(e_2(m))C_1(t, K) \end{aligned} \quad (60)$$

<sup>19</sup> Johnson and Stulz’s (1987) approach is easily understood using our methodology. Our equation (57) corresponds to their equation (1) where,

$$e_2(T) \equiv \begin{cases} 1 & \text{if } V(T) \geq S(T) - X > 0 \\ \frac{V(T)}{S(T) - X} & \text{if } S(T) - X > V(T) \geq 0 \end{cases}$$

The notation  $V(T)$ ,  $S(T)$ , and  $X$  are from Johnson and Stulz (1987). In addition, Johnson and Stulz assume constant interest rates. Unlike our simple model, they allow  $e_2(T)$  to be random and to depend on the asset underlying the option ( $S(T)$ ). This correlation is easily included within our framework. With the  $e_2(T)$  process as above, Johnson and Stulz (1987) is a special case of our more general methodology.

where

$$\tilde{E}_t(e_2(m)) = \begin{cases} \delta_2 & \text{if } t \geq \tau_2^* \\ e^{-\lambda_2 \mu_2(m-t)} + \delta_2(1 - e^{-\lambda_2 \mu_2(m-t)}) & \text{if } t < \tau_2^* \end{cases} \quad (61)$$

Expression (60) provides (along with expression (54)) a simple closed-form solution for this option's value. Again, as with nonvulnerable options, to compute this value we only need to slightly modify software written for default-free debt options. This is the continuous-time analogue of expression (25).

We can alternatively use expression (49) for  $v_2(t, T)$  to write

$$\tilde{E}_t(e_2(m)) = v_2(t, m)/p_0(t, m), \quad (62)$$

giving an alternative expression

$$C_2(t, K) = (v_2(t, m)/p_0(t, m))C_1(t, K). \quad (63)$$

This is the continuous-time analogue of expression (26). In the case of no default for the option writer, expression (63) collapses to  $C_1(t, K)$ .

#### F. Equity Derivatives

This section demonstrates how to augment the previous economy to include trading in common equities on XYZ. This extension is significant in that it would allow, for example, the pricing and hedging of convertible XYZ debt without using the compound options approach of Merton (1974, 1977). The discrete-time model can be augmented in a similar fashion.

Let the common equity for firm XYZ trade and its time  $t$  price be denoted by  $S(t)$ . We assume its stochastic process satisfies.

$$S(t) = \begin{cases} S(0)e^{\int_0^t \xi(s) ds - (1/2)[\eta_1^2 + \eta_2^2]t + \eta_1 W_1(t) + \eta_2 W_2(t) + \lambda_1 t} & \text{if } t < \tau_1^* \\ 0 & \text{if } t \geq \tau_1^* \end{cases} \quad (64)$$

where  $\xi(s)$  satisfies some smoothness and boundedness conditions,<sup>20</sup> and  $\eta_1$  and  $\eta_2$  are constants.

Prior to bankruptcy, expression (64) is a geometric Brownian motion with instantaneous expected return  $\xi(s)ds$ .<sup>21</sup> The volatility of this stock is generated by changes in the first Brownian motion,  $W_1(t)$ , which also affects the term structure of interest rates, and changes in the second Brownian motion,  $W_2(t)$ , which is unique to the stock. The instantaneous volatility is

<sup>20</sup> Let  $(G_t: t \in [0, \tau])$  be the augmented filtration generated by  $W_1$ ,  $N_1$ , and  $W_2$  where  $\{W_2(t): t \in [0, \tau]\}$  is a Brownian motion independent of  $\{W_1(t): t \in [0, \tau]\}$  and  $\{N_1(t): t \in [0, \tau]\}$  where  $N_1(t) \equiv 1(t \geq \tau_1^*)$ . Then, we require that  $\xi: \Omega \times [0, \tau] \rightarrow \mathfrak{R}$  is predictable with respect to  $(G_t: t \in [0, \tau])$  and uniformly bounded.

<sup>21</sup> The  $\lambda_1 t$  term appears in the first line of expression (64) so that  $E_0(dS(t)/S(t)) = \xi(s)ds$ , see the stochastic differential equation (E) in the appendix.

$(\sqrt{\eta_1^2 + \eta_2^2})$ . After bankruptcy, the XYZ stock has zero value. This occurs when XYZ debt defaults and pays off  $\delta_1 < 1$  dollars per dollar promised.<sup>22</sup>

To use the risk-neutral valuation methodology, it is shown in the Appendix that there exists a unique martingale probability such that  $S(t)/B(t)$ ,  $p_0(t, T)/B(t)$ , and  $v_1(t, T)/B(t)$  are martingales. Under this martingale probability, prior to bankruptcy ( $t < \tau_1^*$ ),  $S(t)$  follows a geometric Brownian motion with modified drift  $(r_0(t) + \lambda_1 \mu_1)$ . After bankruptcy ( $t \geq \tau_1^*$ ),  $S(t)$  is again zero.

The risk-neutral valuation methodology can now be used to price derivative securities involving XYZ stock. For example, consider a European-type call option on the stock with exercise date  $T$  and exercise price  $K$ . Denoting its time  $t$  value as  $C(t)$ , its payoff at time  $T$  is

$$C(T) = \max[S(T) - K, 0]. \quad (65)$$

Its time 0 value is

$$C(0) = \tilde{E}_0 \left( \max \left( \frac{S(T) - K}{B(T)}, 0 \right) \right). \quad (66)$$

Using Amin and Jarrow (1992), it is shown in the appendix that

$$\begin{aligned} C(0) &= [S(0)e^{\lambda_1 \mu_1 T} \Phi(g) - Kp_0^*(0, T)\Phi(g - h)]e^{-\lambda_1 \mu_1 T} \\ &= S(0)\Phi(g) - Kp_0^*(0, T)\Phi(g - h) \end{aligned} \quad (67)$$

where

$$p_0^*(0, T) = p_0(0, T)e^{-\lambda_1 \mu_1 T} = \tilde{E}_0 \left( e^{-\int_0^T [r_0(s) + \lambda_1 \mu_1] ds} \right), \quad (68a)$$

$$g = \frac{\log(S(0)/Kp_0^*(0, T)) + (1/2)h^2}{h}, \quad (68b)$$

and

$$h^2 = [\eta_1^2 + \eta_2^2]T - 2\eta_1 \int_0^T a(t, T) dt + \int_0^T a(t, T)^2 dt. \quad (68c)$$

This is the generalization under stochastic interest rates of Merton's (1976; (17), p. 135) formula for the value of a call option on a stock that can go bankrupt.<sup>23</sup> This is the Black-Scholes equation with volatility ( $h$ ) and the interest rate factor " $e^{-rT}$ " replaced by  $p_0^*(0, T) = p_0(0, T)e^{-\lambda_1 \mu_1 T}$ . This value

<sup>22</sup> Deviations from absolute priority rules typically imply that XYZ stock does not have zero value after bankruptcy. This can be incorporated into the above model (64) by letting  $S(t)$  have a constant residual value after bankruptcy. The analysis with this extension follows in a straightforward manner. Alternatively, after bankruptcy, one could allow  $S(t)$  to follow a different (lower valued) stochastic process. This generalization is left for future research.

<sup>23</sup> For put options, a similar analysis gives

$$Put(0) = Kp_0^*(0, T)\Phi(-g + h) - S(0)\Phi(-g) + Kp_0(0, T)[1 - e^{-\lambda_1 \mu_1 T}]$$

and the usual put-call parity result holds.

reflects the credit risk spread on XYZ debt as determined in the market (see expression (45d)). In equation (67), the option value depends upon the pseudoprobability of default. In Merton (1976), the risk of default is assumed to be fully diversifiable implying that  $\mu_1 \equiv 1$ .

This application gives additional markets in which we can estimate the default pseudoprobability ( $\lambda_1 \mu_1$ ). In the equity option market, it is reflected in the value of the option where it can be implicitly estimated.

### G. Generalizations and Extensions

The above continuous-time economy can be generalized in numerous ways. First, vector stochastic processes for  $W_1(t)$  and the bankruptcy processes can be included. These would give a multiple factor model for the default-free term structure of interest rates, and it would allow different credit classes for firm XYZ.

Secondly, as with the discrete-time model, the bankruptcy process can be correlated with the default-free term structure. This could be handled, for example, by making the pseudodefault probability ( $\lambda_1 \mu_1$ ) a function of the spot rate process. These generalizations follow in a straightforward manner using the martingale pricing technology. Computations, however, become more complicated and numerical approximation procedures need to be employed.

The methodology can be extended to the pricing and hedging of over-the-counter foreign currency derivatives. It also provides a general framework for risk management, as it directly addresses market risk and credit risk. Drawing on the result described by equation (19) for credit risky bonds, the methodology can be applied to credit-linked notes, credit swaps, and over-the-counter derivatives.

## IV. Conclusion

This article presents a technique for valuing options on a term structure of securities subject to credit risk. Both a stochastic process for the evolution of the default-free term structure and the term structure for risky debt are exogenously specified. Arbitrage-free dynamics for these term structures and a risk-neutral valuation procedure are derived. This methodology is applied to corporate debt, but the technique is applicable to other securities as well.

## Appendix

*Proof of Lemma 1:* Define  $N_1(t) \equiv 1(t \geq \tau_1^*)$ . Under Assumptions 1 to 5,

$$\begin{aligned}\tilde{E}_t(e_1(T)) &= 1(\tau_1^* \leq t) \delta_1 + 1(\tau_1^* > t) [e^{-\lambda_1 \mu_1 (T-t)} + \delta_1 (1 - e^{-\lambda_1 \mu_1 (T-t)})] \\ \tilde{E}_t(e_1(T)) &= N_1(t) e^{-\lambda_1 \mu_1 (T-t)} [\delta_1 - 1] + [e^{-\lambda_1 \mu_1 (T-t)} + \delta_1 (1 - e^{-\lambda_1 \mu_1 (T-t)})].\end{aligned}$$

Thus,

$$d\tilde{E}_t(e_1(T)) = e^{-\lambda_1 \mu_1(T-t)}(\delta_1 - 1)[dN_1(t) - 1(t \leq \tau_1^*)\lambda_1 \mu_1 dt].$$

or

$$d\tilde{E}_t(e_1(T)) = \tilde{E}_t(e_1(T)) \left( \frac{e^{-\lambda_1 \mu_1(T-t)}(\delta_1 - 1)}{e^{-\lambda_1 \mu_1(T-t)} + \delta_1(1 - e^{-\lambda_1 \mu_1(T-t)})} \right) d\tilde{N}_1(t),$$

where  $d\tilde{N}_1(t) = dN_1(t) - \lambda_1 \mu_1 dt$ . Next,  $v_1(t, T) = p_0(t, T)\tilde{E}_t(e_1(T))$ . Using Ito's lemma and Jacod and Shirayaev (1987; 4.49, p. 52), yields

$$dv_1(t, T) = dp_0(t, T)\tilde{E}_t(e_1(T)) + p_0(t, T)d\tilde{E}_t(e_1(T)).$$

Substitution and simplification generates

$$\begin{aligned} dv_1(t, T) = & v_1(t, T)(r_0(t)dt + a(t, T)d\tilde{W}_1(t)) \\ & + v_1(t, T) \left[ \frac{e^{-\lambda_1 \mu_1(T-t)}(\delta_1 - 1)}{(e^{-\lambda_1 \mu_1(T-t)} + \delta_1(1 - e^{-\lambda_1 \mu_1(T-t)}))} \right] d\tilde{N}_1(t) \end{aligned}$$

where  $d\tilde{W}_1(t) = dW_1(t) - \gamma_1(t)dt$ . Comparison of expression (42) under expression (45b) gives the result. Q.E.D.

*Proof of existence and uniqueness of a martingale probability for XYZ equities:* Let  $(\Omega, \mathcal{Q}, F)$  be the probability space. There exists a unique measure  $\tilde{Q}$  making  $v_1(t, T)/B(t)$ ,  $p_0(t, T)/B(t)$  and  $S(t)/B(t)$  martingales. The probability measure  $\tilde{Q}$  is given by

$$\begin{aligned} d\tilde{Q}/dQ = \exp \bigg\{ & \int_0^\tau \gamma_1(s) dW_1(s) + \int_0^\tau \gamma_2(s) dW_2(s) - (1/2) \int_0^\tau \gamma_1^2(s) ds \\ & + \int_0^\tau \log \mu_1(s) dN_1(s) + \int_0^\tau (1 - \mu_1(s))\lambda_1(s) ds \bigg\}. \end{aligned}$$

This can be seen as follows. First, define

$$\tilde{W}_2(t) \equiv W_2(t) - \int_0^t \gamma_2(s) ds \quad (\text{A})$$

where

$\gamma_2: \Omega \times [0, \tau] \rightarrow \Re$  is predictable with respect to  $(G_t: t \in [0, \tau])$  and uniformly bounded.  $(G_t: t \in [0, \tau])$  is defined in footnote 15.

Using equation (A), we can write  $S(t)/B(t)$  as

$$\begin{aligned} d(S(t)/B(t)) = & (S(t-)/B(t))(\xi(t) + \gamma_1(s)\eta_1 + \gamma_2(s)\eta_2 + \lambda_1 1(t \leq \tau_1^*) \\ & - \lambda_1 \mu_1 1(t \leq \tau_1^*) - r_0(t))dt + (S(t-)/B(t)) \\ & \times (\eta_1 d\tilde{W}_1(t) + \eta_2 d\tilde{W}_2(t)) \\ & - (S(t-)/B(s)) [d\tilde{N}_1(t) - \lambda_1 \mu_1 1(t \leq \tau_1^*)dt] \end{aligned} \quad (\text{B})$$

Selecting  $(\gamma_1(t), \gamma_2(t), \mu_1(t))$  to satisfy expression (45) and

$$\xi(t) - r_0(t) + \gamma_1(s)\eta_1 + \gamma_2(s)\eta_2 + \lambda_1 1(t \leq \tau_1^*) - \lambda_1 \mu_1 1(t \leq \tau_1^*) = 0 \quad (C)$$

makes  $p_0(t, T)/B(t)$ ,  $v_1(t, T)/B(t)$ , and  $S(t)/B(t)$  martingales under  $\tilde{Q}$ . In this solution,  $(\gamma_1(t), \mu_1(t))$  are predictable with respect to the augmented filtration  $(G_t: t \in [0, \tau])$ . For example, under equation (C), we can rewrite expression (64) as

$$S(t) = \begin{cases} S(0)e^{\int_0^t r_0(s) ds - (1/2)[\eta_1^2 + \eta_2^2]t + \lambda_1 \mu_1 t + \eta_1 \tilde{W}_1(t) + \eta_2 \tilde{W}_2(t)} & \text{if } t < \tau_1^* \\ 0 & \text{if } t \geq \tau_1^* \end{cases} \quad (D)$$

which is a martingale (when divided by  $B(t)$ ) under  $\tilde{Q}$ .

In stochastic differential form,

$$\frac{dS(t)}{S(t-)} = r_0(t)dt + \eta_1 d\tilde{W}_1(t) + \eta_2 d\tilde{W}_2(t) - [dN_1(t) - \lambda_1 \mu_1 dt]. \quad (E)$$

*Proof of Expression (67):*

$$\bar{S}(t) \equiv S(0)e^{\int_0^t r_0(s) ds - (1/2)[\eta_1^2 + \eta_2^2]t + \lambda_1 \mu_1 t + \eta_1 \tilde{W}_1(t) + \eta_2 \tilde{W}_2(t)} \quad (F)$$

for all  $t \in [0, \tau]$ . The stochastic process  $\bar{S}(t)$  follows a geometric Brownian motion with instantaneous drift  $(r_0(t) + \lambda_1 \mu_1)$ .

Substitution of equation (F) into equation (64), and algebra yields

$$\begin{aligned} C(0) &= \tilde{E} \left( \max \left( \frac{\bar{S}(T) - K}{B(T)}, 0 \right) \right) \tilde{Q}(T < \tau_1^*) \\ &= \tilde{E} \left( \max \left( \frac{\bar{S}(T) - K}{B(T)}, 0 \right) \right) e^{-\lambda_1 \mu_1 T}. \end{aligned} \quad (G)$$

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