Bond futures: Delivery Option with Term Structure Modelling

Model development

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Abstract

Bond futures are characterised by a set of underlying bonds; the short party has the option to deliver at expiry any of those underlying bonds. Consequently, bond futures embed a choice option between bonds with different maturities and coupons. The delivery mechanism also incorporates conversion factors that create an implicit strike. The option is impacted by different maturities and different moneyness for each bond. It is important to take into account the full term structure of volatility with smile. A recent paper Bang and Daboussi (2022) developed such an approach for swap rate based products like CMS. In this paper we extend their approach to cover futures and apply it to the specific case of bond futures. The method allows the analysis of the impact of smile, term structure of volatility and correlations between rates on the delivery option and convexity adjustment values. All of them have an impact on the valuation and risk management of bond futures.

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1 Introduction

The recent paper Bang and Daboussi (2022)[1] describes a way to price swap rates dependent products with a method that replicates the market marginal distribution for each swap rate based on the swaption market and links the rates with a user selected copula. The paper is written in the OIS discounted framework and applied to OIS-linked swap rates.

The method is very generic and its first application was to CMS. It describes the (explicit) connection between swap rates and discount factors (at the swap maturities) and so can be viewed as a full term structure model (on discrete curve points). The method preserves the full market volatility smile, the only element not directly obtained from market prices is the copula between swap rates.

In this paper, we extend the method to the pricing of futures – i.e. expectation without discounting – and apply the results to bond futures. We provide the main results of the above paper in this new setting but not their proofs. We refer to the original paper for the details or to Henrard (2022) for an implementation note with more generic accrual factors and dates.

Bond futures have to be seen in this context as exotic options due to the delivery mechanism which allows for the delivery of several bonds with different maturities. In the case of the 10-year US Treasury futures (TY), the delivery basket is any treasury note with a maturity between 6.5 years and 10 years. This creates a real optionality where the full term structure of rates need to be modelled in a coherent way. The bond futures are liquid products with European optionality for which the full term structure of rates need to be modelled and for which the methodology developed in the above mentioned article provides real benefit with respect to a one-factor model.

In practice the optionality embedded in the bond futures goes well beyond the delivery option. There is a "when-issued" feature as the bonds in the basket are the bonds in existence at the expiry of the futures, not on the trade date. Bonds, with unknown features – in particular unknown coupon level – can be added in the basket between the trade and the expiry. There are also further options for futures on US Treasuries with early exercise options and wild card options. Those further options are not discussed here.

The valuation of the bond futures delivery option has been studied in the literature. In Grieves and Marcus (2005), a simplified setting with only two bonds, a flat yield curve and the two bond prices following geometric Brownian motion with a known constant volatility for their ratio is used. Grieves et al. (2010) provides empirical analysis of the PVBP using the previous developments and shows that the two-bonds only approach is lacking in some circumstances. In Henrard (2006) the pricing is done in a one-factor Hull-White with the full basket and actual yield

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[1]This author thanks Dominique Bang and Elias Daboussi for fruitful discussions related to the referenced paper and the draft of this paper.

curves. We will compare some of the results obtained here to that approach. In this note, we use the notations from Henrard (2014) for the curves used in collateral discounting. We use a deterministic spread between the collateral discounting curves and the bond discounting and repo curves.

Our goal is to describe the valuation of the futures and their delivery option with a known rate term structure and volatility smile structure. The volatility market on government bonds is not very liquid and obtaining this term structure itself can be a challenge.

The inputs to the model are the volatility smiles at maturities in line with some given frequency corresponding to the volatility instrument payment frequency. Typically this frequency is annual for overnight-indexed swaps. But for government bonds, this frequency varies; it is annual for EUR bonds and semi-annual for US bonds. The pricing mechanism provides an interpolation mechanism on the discounting curves. We calibrate on annual frequency with exact tenors and then interpolate to get the exact bonds frequency and payment dates.

Bond futures are based on *conversion factors* computed from a *reference or nominal rate*. Those terms are explained below. The reference rate is 6% for US Treasury futures. The rate is different for some other government bonds futures. The German bund long futures have a reference rate of 4%; the UK Gilt futures have some reference rates at 3% and some at 4%. With yield on government bonds close to 0% for a long time, the delivery option appeared almost irrelevant. With the rate hikes in all the major currencies, the delivery option is again becoming very relevant.

In a simplified approach, the delivery option appears like a choice option with a strike at the reference rate. When the yield is below the reference rate, the shortest duration bond is delivered and when the yield is above, the longest duration is delivered. When looking at the details, the optimal delivery depends also on the yield curve shape. A precise pricing of the option requires a correct starting forward curve and a term structure of the different rate dynamics and their correlations.

The goal of this note is to adapt the flexible term structure model referenced above to the context of bond futures and analyse to which extent this term structure impacts the delivery option pricing. The methods developed here could also be applied to the pricing of swap futures, e.g. its convexity adjustment estimation.

The method allows the analysis of the impact of smile, term structure of volatility and correlations between rates on the delivery option and convexity adjustment values. All of them have an impact on the valuation and risk management of bond futures.

2 Notations and curves

The discount factors associated to the government and repo curves for a discounting from payment date t to valuation date s are denoted $P^G(s,t)$ and $P^R(s,t)$. The pseudo discount factors curves associated to the overnight collateral are denoted $P^c(s,t)$.

When the discounting curve $P^X(t,.)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f^X(t,u)$ such that

$$P^X(t,u) = \exp\left(-\int_t^u f^X(t,s)ds\right). \tag{1}$$

As a starting point we suppose that the continuously compounded spread between government/repo and OIS is deterministic.

The continuously compounded rates for government, repo and OIS curves are denoted by $f^G(t)$, $f^R(t)$ and $f^c(t)$ and the spreads to OIS by $s^G(t)$ and $s^R(t)$. This means that

$$\begin{aligned} P^X(t,u) &= \exp\left(-\int_t^u f^X(t,\tau)d\tau\right) = \exp\left(-\int_t^u f^c(t,\tau) + s^X(\tau)d\tau\right) \\ &= \exp\left(-\int_t^u f^c(t,\tau)d\tau\right)\exp\left(-\int_t^u s^X(\tau)d\tau\right) = P^c(t,u)\frac{P^{s^X}(0,u)}{P^{s^X}(0,t)} \end{aligned} \tag{2}$$

Our model gives us the OIS discount factors; we get the government and repo discount factors through a simple multiplication by a deterministic factor. Note that in the USD-SOFR case, the spread between OIS and term repo rate is in theory $s^R = 0$ has described in Henrard (2018).

The OIS rate for a tenor of p periods, an effective date t_0 and a maturity date t_p is given in t by

$$\text{Swap}^p(t) = \frac{P^c(t,t_0) - P^c(t,t_p)}{A^p(t)}$$

where $A^p(t)$ is the annuity given by

$$A^p(t) = \sum_{i=1}^{p} \delta_i P^c(t,t_i).$$

The collateral account for rate c is denoted B^c, i.e.

$$B^c(u) = \exp\left(\int_0^u c(s)ds\right).$$

As we work with only one collateral/overnight rate, we ignore the dependency on the rate c.

The dates of interest, on top of the swap/bond dates $(t_i)_{i=1,\ldots,n}$ are the option expiry date $\theta \leq t_0$ and the payment date $\tau \geq \theta$.

The approach uses several measures associated to quantity N. The notations are

- $\mathbb{Q}^N$ for the measure.
- $\mathrm{E}^N[.]$ the associated expectations.
- $\mathrm{V}^N[.] = \mathrm{E}^N\left[(. - \mathrm{E}^N[.])^2\right]$ the the associated variance.
- $\Phi_p^N(K) = \mathbb{Q}^N(\mathrm{Swap}^p(\theta) \leq K)$ for $i = 1, \dots n$ (resp. $\rho_p^N(K) = \partial\Phi_p^N(K)/\partial K$) the cumulative (resp. probability) density function for the tenor p swap rate.
- $\Phi_0^N(K) = \mathbb{Q}^N(B(\theta) \leq K)$ (resp. $\rho_0^N(K) = \partial\Phi_0^N(K)/\partial K$) the cumulative (resp. probability) density function for the collateral account.
- $\pi_p^N(K) = \mathrm{E}^N\left[(K - \mathrm{Swap}^p(\theta))^+\right]$ the un-discounted put/receiver price.
- $\pi_0^N(K) = \mathrm{E}^N\left[(K - B(\theta))^+\right]$ the un-discounted put/floor price.

To shorten some notation, we use the index 0 in relation to the cash account up to the expiry date.

3 Bond futures

3.1 Invoicing

Suppose there are m bonds in the delivery basket and denote t_0 the delivery date. Let $\mathrm{AccruedInterest}_i(t)$ denote the accrued interest of bond i for delivery date t. The conversion factor associated with each bond is denoted K_i. The bond future notice takes place on $\theta \leq t_0$. The time t futures price is denoted by F_t. Let i^* be the index of the bond delivered. The short party pays the amount

$$F_\theta.K_{i^*} + \mathrm{AccruedInterest}_{i^*}(t_0)$$

on the delivery date.

3.2 Price at expiry

The future price at notice date will reflect the delivery option of the short party. Let $\mathrm{DirtyPrice}_i(t, t_0)$ denote the forward dirty price of the bond i in t for delivery on t_0. On the notice date θ, for a delivery in t_0 (typically the spot date in the relevant market), the relation is

$$\mathrm{DirtyPrice}_{i^*}(\theta, t_0) = F(\theta)K_{i^*} + \mathrm{AccruedInterest}_{i^*}(t_0).$$

The other bonds are more expensive and for all $1 \leq i \leq m$.

$$\mathrm{DirtyPrice}_i(\theta, t_0) \geq F(\theta)K_i + \mathrm{AccruedInterest}_i(t_0).$$

The equality and the inequalities are summarised by

$$\max_{1\leq i\leq m}\left(F(\theta)K_i + \mathrm{AccruedInterest}_i(t_0) - \mathrm{DirtyPrice}_i(\theta, t_0)\right) = 0.$$

This is equivalent to

$$F(\theta) = \min_{1 \le i \le m} \frac{1}{K_i} \left(\text{DirtyPrice}_i(\theta, t_0) - \text{AccruedInterest}_i(t_0) \right). \tag{3}$$

This is the implicit option we have to value to obtain the fair bond futures price.

3.3 Price today

The generic price of a future at date s with price $F(t)$ in t is given by

$$F(s) = \mathrm{E}^{\mathbb{X}} \left[F(t) | \mathcal{F}_s \right] \tag{4}$$

where $\mathbb{X}$ is the collateral account measure associated to the currency of the futures.

In this note, we simplify slightly the issue by considering that the expiry date θ is equal to the bond delivery date and to the swap rate effective date t_0.

As our swap-like model will be written in the θ-forward c-collateral measure, we write the futures price equation also in this measure. The future price is set in term of the bonds at the expiry date θ for delivery also in θ.

$$F(0) = \mathrm{E}^{\mathbb{X}} \left[F(\theta) | \mathcal{F}_0 \right] = B^c(0) \, \mathrm{E}^{\mathbb{X}} \left[\frac{B^c(\theta)}{B^c(\theta)} F(\theta) \right] = P^c(0, \theta) \, \mathrm{E}^{\theta, c} \left[\frac{B^c(\theta)}{P^c(\theta, \theta)} F(\theta) \right]$$

4 Swap rates set-up

The *Annuity Due* is defined by

$$\bar{A}^p(t) = \delta_p P^c(t, t_0) + A^{p-1}(t).$$

It corresponds to the fixed annuity of an in arrears Swap where payments are made at the beginning of each period. Note that we use the first δ on the "wrong period" to facilitate technical lemmas.

With this definition we also have

$$A^p(t) \, \text{Swap}^p(t) = P(t, t_0) - P(t, t_p)$$

$$A^p(t) = A^{p-1}(t) + \delta_p P^c(t, t_p) = A^{p-1}(t) + \delta_p \left(P(t, t_0) - A^p(t) \, \text{Swap}^p(t) \right)$$

Hence

$$A^p(t) = \frac{\delta_p P(t, t_0) + A^{p-1}(t)}{1 + \delta_p \, \text{Swap}^p(t)} = \frac{\bar{A}^p(t)}{1 + \delta_p \, \text{Swap}^p(t)}$$

We can also show that

$$A^p(t) = P^c(t, t_0) \sum_{i=1}^{p} \delta_i \prod_{j=i}^{p} \frac{1}{1 + \delta_j \, \text{Swap}^j(t)} = P^c(t, t_0) A^{*,p}(\text{Swap}^1(t), \ldots, \text{Swap}^p(t)). \tag{5}$$

The annuity can be written as an explicit function of the swap rates (and some discounting from settlement).

5 Measures

Several measures are used in the analysis.

- $\mathbb{Q}^{\mathbb{X}}$ the measure associated to the c cash-account
- $\mathbb{Q}^{\theta,c}$, the θ-forward c-collateral measure associated to $P^c(.,\theta)$.
- $\mathbb{Q}^{A^p}$, the physical annuity of tenor p measure associated to $A^p(.)$
- $\mathbb{Q}^{\bar{A}^p}$, the annuity due of tenor p measure associated to $\bar{A}^p(.)$

The generic change of measure to the θ-forward c-collateral measure is

$$\mathrm{E}^N[Y_t] = \mathrm{E}^N\left[(N_t)^{-1}N_tY_t\right] = P(0,\theta)\,\mathrm{E}^{\theta,c}\left[(P(t,\theta))^{-1}N_tY_t\right]$$

5.1 Annuity Due measure

Lemma 1 (Annuity due change of measure) *For any $1 \le p \le n$, the Radon-Nikodym derivative between the due annuity and the physical annuity is a local function of* $\mathrm{Swap}^p(\theta)$*:*

$$\frac{d\mathbb{Q}^{\bar{A}^p}}{d\mathbb{Q}^{A^p}} = \frac{\bar{A}^p(\theta)}{A^p(\theta)}\frac{A^p(0)}{\bar{A}^p(0)} = \frac{1+\delta_p\,\mathrm{Swap}^p(\theta)}{1+\delta_p\,\mathrm{Swap}^p(0)}$$

Also

$$\begin{aligned}
\rho_p^{\bar{A}^p}(K) &= \frac{1+\delta_pK}{1+\delta_p\,\mathrm{Swap}^p(0)}\rho_p^{A^p}(K) \\
\Phi_p^{\bar{A}^p}(K) &= \frac{1}{1+\delta_p\,\mathrm{Swap}^p(0)}\left((1+\delta_pK)\Phi_p^{A^p}(K) - \delta_p\pi_p^{A^p}(K)\right) &(6)\\
\pi_p^{\bar{A}^p}(K) &= \frac{1}{1+\delta_p\,\mathrm{Swap}^p(0)}\left((1+\delta_pK)\pi_p^{A^p}(K) - \delta_p\,\mathrm{E}^{A^p}\left[\left((K-\mathrm{Swap}^p(\theta))^+\right)^2\right]\right) &(7)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}^{\bar{A}^p}\left[\frac{\mathbb{1}(\mathrm{Swap}^p(\theta)\le K)}{1+\delta_p\,\mathrm{Swap}^p(\theta)}\right] &= \frac{\Phi_p^{A^p}(K)}{1+\delta_p\,\mathrm{Swap}^p(0)} &(8)\\
\mathrm{E}^{\bar{A}^p}\left[\mathrm{Swap}^p(\theta)\right] &= \mathrm{Swap}^p(0) + \frac{\delta_p}{1+\delta_p\,\mathrm{Swap}^p(0)}\mathrm{V}^{A^p}\left[\mathrm{Swap}^p(\theta)\right] &(9)
\end{aligned}$$

We introduce a function Ω_p that characterizes the distribution of $\mathrm{Swap}^p(t_0)$ under A^p (or $\bar{A}^p$) and that will be used in Algorithm 2 when sampling Swap rates in the t_0-forward measure for the purpose of Monte Carlo pricing.

Definition 1 *The function $\Omega_p : [0,1] \to [0,1]$ is defined by*

$$\Omega_p = \Phi_p^{A^p} \circ \left(\Phi_p^{\bar{A}^p}\right)^{-1}$$

5.2 Forward measure

Lemma 2 *For any $p \geq 1$, the Radon-Nikodym derivative $d\mathbb{Q}^{t_0}/dQ^{\bar{A}^p}$ is a functional of* $(\mathrm{Swap}^1(\theta), \dots, \mathrm{Swap}^{p-1}(\theta))$

$$\frac{d\mathbb{Q}^{t_0}}{dQ^{\bar{A}^p}} = \frac{\bar{A}^p(0)}{P^c(0,t_0)} \frac{P(\theta, t_0)}{\bar{A}^p(\theta)}$$

with

$$\begin{aligned} \bar{A}^p(\theta) &= P^c(\theta, t_0)\bar{A}^{*,p}(\mathrm{Swap}^1(\theta), \dots, \mathrm{Swap}^{p-1}(\theta)) & (10)\\ &= P^c(\theta, t_0)\left(\delta_p + A^{*,p-1}(\theta)(\mathrm{Swap}^1(\theta), \dots, \mathrm{Swap}^{p-1}(\theta))\right). & (11) \end{aligned}$$

The cumulative is then

$$\Phi_p^{\bar{A}^p}(K) = \frac{P^c(0,t_0)}{\bar{A}^p(0)} \mathrm{E}^{t_0,c}\left[\bar{A}^{*,p}(\mathrm{Swap}^1(\theta), \dots, \mathrm{Swap}^{p-1}(\theta))\mathbb{1}(\mathrm{Swap}^p(\theta) \leq K)\right]$$

5.3 Cap/floor in arrears

In the new overnight world, some cap/floor are written as cap/floor "in-arrears", i.e. the option is written on the compounded rate at the end of the accrual period. If we take a period that starts today up to date θ, paid in $\bar{t}_0 \geq \theta$ and with a strike K and an accrual factor δ, the price of a caplet, in the $\bar{t}_0$-forward c-collateral measure, is

$$\mathrm{Floor}(K, \theta) = P^c(0, t_0)\delta\, \mathrm{E}^{\bar{t}_0,c}\left[\max(K - R(\theta), 0)\right]$$

where $R(\theta) = (B(\theta) - 1)/\delta$.

This means that the short term cap/floor market gives us the distribution of our new term $B(\theta)$ directly in the measure we will use for the pricing. The smile can be described through a SABR model as described in Willems (2020).

What we need is the price of an option on B with strike K^B

$$P^c(0, t_0)\pi_0^{\bar{t}_0,c} = P^c(0, t_0)\, \mathrm{E}^{\bar{t}_0,c}\left[\max(K^B - B(\theta), 0)\right] = \mathrm{Floor}((K^B - 1)/\delta, \theta).$$

For the floor part, we ignore the fact that t_0 and $\bar{t}_0$ may be different (by one day in USD but the same in EUR) and the distinction between $\bar{t}_0$ and θ. This will impact (in an almost negligible way) the convexity adjustment coming from $B(\theta)$.

Note that in practice we don't have caplet with exactly the required maturity date θ. The market trade caps, i.e. strips of caplet, with standard tenors, typically one year is the shorter one. In our numerical examples, we ignore the practical details on how to extract from the market the relevant information and suppose that we have directly the prices of the in-arrears caplet starting today and with expiry θ.

6 Swap rates and cash account joint distribution

Cash account and swap rates joint distribution is described by a copula function C

$$\mathrm{E}^{t_0,c}\left[\mathbb{1}(B(t_0) \leq K_0) \prod_{i=1}^{n} \mathbb{1}(\mathrm{Swap}^i(\theta) \leq K_i)\right] = C(\Phi_0^{t_0}(K_0), \Phi_1^{t_0}(K_1), \ldots, \Phi_n^{t_0}(K_n)).$$

Definition 2 (Bank Account and Swaps density) *We define $(U_{t_0}^i)_{0\leq i\leq n}$ with $U_{t_0}^0 = \Phi_0^{t_0}(B(\theta))$ and $U_{t_0}^i = \Phi_i^{t_0}(\mathrm{Swap}^i(\theta))$ for $1 \leq i \leq n$ the vector of uniform random variables in the t_0-forward measure correlated according to the copula function $C(u^0, u^1, \ldots, u^n)$.*

Lemma 3 (Recursive density) *The measure associated to $\bar{A}_1$ is the same as the one associated to the t_0-forward measure and*

$$\left(\Phi_1^{t_0}\right)^{-1}(q) = \left(\Phi_1^{\bar{A}_1}\right)^{-1}(q)$$

For $p \geq 2$, the following recursive relation holds

$$\left(\Phi_p^{t_0}\right)^{-1}(q) = \left(\Phi_p^{\bar{A}_p}\right)^{-1}(Z^p(q))$$

where

$$Z^p(q) = \frac{1}{\mathrm{E}^{t_0}\left[\frac{\bar{A}^p(\theta)}{P^c(\theta,t_0)}\right]} \mathrm{E}^{t_0}\left[\bar{A}^{*,p}(\mathrm{Swap}^1(\theta), \ldots, \mathrm{Swap}^{p-1}(\theta)) c_p\left(U_{t_0}^0, U_{t_0}^1, \cdots, U_{t_0}^{p-1}, q\right)\right] \tag{12}$$

and

$$c_p\left(u^0, u^1, \cdots, u^{p-1}, u^p\right) = \mathrm{E}^{t_0}\left[\mathbb{1}(U_{t_0}^p \leq u^p) \middle| U_{t_0}^0 = u^0, U_{t_0}^1 = u^1, \ldots, U_{t_0}^{p-1} = u^{p-1}\right]$$

The cumulative density function of $\mathrm{Swap}^p(t_0)$ under the t_0-forward measure may be computed iteratively using Equation (12) as it has dependency only on the previous swap rates $(\mathrm{Swap}^1(\theta), \ldots, \mathrm{Swap}^{p-1}(\theta))$.

Note that we write Z using the expected value of the Randon-Nikodym derivatives instead of its known value. This is to be used in numerical implementation. If we used the same simulated values for the two expectation, we avoid probability above one.

Proof: We have

$$
\begin{aligned}
\Phi_p^{\bar{A}_p}(K) &= \mathrm{E}^{\bar{A}_p}\left[\mathbb{1}(\mathrm{Swap}^p(\theta) \le K)\right] \\
&= \frac{P^c(0,t_0)}{\bar{A}^p(0)} \mathrm{E}^{t_0}\left[\frac{\bar{A}^p(\theta)}{P^c(\theta,t_0)} \mathbb{1}(\mathrm{Swap}^p(\theta) \le K)\right] \\
&= \frac{1}{\mathrm{E}^{t_0}\left[\frac{\bar{A}^p(\theta)}{P^c(\theta,t_0)}\right]} \mathrm{E}^{t_0}\left[\frac{\bar{A}^p(\theta)}{P^c(\theta,t_0)} \mathbb{1}((\Phi_p^{t_0})^{-1}(U_{t_0}^p) \le K)\right] \\
&= \frac{1}{\mathrm{E}^{t_0}\left[\frac{\bar{A}^p(\theta)}{P^c(\theta,t_0)}\right]} \mathrm{E}^{t_0}\left[\frac{\bar{A}^p(\theta)}{P^c(\theta,t_0)} \mathrm{E}^{t_0}\left[\mathbb{1}(U_{t_0}^p \le \Phi_p^{t_0}(K)) \middle| U_{t_0}^0, U_{t_0}^1, \cdots, U_{t_0}^{p-1}\right]\right] \\
&= \frac{1}{\mathrm{E}^{t_0}\left[\frac{\bar{A}^p(\theta)}{P^c(\theta,t_0)}\right]} \mathrm{E}^{t_0}\left[\frac{\bar{A}^p(\theta)}{P^c(\theta,t_0)} c_p\left(U_{t_0}^0, U_{t_0}^1, \cdots, U_{t_0}^{p-1}, \Phi_p^{t_0}(K)\right)\right]
\end{aligned}
$$

where in the fourth step, we have used the $(U_{t_0}^0, U_{t_0}^1, \cdots, U_{t_0}^{p-1})$-measurability of $\bar{A}^{*,p}(\theta)$. If we set $q = \Phi_p^{t_0}(K)$ and we apply $(\Phi_p^{\bar{A}_p})^{-1}$ on both side of the equality, we obtain the result. □

Consider the vector U_{t_0} of uniform variables in the t_0-forward measure with copula C. For that variable, we generate a sample of dimension N_s: $(U^{0,i}, U^{1,i}, \cdots, U^{n,i})_{i=1,\ldots,N_s}$. The following direct algorithm generates paths for the swap rates in the t_0-forward measure.

Algorithm 1 *Samples generation of by direct method.*

Cash account

For $i = 1$ to N_s (iteration on paths)

$$B^i = \left(\Phi_0^{t_0}\right)^{-1}(U^{0,i})$$

End i

For $p = 1$ to n (iteration on swap tenors)

For $i = 1$ to N_s (iteration on paths)

$$\xi^{p,i} = \frac{\bar{A}^{*,p}(\mathrm{Swap}^{1,i}, \ldots, \mathrm{Swap}^{p-1,i})}{\sum_{k=1}^{N_s} \bar{A}^{*,p}(\mathrm{Swap}^{1,k}, \ldots, \mathrm{Swap}^{p-1,k})}$$

$$\mathrm{Swap}^{p,i} = \left(\Phi_p^{\bar{A}_p}\right)^{-1}(Z^{p,i}) \textit{ with } Z^{p,i} = \textstyle\sum_{j=1}^{N_s} \xi^{p,j} c_p\left(U^{0,j}, U^{1,j}, \cdots, U^{p-1,j}, U^{p,i}\right)$$

End i

End p

Note that $\xi^{p,1} = 1/N_s$.

Lemma 4 (Re-sampling) *Consider N_s realisations of the vectors $U_{t_0}^i$: $(U^{0,i}, U^{1,i}, \ldots, U^{n,i})_{1\le i\le N_s}$. For $p \ge 1$, assume $(\mathrm{Swap}^{1,i}, \ldots, \mathrm{Swap}^{p-1,i})_{1\le i\le N_s}$ to be known and set $\xi^{p,i}$ according to the equation in Algorithm 1. Denote σ_p a permutation that orders realizations of U^p: $U^{p,\sigma_p(1)} \le U^{p,\sigma_p(2)} \le \cdots \le U^{p,\sigma_p(N_s)}$. Consider the sampling*

$$\mathrm{Swap}^{p,i} = \frac{1}{\delta_p}\left(\frac{\xi^{p,i}(1 + \delta_p \mathrm{Swap}^p(0))}{\Omega_p(q_{\sigma_p^{-1}(i)}) - \Omega_p(q_{\sigma_p^{-1}(i)-1})} - 1\right)$$

with $q_i = \sum_{j=1}^{i} \xi^{p,\sigma_p(j)}$.

Then the set of nodes $(\mathrm{Swap}^{p,i})_{1\le i\le N_s}$ *enjoys the following properties*

1. *It preserve the overall correlation structure defined by the Copula* C*.*
2. *It ensures the empirical CDF of* $\mathrm{Swap}^p(\theta)$ *in both* Annuity *and* Annuity Due *measures are consistent with their theoretical values for the set of* N_s *points* $K_i = \left(\Phi_p^{\bar{A}_p}\right)^{-1}(q_i)$*.*
3. *Receivers swaptions are guaranteed to be empirically repriced for the same set of strikes.*

Algorithm 2 *Quadrature nodes consistent with measures* A *and* $\bar{A}$*.*

Cash account

For $i = 1$ *to* N_s *(iteration on paths)*

$$B^i = \left(\Phi_0^{t_0}\right)^{-1}(U^{0,i})$$

End i

For $p = 1$ *to* n *(iteration on swap tenor)*

Set permutation σ_p *such that* $U^{p,\sigma_p(1)} \le U^{p,\sigma_p(2)} \le U^{p,\sigma_p(N_s)}$

For $i = 1$ *to* N_s *(iteration on paths)*

$$\xi^{p,i} = \frac{\bar{A}^{*,p}(\mathrm{Swap}^{1,i},\ldots,\mathrm{Swap}^{p-1,i})}{\sum_{k=1}^{N_s} \bar{A}^{*,p}(\mathrm{Swap}^{1,k},\ldots,\mathrm{Swap}^{p-1,k})}$$

$$\mathrm{Swap}^{p,i} = \frac{1}{\delta_p}\left(\frac{\xi^{p,i}(1+\delta_p\,\mathrm{Swap}^p(0))}{\Omega_p\left(\sum_{j=1}^{\sigma_p^{-1}(i)} \xi^{p,\sigma_p(j)}\right) - \Omega_p\left(\sum_{j=1}^{\sigma_p^{-1}(i)-1} \xi^{p,\sigma_p(j)}\right)} - 1\right)$$

End i

End p

Note 1: A reordering mechanism could also be used for the cash account B^i instead of the "simple" density inverse.

Note 2: $\xi^{1,i} = 1/N_s$ and $\sum_{j=1}^{\sigma_1^{-1}(i)} \xi^{1,\sigma_p(j)} = \sigma_1^{-1}(i)/N_s$. The $\mathrm{Swap}^{1,i}$ values are the initial value $\mathrm{Swap}^1(0)$ adjusted by some change of measure embedded in Ω_p at regularly spaced points k/N_s.

Note 3: The random draw is used only in ordering the $U^{p,\cdot}$. When ordered, the condition $\mathrm{Swap}^{p,j} \le K$ is embedded in the partial sums $\sum_{j=1}^{\sigma_p^{-1}(i)}$.

The value of a swap rate/bond dependent futures with expiry in θ is

$$P^c(0,\theta)\,\mathrm{E}^{\theta,c}\left[B(\theta)f(\mathrm{Swap}^1(\theta),\ldots,\mathrm{Swap}^n(\theta))\right].$$

In the formula, f is the min from Formula (3) and the conversion between the OIS discount factors and the government/repo rates from Formula (2).

Using one of the above algorithms to generate the simulations B^i and $\mathrm{Swap}^{p,i}$, the value is

$$P^c(0,\theta)\frac{1}{N_s}\sum_{i=1}^{N_s} B^i f(\mathrm{Swap}^{1,i},\ldots,\mathrm{Swap}^{n,i}).$$

7 Numerical examples

We have implemented the above model and algorithm. We present several examples of results obtained from it. On the rate side, we use flat rate curves at different levels. On the volatility side, we use a shifted SABR model for the pricing of vanilla swaptions and cap/floors. The SABR surface have constant α, β, ν, ρ and shift parameters and linear interpolation. The starting parameters are $\alpha = 0.02$, $\beta = 0.00$, $\nu = 0.50$, and $\rho = -0.25$.

In our examples, we use a synthetic bond futures with four underlyings bonds. The valuation date is 2022-01-30 and the futures expiry is 2022-11-30 (December contract, 10 months in the future). The bonds have maturities 2029-08-15, 2030-08-15, 2031-08-15, and 2032-08-15 (from a little bit above 6.5 years to a little bit less than 10 years). The coupons of the underlying bonds are all the same, the coupon selection is different for each test.

In most of the tests, we use a correlation matrix inspired by the one used in Bang and Daboussi (2022), to which we have added the correlation to the cash account. The matrix is described in Figure 1.

	Cash Accoun	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Cash Accoun	1.00	0.60	0.59	0.58	0.57	0.56	0.55	0.54	0.53	0.52	0.51
1Y	0.60	1.00	0.97	0.94	0.91	0.88	0.85	0.83	0.80	0.75	0.70
2Y	0.59	0.97	1.00	0.97	0.94	0.91	0.88	0.85	0.83	0.80	0.75
3Y	0.58	0.94	0.97	1.00	0.97	0.94	0.91	0.88	0.85	0.83	0.80
4Y	0.57	0.91	0.94	0.97	1.00	0.97	0.94	0.91	0.88	0.85	0.83
5Y	0.56	0.88	0.91	0.94	0.97	1.00	0.97	0.94	0.91	0.88	0.85
6Y	0.55	0.85	0.88	0.91	0.94	0.97	1.00	0.97	0.94	0.91	0.88
7Y	0.54	0.83	0.85	0.88	0.91	0.94	0.97	1.00	0.97	0.94	0.91
8Y	0.53	0.80	0.83	0.85	0.88	0.91	0.94	0.97	1.00	0.97	0.94
9Y	0.52	0.75	0.80	0.83	0.85	0.88	0.91	0.94	0.97	1.00	0.97
10Y	0.51	0.70	0.75	0.80	0.83	0.85	0.88	0.91	0.94	0.97	1.00

Figure 1: Test correlation matrix.

7.1 SABR sensitivities

In this first example, the rate curve is flat at 4.50% and all the underlying bonds coupons are also at 4.50%. The sensitivities to the α parameters, split by expiry and tenors, is provided in Table 1. The sensitivity is for a position long one contract and is express in USD as the derivative with respect to α.

Globally the sensitivity is negative as the party long the futures is short the delivery option. As expected, sensitivities are between the 6-month and 1-year expiries, with more sensitivities on the 1-year tenor.

The bond features that are important for the cheapest-to-deliver (CTD) bond to change are, in a simplified way, its total volatility, which can be viewed as its duration multiplied by its yield/rate volatility. When the figure is higher, there will be more change from one bond to another.

Expiry	Tenor 1Y–4Y	5Y	6Y	7Y	8Y	9Y	10Y	Total
0.00	0	0	0	0	0	0	0	0
0.50	-146	-78	6,858	20,799	-13,011	-25,473	-42,165	-53,216
1.00	-284	-152	13,329	40,424	-25,287	-49,507	-81,951	-103,428
2.00	0	0	0	0	0	0	0	0
3.00	0	0	0	0	0	0	0	0
Total	-431	-230	20,187	61,222	-38,297	-74,980	-124,116	-156,644

Table 1: SABR α sensitivity

In the sensitivity bucketing by tenor, we can see that the 6-year and 7-year tenor risks are positive and the 8-year to 10-year tenors risks are negative. The current rate is 4.5% and is below the reference rate of 6.00%. The shorter maturity bond will be the CTD in absence of market change. Adding volatility to the short part of the delivery spectrum delay the change of CTD from the current short term one to the longer term ones. Increasing the longer term rate volatilities bring them faster to the CTD status.

We can compare those figures to a 10-year tenor swaptions with same expiry and at-the-money coupon. The the swaption, the sensitivity appears only in the 10-year tenor and is 96,456 at the 6-month expiry and 187,467 at the 1-year expiry for a total of 283,923. The swaption volatility risk is roughly twice the bond futures volatility risk.

The ν and ρ sensitivities are provided in Table 2 and 3.

Expiry	Tenor 1Y–4Y	5Y	6Y	7Y	8Y	9Y	10Y	Total
0.50	0	0	16	48	-26	-53	-87	-102
1.00	0	0	32	94	-50	-103	-170	-198
Total	-1	-1	48	143	-76	-156	-258	-300

Table 2: SABR ν sensitivities

Expiry	Tenor 1Y–4Y	5Y	6Y	7Y	8Y	9Y	10Y	Total
0.50	-1	-1	4	15	4	-30	-103	-112
1.00	-4	-2	7	29	8	-58	-199	-218
Total	-4	-3	11	44	12	-87	-302	-330

Table 3: SABR ρ sensitivities

7.2 Convexity adjustment

Even in absence of delivery optionality, the volatility has an impact through the "convexity adjustment". We measure the impact of volatility in the case of a unique underlying. For this we use a unique bond with a maturity 2032-08-15 and a coupon of 4.50%. The dates and rates are the same as in the previous example. We use a correlation between the different rates of 99.9%. This is to compare more easily to the Hull-White one-factor model. The impact of correlation change is analysed in Section 7.4.

The total α sensitivity – not detailing the bucketing – is -8,131 USD for the bond future using the approach described in this note. For the Hull-White model, using the formula developed in Henrard (2006), the σ sensitivity is -12,292 USD. We obtain a similar order of magnitude volatility risk in both cases. For expiries beyond a couple of months, the convexity adjustment in this model is non-negligible and is comparable to the adjustments seen in other models.

7.3 Rate level impact

As indicated previously, the futures reference rate – 6.00% in the USD bond futures case – act as some type of strike rate. We look at the impact of the curve rate level on the volatility risk. We see clearly a higher sensitivity around the 6.00% mark. A similar change of risk level would be visible on swaptions with a 6.00% strike.

Rate	Alpha	Nu	Rho
2.00%	-86,249	-269	-471
2.50%	-97,676	-259	-460
3.00%	-112,435	-259	-451
3.50%	-122,161	-257	-427
4.00%	-134,125	-273	-375
4.50%	-139,189	-281	-302
5.00%	-138,288	-302	-237
5.50%	-138,886	-355	-150
6.00%	-133,289	-343	-101
6.50%	-125,723	-391	-9
7.00%	-119,549	-389	47
7.50%	-108,662	-396	83
8.00%	-101,440	-410	110
8.50%	-90,315	-430	127
9.00%	-78,260	-417	149
9.50%	-73,142	-413	149
10.00%	-62,508	-396	141

Table 4: Sensitivities to the different SABR parameters are different rate levels.

7.4 Correlation impact

The actual value of the delivery option depends not only on the rate level, the rates and the volatility but also on the correlations. If rates are strongly correlated, the curve move roughly in parallel and exercise happen only when the rate level reaches the reference rate. When rates are de-correlated, the changing shape plays a role and the change of cheapest-to-deliver can happen more easily. To represent this we run the pricing of the same bond futures with the same starting yield curve but different correlation structures. In all case, we have an expiry in 10 months, a starting yield curve flat at 2%, underlyings with coupons at 6%, and SABR with $\alpha = 0.02$. We change the correlation and notice in each case the portion (in %) of the scenarios in which each of the 4 underlying bonds is exercised.

Correlation	Price	Bond 1	Bond 2	Bond 3	Bond 4
99.9%	124.54	96.77	0.73	0.43	2.07
As above	124.17	81.36	6.84	4.75	7.05
75.0%	122.58	58.00	21.18	12.42	8.40

Table 5: Correlation impact

As can be seen a de-correlation of the rates creates more changes in CTD and a higher option price (lower futures price). With low correlation, even far away from the reference rate, the change in CTD is not negligible at all.

8 Conclusion

Bond futures delivery mechanisms is complex. Valuing the optionality in the quality option is significantly more involved from a technical perspective than a swaption or bond option. It is to some extend similar to the a CMS spread option as it involves several maturities.

The recent moves in government rates have brought the bond futures closer to the "strike" rates than in the last decade. The delivery optionality may have been "forgot" by market participants over that period.

We have extended previous results initially developed for CMS products to adapt them to futures and the delivery option of bond futures in particular. The approach allows to estimate the impact of the full smile of all maturities and the interaction between different rates on the convexity adjustment and the delivery optionality.

Numerical examples show the impacts of those different components. They revile that volatility term structure, smile, and correlations have all a significant impacts in some circumstances. Trading bond futures, specially on a relative basis, cannot be envisaged anymore without reassessing those issues in detail.

A Implementation details

A.1 Cumulative density

The price of a vanilla receiver (put) swaption is

$$\text{Swaption}(K,\theta) = A^p(0)\,\mathrm{E}^{A^p}\left[(K - \text{Swap}^p(\theta)\right] = A^p(0)\pi_p^{A^p}(K).$$

The cumulative density in the A^p-associated measure is the price of a digital and can be approximated with vanilla receiver with

$$\begin{aligned}
\Phi_p^{A^p}(K) &= \mathrm{E}^{A^p}\left[\mathbb{1}(\text{Swap}^p(\theta) \leq K)\right] \\
&\simeq \mathrm{E}^{A^p}\left[\frac{1}{K_+ - K_-}\left((K_+ - \text{Swap}^p(\theta))^+ - (K_- - \text{Swap}^p(\theta))^+\right)\right] \\
&= \frac{1}{A^p(0)}\frac{1}{K_+ - K_-}\left(\text{Swaption}(K_+,\theta) - \text{Swaption}(K_-,\theta)\right)
\end{aligned}$$

The cumulative density in the $\bar{A}^p$-associated measure is given (see Equation (6)) by

$$\Phi_p^{\bar{A}^p}(K) = \frac{1}{1 + \delta_p\,\text{Swap}^p(0)}\left((1+\delta_p K)\Phi_p^{A^p}(K) - \delta_p \pi_p^{A^p}(K)\right).$$

The two cumulative densities can be easily tabulated for a set of strikes and obtained through linear interpolation.

The inverse function $(\Phi_p^{\bar{A}^p})^{-1}$ can be obtained from the same table read in the opposite direction also by linear interpolation. We use a linear interpolation as the inverse of a linear interpolation is also a linear interpolation. TODO: Impact of interpolation and extrapolation?

The function Ω_p is obtained by the composition of the above and limits in 0 and 1. No extrapolation is needed as we know the extreme values at 0 and 1.

The swap depends of the difference of Ω_p at two close points, given by two sums of $\xi^{p,\sigma_p(j)}$; if a linear interpolation is used in the discretisation of Ω_p, the swap rates will cluster around the values of the constant slopes between the discretisation nodes. A smoother interpolator is required to avoid steps in the value profile.

A.2 Annuities

The annuities as functions of the swap rates are given (see Equation (5)) by

$$A^{*,p}(\text{Swap}^1(\theta),\ldots,\text{Swap}^p(\theta)) = \sum_{i=1}^{p}\delta_i\prod_{j=i}^{p}\frac{1}{1+\delta_j\,\text{Swap}^j(\theta)}$$

and

$$\bar{A}^{*,p}(\text{Swap}^1(\theta),\ldots,\text{Swap}^{p-1}(\theta)) = \delta_p + A^{*,p-1}(\text{Swap}^1(\theta),\ldots,\text{Swap}^{p-1}(\theta)).$$

The discount factors at the swap maturities are given by

$$\frac{P^c(\theta, t_p)}{P^c(\theta, t_0)} = 1 - A^{*,p}(\text{Swap}^1(\theta), \ldots, \text{Swap}^p(\theta))\,\text{Swap}^p(\theta).$$

If we use log-linear interpolation on discount factors for $t_i < t < t_{i+1}$, the discount factors at intermediary times are given by

$$\frac{P^c(\theta, t)}{P^c(\theta, t_0)} = \frac{P^c(\theta, t_i)}{P^c(\theta, t_0)} \left(\frac{P^c(\theta, t_{i+1})}{P^c(\theta, t_i)} \right)^{\frac{t-t_i}{t_{i+1}-t_i}}$$

A.3 Algorithmic differentiation

Our implementation relies on Algorithmic Differentiation as described in Henrard (2017).

U: no dependency on rates or volatilities (only correlations)

$\left(\Phi_p^{\bar{A}_p}\right)^{-1}$: y-values are swap rates and volatility dependent

Ω_p: y-values are swap rates and volatility dependent. The y-values are themselves dependent on $\Phi_p^{A_p}$ and $\left(\Phi_p^{\bar{A}_p}\right)^{-1}$.

This means that in the calibration step we compute not only the present values of the digital and vanilla swaptions, but also their dependency on the market parameters (SABR parameters in our case) to be able to perform the AD step.

The approximate computation times on the author laptop are:

- Calibration with swaps up to 10 years and 300 points for ϕ: 500 ms
- Pricing for 10 years bond futures with 4 underlyings and 10,000 Monte Carlo simulations: 250 ms
- Pricing as above and SABR sensitivities where the SABR surfaces have 10 tenors and 5 expiries for the 4 parameters: 1200 ms

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