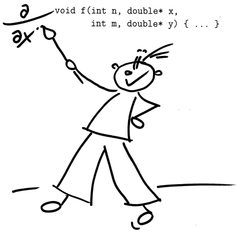

(Adjoint) Algorithmic Differentiation [(A)AD]

A Hands-On Introduction

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and

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Progress

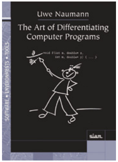
The Art of Differentiating Computer Programs

Bumping?

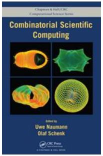
...

For differentiation, is there anything else?
Perturbing the inputs – can't imagine this fails.
I pick a small Epsilon, and I wonder ...

...



from: "Optimality" (Lyrics: Naumann; Music: Think of Fool's Garden's "Lemon Tree") in Naumann: The Art of Differentiating Computer Programs. An Introduction to Algorithmic Differentiation. Number 24 in Software , Environments, and Tools, SIAM, 2012. Page xvii



- inspired by sensitivity analysis, uncertainty quantification, calibration / optimization
- finite differences (first- and second-order), symbolic, algorithmic
- implementation by overloading, source trafo, hand-coding
- real code
- sensitivity analysis as modelling and software engineering tool
- what matters
  - user expertise
  - tool quality
  - tool sustainability and support

Let $y = F(\mathbf{x}),\ F : \mathbb{R}^n \rightarrow \mathbb{R}$:

1. tangent AD: $y^{(1)} = \nabla F \cdot \mathbf{x}^{(1)} \Rightarrow \nabla F$ at $O(n) \cdot \text{Cost}(F)$
2. adjoint AD: $\mathbf{x}_{(1)} = \nabla F^T \cdot y_{(1)} \Rightarrow \nabla F$ at $O(1) \cdot \text{Cost}(F)$
3. 2nd-order tangent AD: $y^{(1,2)} = {\mathbf{x}^{(1)}}^T \cdot \nabla^2 F \cdot \mathbf{x}^{(2)} \Rightarrow \nabla^2 F$ at $O(n^2) \cdot \text{Cost}(F)$
4. 2nd-order adjoint AD: $\mathbf{x}^{(2)}_{(1)} = y_{(1)} \cdot \nabla F^2 \cdot \mathbf{x}^{(2)} \Rightarrow \nabla^2 F$ at $O(n) \cdot \text{Cost}(F)$ and $\nabla^2 F \cdot \mathbf{x}^{(2)}$ at $O(1) \cdot \text{Cost}(F)$

Aims of this Course

You will learn how to

- implement tangent and adjoint versions of a Monte Carlo / Euler-Maruyama solver for parameterized scalar SDEs
- ensure feasibility of adjoint Monte Carlo simulation through pathwise adjoints
- “get serious” with AAD (tools, checkpointing, symbolic adjoints, design patterns, ...)

We are looking for the expected value $\mathbb{E}(x)$ of the solution $x(\mathbf{p},T), T>0$ of the scalar stochastic initial value problem

$$dx = f(x(\mathbf{p},t),\mathbf{p},t))dt + g(x(\mathbf{p},t),\mathbf{p},t)dW$$

with Brownian Motion dW and for $x(\mathbf{p},0)=x^0$.

Forward finite differences in time with time step $0<\delta t \ll 1$ yield the explicit Euler-Maruyama evolution

$$x^{i+1} := x^i + \delta t \cdot f(x^i,\mathbf{p},i\cdot\delta t) + \sqrt{\delta t}\cdot g(x^i,\mathbf{p},i\cdot\delta t)\cdot dW^i$$

for $i=0,\ldots,n-1$, target time $T=n\cdot\delta t$, parameter vector $\mathbf{p}\in\mathbb{R}^l$, and with random numbers dW^i drawn from the standard normal distribution $N(0,1)$.

The solution $\mathbb{E}(x)$ is approximated using Monte Carlo simulation over (a sufficiently large number of) Euler-Maruyama paths.

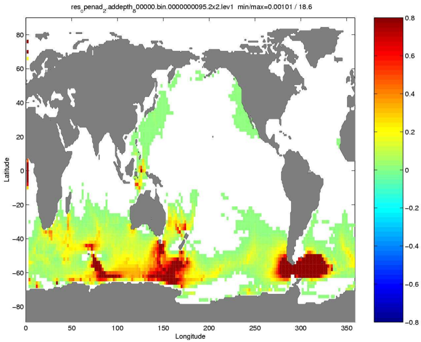
We are interested in sensitivities of the final state $\mathbb{E}(x)$ wrt. $\mathbf{p}$.

Race (Euler-Maruyama $m = 10^4, n = 10^2, l = 10^2$)

- primal: `primal.cpp` (inspect)
- bumping: `fd.cpp` (inspect)
- tangent: `tangent.cpp` (live)
- vector tangent: `tangent_vector.cpp` (inspect)
- adjoint: `adjoint.cpp` (live)
- pathwise adjoint: `adjoint_pathwise.cpp` (inspect)

mode	run time (s)	memory size (b)	accuracy
bump	$10.9 \sim O(l)$	$\sim P$	-
tangent	$21.5 \sim O(2 \cdot l)$	$\sim 2 \cdot P$	+
vector tangent	$13.6 \sim O(2 \cdot l)$	$\sim P + P \cdot l$	+
adjoint	$0.3 \sim O(1)$	$\sim 2 \cdot P + 2 \cdot m \cdot n \cdot 8$	+
pathwise adjoint	$0.5 \sim O(1)$	$\sim 2 \cdot P + 2 \cdot (m + n) \cdot 8$	+

where P denotes the memory requirement of the primal code.



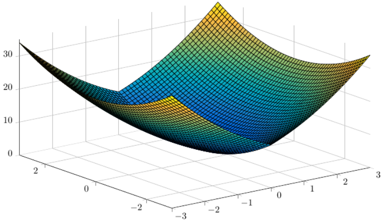
MITgcm, (EAPS, MIT)

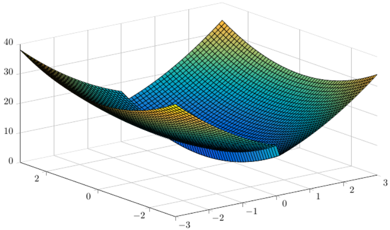
in collaboration with ANL, MIT, Rice, UColorado

J. Utke, U.N. et al: OpenAD/F: A modular, open-source tool for automatic differentiation of Fortran codes . ACM TOMS 34(4), 2008.

Plot: A tangent computation / finite difference approximation for 64,800 grid points at 1 min each would keep us waiting for a month and a half ... :-(((We can do it in less than 10 minutes thanks to adjoints computed by a differentiated version of the MITgcm :-)

- continuity
- differentiability?





- gradient, Jacobian, Hessian, higher-order derivative tensors
- Taylor expansion
- chain rule

Progress

1. The given implementation of $F : \mathbb{R}^n \rightarrow \mathbb{R}^m : \mathbf{y} = F(\mathbf{x})$, can be decomposed into a single assignment code (SAC)

$$\begin{aligned} v_i &= \varphi_i(x_i) = x_i & i &= 0, \ldots, n-1 \\ v_j &= \varphi_j\left((v_k)_{k \prec j}\right) & j &= n, \ldots, n+q-1 \\ y_k &= \varphi_{n+q+k}(v_{n+p+k}) = v_{n+p+k} & k &= 0, \ldots, m-1 \end{aligned}$$

   where $q = p + m$ and $k \prec j$ denotes a direct dependence of v_j on v_k as an argument of φ_j.

2. All elemental functions φ_j possess continuous partial derivatives

$$d_{j,i} \equiv \frac{d\varphi_j}{dv_i}(v_k)_{k \prec j}$$

   with respect to their arguments $(v_k)_{k \prec j}$ at all points of interest.

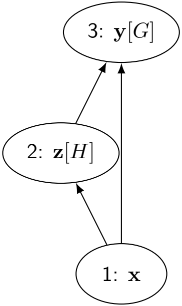
3. A linearized SAC (lSAC) is obtained by augmenting the elemental assignments with computations of the local partial derivatives $d_{j,i}$.
4. The SAC induces a directed acyclic graph (DAG) $G = G(F) = (V, E)$ with integer vertices $V = \{0, \ldots, n+q\}$ and edges $V \times V \supseteq E = \{(i,j) : i \prec j\}$.
5. The set of vertices representing the n inputs is denoted as $X \subseteq V$. The m outputs are collected in $Y \subseteq V$. All remaining intermediate vertices belong to $Z \subsetneq V$.
6. A linearized DAG (lDAG) is obtained by attaching the $d_{j,i}$ to the corresponding edges (i,j) in the DAG.

First-Order (A)AD

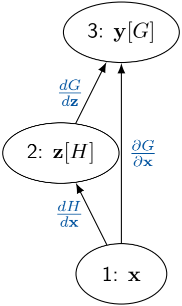
Prerequisites: Chain Rule on IDAG

SAC:
$$\begin{aligned} \mathbf{z} &:= H(\mathbf{x}) \\ \mathbf{y} &:= G(\mathbf{z}, \mathbf{x}) \end{aligned}$$

DAG:



IDAG:



$$\nabla F(\mathbf{x}) \equiv \frac{d\mathbf{y}}{d\mathbf{x}} = \sum_{\text{path} \in \text{IDAG}} \prod_{(i,j) \in \text{path}} d_{j,i}$$

- U. N.: *Optimal Jacobian accumulation is NP-complete.* Math. Prog. 112(2):427–441, Springer, 2008.

  Proof by reduction from ENSEMBLE COMPUTATION

- U. N.: *Optimal accumulation of Jacobian matrices by elimination methods on the dual computational graph.* Math. Prog. 99(3):399–421, Springer, 2004.

  Example: bat graph in STCE logo

- A. Griewank and U. N.: *Accumulating Jacobians as chained sparse matrix products.* Math. Prog. 95(3):555–571, Springer, 2003.

  Example: $\mathbb{R}^4 \to \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}^4$

Tangents

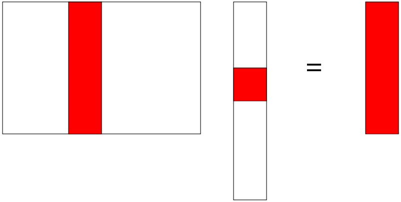
Mathematician's View

A first-order tangent model $F^{(1)} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^m$,

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{y}^{(1)} \end{pmatrix} = F^{(1)}(\mathbf{x}, \mathbf{x}^{(1)}),$$

defines a directional derivative alongside with the function value:

$$\mathbf{y} = F(\mathbf{x})$$
$$\mathbf{y}^{(1)} = \nabla F(\mathbf{x}) \cdot \mathbf{x}^{(1)}$$



... definition of the whole Jacobian column-wise by input directions $\mathbf{x}^{(1)} \in \mathbb{R}^n$ equal to the Cartesian basis vectors in $\mathbb{R}^n$.

A first-order tangent code $F^{(1)} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{\tilde{m}}$

$$\begin{pmatrix} \mathbf{z} \\ \mathbf{z}^{(1)} \\ \tilde{\mathbf{z}} \\ \mathbf{y} \\ \mathbf{y}^{(1)} \\ \tilde{\mathbf{y}} \end{pmatrix} := F^{(1)}(\mathbf{x}, \mathbf{x}^{(1)}, \tilde{\mathbf{x}}, \mathbf{z}, \mathbf{z}^{(1)}, \tilde{\mathbf{z}}),$$

computes a Jacobian $\times$ vector product alongside with the function value:

$$\mathbb{R}^m \times \mathbb{R}^{\tilde{m}} \ni \begin{pmatrix} \mathbf{z} \\ \tilde{\mathbf{z}} \\ \mathbf{y} \\ \tilde{\mathbf{y}} \end{pmatrix} := F(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{z}, \tilde{\mathbf{z}})$$

$$\mathbb{R}^m \ni \begin{pmatrix} \mathbf{z}^{(1)} \\ \mathbf{y}^{(1)} \end{pmatrix} := \nabla F(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{z}, \tilde{\mathbf{z}}) \cdot \begin{pmatrix} \mathbf{x}^{(1)} \\ \mathbf{z}^{(1)} \end{pmatrix}$$

Variables for which derivatives are computed are referred to as active; $\mathbf{x}$ and $\mathbf{z}$ are active inputs; $\mathbf{z}$ and $\mathbf{y}$ are active outputs.

Variables which depend on active inputs are referred to as varied.

Variables for which no derivatives are computed are referred to as passive; $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{z}}$ are passive inputs; $\tilde{\mathbf{z}}$ and $\tilde{\mathbf{y}}$ are passive outputs.

Variables on which active outputs depend are referred to as useful.

Active variables are both varied and useful.

The whole (dense) Jacobian can be *harvested* column-wise from the active output directions $(\mathbf{z}^{(1)}, \mathbf{y}^{(1)})^T \in \mathbb{R}^m$ by *seeding* active input directions $(\mathbf{x}^{(1)}, \mathbf{z}^{(1)})^T \in \mathbb{R}^n$ with the Cartesian basis vectors in $\mathbb{R}^n$.

A first-order tangent code $F^{(1)} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^m$,

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{y}^{(1)} \end{pmatrix} := F^{(1)}(\mathbf{x}, \mathbf{x}^{(1)}),$$

computes a Jacobian $\times$ vector product alongside with the function value:

$$\begin{aligned} \mathbf{y} &:= F(\mathbf{x}) \\ \mathbf{y}^{(1)} &:= \nabla F(\mathbf{x}) \cdot \mathbf{x}^{(1)} \end{aligned}$$

For $i = 0, \ldots, n-1$

$$\begin{pmatrix} v_i \\ v_i^{(1)} \end{pmatrix} := \begin{pmatrix} x_i \\ x_i^{(1)} \end{pmatrix} \qquad \text{(seed)}$$

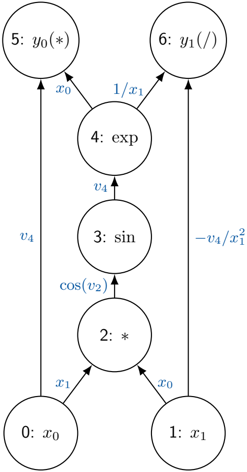
For $i = n, \ldots, q-1$

$$\begin{pmatrix} v_i \\ v_i^{(1)} \end{pmatrix} := \begin{pmatrix} \varphi_i(v_k)_{k \prec i} \\ \sum_{j \prec i} \frac{d\varphi_i(v_k)_{k \prec i}}{dv_j} \cdot v_j^{(1)} \end{pmatrix} \qquad \text{(propagate)}$$

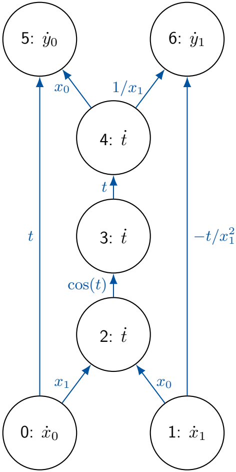
For $i = 0, \ldots, m-1$

$$\begin{pmatrix} y_i \\ y_i^{(1)} \end{pmatrix} := \begin{pmatrix} v_{n+p+i} \\ v_{n+p+i}^{(1)} \end{pmatrix} \qquad \text{(harvest)}$$

Tangent assignments augment primal ...



$$\dot{t} = \dot{x}_0 \cdot x_1 + x_0 \cdot \dot{x}_1$$
$$t := x_0 \cdot x_1$$
$$\dot{t} = \cos(t) \cdot \dot{t}$$
$$t := \sin(t)$$
$$t := e^t$$
$$\dot{t} = t \cdot \dot{t}$$
$$\dot{y}_0 = \dot{x}_0 \cdot t + x_0 \cdot \dot{t}$$
$$y_0 := x_0 \cdot t$$
$$\dot{y}_1 = \dot{t}/x_1 - t \cdot \dot{x}_1/x_1^2$$
$$y_1 := t/x_1$$



Euler-Maruyama live ...

Vector Tangents

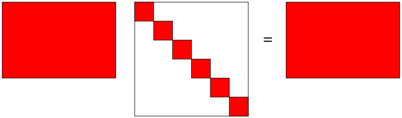
Computer Scientist's View (Simplified)

A first-order vector tangent code $F^{(1)} : \mathbb{R}^n \times \mathbb{R}^{n \times l} \to \mathbb{R}^m \times \mathbb{R}^{m \times l}$,

$$\begin{pmatrix} \mathbf{y} \\ Y^{(1)} \end{pmatrix} := F^{(1)}(\mathbf{x}, X^{(1)}),$$

computes a Jacobian $\times$ matrix product alongside with the function value:

$$\mathbf{y} := F(\mathbf{x})$$
$$Y^{(1)} := \nabla F(\mathbf{x}) \cdot X^{(1)}$$



... harvesting of the whole Jacobian by seeding input directions $X^{(1)}[i] \in \mathbb{R}^n$, $i = 0, \ldots, n-1$, with the Cartesian basis vectors in $\mathbb{R}^n$. Note concurrency!

Euler-Maruyama live ...

Adjoints

The Jacobian is a linear operator $\nabla F : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Its adjoint is defined as $(\nabla F)^* : \mathbb{R}^m \rightarrow \mathbb{R}^n$ where

$$< (\nabla F)^* \cdot \mathbf{y}_{(1)}, \mathbf{x}^{(1)} >_{\mathbf{R}^n} = < \mathbf{y}_{(1)}, \nabla F \cdot \mathbf{x}^{(1)} >_{\mathbf{R}^m} \quad ,$$

and where $< .,. >_{\mathbf{R}^n}$ and $< .,. >_{\mathbf{R}^m}$ denote appropriate scalar products in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

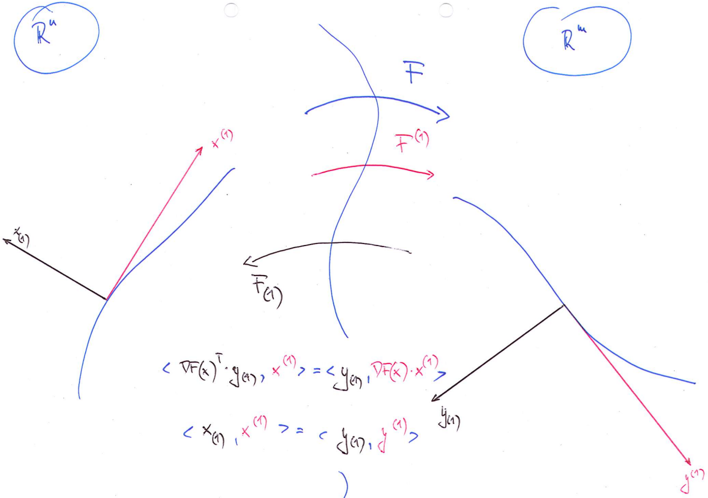
Theorem

$(\nabla F)^* = (\nabla F)^T$.

$$< \underbrace{(\nabla F)^T \cdot \mathbf{y}_{(1)}}_{[=:\mathbf{x}_{(1)}]}, \mathbf{x}^{(1)} >_{\mathbf{R}^n} = < \mathbf{y}_{(1)}, \underbrace{\nabla F \cdot \mathbf{x}^{(1)}}_{[=:\mathbf{y}^{(1)}]} >_{\mathbf{R}^m}$$

Note invariant at each point in the program execution $\rightarrow$ validation.

Adjoints



$$\langle DF(x)^T \cdot y_{(1)}, x^{(1)} \rangle = \langle y_{(1)}, DF(x) \cdot x^{(1)} \rangle$$

$$\langle x_{(1)}, x^{(1)} \rangle = \langle y_{(1)}, y^{(1)} \rangle$$

E.g. $y = \sin(x)$; $y = x_0 \cdot x_1$; ...

Adjoints

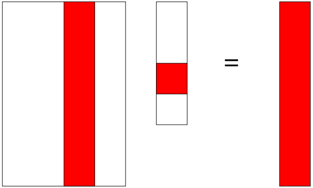
Mathematician's View

A first-order adjoint model $F_{(1)} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m \times \mathbb{R}^n$,

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{x}_{(1)} \end{pmatrix} = F_{(1)}(\mathbf{x}, \mathbf{y}_{(1)}),$$

defines an adjoint directional derivative alongside with the function value:

$$\begin{aligned} \mathbf{y} &= F(\mathbf{x}) \\ \mathbf{x}_{(1)} &= \nabla F(\mathbf{x})^T \cdot \mathbf{y}_{(1)} \end{aligned}$$



... definition of the whole Jacobian row-wise through input directions $\mathbf{y}_{(1)} \in \mathbb{R}^m$ equal to the Cartesian basis vectors in $\mathbb{R}^m$.

In

$$\left(\frac{dF}{d\mathbf{x}}\right)^T \cdot \mathbf{y}_{(1)}$$

the subscript on $\mathbf{y}$ denotes the first directional differentiation of F performed in adjoint mode in direction $\mathbf{y}_{(1)} \in \mathbb{R}^m$.

Enumeration of derivatives and distinction of super- and subscripts will become relevant in the discussion of higher derivatives computed by combinations of tangent and adjoint modes.

$$F_{(1)}: \mathbb{R}^n \times \mathbb{R}^{n_{\mathbf{x}}} \times \mathbb{R}^m \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{m_{\mathbf{y}}} \times \mathbb{R}^{\tilde{m}},$$

$$\begin{pmatrix} \mathbf{z} & \tilde{\mathbf{z}} & \mathbf{y} & \tilde{\mathbf{y}} & \mathbf{x}_{(1)} & \mathbf{z}_{(1)} & \mathbf{y}_{(1)} \end{pmatrix}^T := F_{(1)}(\mathbf{x}, \mathbf{x}_{(1)}, \tilde{\mathbf{x}}, \mathbf{z}, \mathbf{z}_{(1)}, \tilde{\mathbf{z}}, \mathbf{y}_{(1)}),$$

computes a shifted transposed Jacobian $\times$ vector product alongside with the function value:

$$\mathbb{R}^m \times \mathbb{R}^m \ni \begin{pmatrix} \mathbf{z} \\ \tilde{\mathbf{z}} \\ \mathbf{y} \\ \tilde{\mathbf{y}} \end{pmatrix} := F(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{z}, \tilde{\mathbf{z}})$$

$$\begin{pmatrix} \mathbf{x}_{(1)} \\ \mathbf{z}_{(1)} \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{(1)} \\ 0 \end{pmatrix} + \nabla F(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{z}, \tilde{\mathbf{z}})^T \cdot \begin{pmatrix} \mathbf{z}_{(1)} \\ \mathbf{y}_{(1)} \end{pmatrix}$$

$$\mathbf{y}_{(1)} := 0$$

The whole (dense) Jacobian can be harvested from the active input adjoints

$$\begin{pmatrix} \mathbf{x}_{(1)} \\ \mathbf{z}_{(1)} \end{pmatrix} \in \mathbb{R}^m$$

row-wise by seeding active output adjoints

$$\begin{pmatrix} \mathbf{z}_{(1)} \\ \mathbf{y}_{(1)} \end{pmatrix} \in \mathbb{R}^m$$

with the Cartesian basis vectors in $\mathbb{R}^m$ and for $\mathbf{x}_{(1)} := 0$ on input.

A first-order adjoint code $F_{(1)} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^n$,

$$\begin{pmatrix} \mathbf{y} \\ \mathbf{x}_{(1)} \end{pmatrix} := F_{(1)}(\mathbf{x}, \mathbf{x}_{(1)}, \mathbf{y}_{(1)}),$$

computes a shifted transposed Jacobian $\times$ vector product alongside with the function value:

$$\begin{aligned} \mathbf{y} &:= F(\mathbf{x}) \\ \mathbf{x}_{(1)} &:= \mathbf{x}_{(1)} + \nabla F(\mathbf{x})^T \cdot \mathbf{y}_{(1)} \end{aligned}$$

... harvesting of the whole Jacobian row-wise by seeding input directions $\mathbf{y}_{(1)} \in \mathbb{R}^m$ with the Cartesian basis vectors in $\mathbb{R}^m$ and for $\mathbf{x}_{(1)} = 0$ on input.

1. Augmented Primal (enable data flow reversal)

For $i = 0, \ldots, n-1$

$$v_i := x_i$$

$$\text{record } i \in V \ (v_{i_{(1)}} := x_{i_{(1)}})$$

For $i = n, \ldots, q-1$

$$v_i := \varphi_i(v_k)_{k \prec i}$$

$$\text{record } i \in V \ (v_{i_{(1)}} := 0)$$

$$\text{For } j \prec i : \text{record } (i,j) \in E \ (d_{j,i} := \frac{d\varphi_i(v_k)_{k \prec i}}{dv_j})$$

For $i = 0, \ldots, m-1$

$$y_i := v_{n+p+i}$$

2. Adjoint

For $i = 0, \ldots, m-1$

$$v_{n+p+i_{(1)}} := y_{i_{(1)}}$$

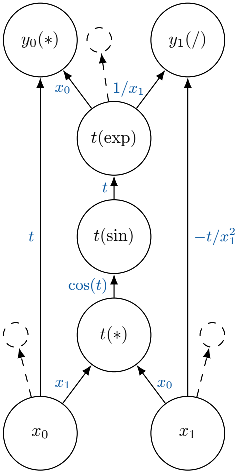
For $i = q-1, \ldots, n$

$$\forall\ (j,i) \in E : v_{j_{(1)}} := v_{j_{(1)}} + v_{i_{(1)}} \cdot d_{i,j}$$

For $i = 0, \ldots, n-1$

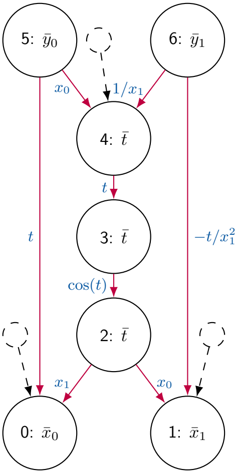
$$x_{i_{(1)}} := v_{i_{(1)}}$$

Mind overwrites and context ...



$t := x_0 \cdot x_1$
push(t); $t := \sin(t)$
$t := e^t$
$y_0 := x_0 \cdot t$
$y_1 := t/x_1$

$\bar{x}_1 \mathrel{+}= -t/x_1^2 \cdot \bar{y}_1$
$\bar{t} \mathrel{+}= 1/x_1 \cdot \bar{y}_1$
$\bar{y}_1 := 0$
$\bar{t} \mathrel{+}= x_0 \cdot \bar{y}_0$
$\bar{x}_0 \mathrel{+}= t \cdot \bar{y}_0$
$\bar{y}_0 := 0$
$\bar{t} := t \cdot \bar{t}$
pop(t); $\bar{t} := \cos(v_2) \cdot \bar{t}$
$\bar{x}_1 \mathrel{+}= x_0 \cdot \bar{t}$
$\bar{x}_0 \mathrel{+}= x_1 \cdot \bar{t}$



Euler-Maruyama live ...

Progress

Initially we consider multivariate scalar functions $y = F(\mathbf{x}) : D_F \subseteq \mathbb{R}^n \rightarrow I_F \subseteq \mathbb{R}$ in order to simplify the notation.

We assume F to be twice continuously differentiable over its domain D_F implying the existence of the Hessian

$$\nabla^2 F(\mathbf{x}) \equiv \frac{d^2 F}{d\mathbf{x}^2}(\mathbf{x}).$$

For multivariate vector functions the Hessian is a three-tensor complicating the notation slightly due to the need for tensor arithmetic; see later.

A second-order *central finite difference* quotient

$$\frac{d^2 F}{dx_i dx_j}(\mathbf{x}^0) \approx \Big[F(\mathbf{x}^0 + (\mathbf{e}_j + \mathbf{e}_i)\cdot h) - F(\mathbf{x}^0 + (\mathbf{e}_j - \mathbf{e}_i)\cdot h) \\ - F(\mathbf{x}^0 + (\mathbf{e}_i - \mathbf{e}_j)\cdot h) + F(\mathbf{x}^0 - (\mathbf{e}_j + \mathbf{e}_i)\cdot h)\Big] / (4\cdot h^2) \tag{1}$$

yields an approximation of the second directional derivative

$$y^{(1,2)} = {\mathbf{x}^{(1)}}^T \cdot \nabla^2 F(\mathbf{x}) \cdot \mathbf{x}^{(2)} \quad (\text{w.l.o.g. } m = 1)$$

as

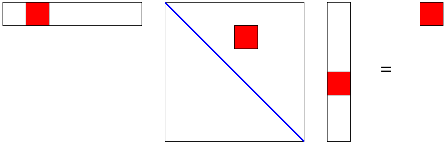
$$\begin{aligned}\frac{d^2 F}{dx_i dx_j}(\mathbf{x}^0) &\approx \frac{\frac{dF}{dx_i}(\mathbf{x}^0 + \mathbf{e}_j\cdot h) - \frac{dF}{dx_i}(\mathbf{x}^0 - \mathbf{e}_j\cdot h)}{2\cdot h} \\ &= \left[\frac{F(\mathbf{x}^0 + \mathbf{e}_j\cdot h + \mathbf{e}_i\cdot h) - F(\mathbf{x}^0 + \mathbf{e}_j\cdot h - \mathbf{e}_i\cdot h)}{2\cdot h}\right. \\ &\quad \left. - \frac{F(\mathbf{x}^0 - \mathbf{e}_j\cdot h + \mathbf{e}_i\cdot h) - F(\mathbf{x}^0 - \mathbf{e}_j\cdot h - \mathbf{e}_i\cdot h)}{2\cdot h}\right] / (2\cdot h).\end{aligned}$$

A second derivative code $F^{(1,2)} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, generated in Tangent-over-Tangent (ToT) mode computes

$$\begin{pmatrix} y \\ y^{(2)} \\ y^{(1)} \\ y^{(1,2)} \end{pmatrix} = F^{(1,2)}(\mathbf{x}, \mathbf{x}^{(2)}, \mathbf{x}^{(1)}, \mathbf{x}^{(1,2)}),$$

as follows:

$$\begin{pmatrix} y \\ y^{(2)} \\ y^{(1)} \\ y^{(1,2)} \end{pmatrix} := \begin{pmatrix} F(\mathbf{x}) \\ \nabla F(\mathbf{x}) \cdot \mathbf{x}^{(2)} \\ \nabla F(\mathbf{x}) \cdot \mathbf{x}^{(1)} \\ {\mathbf{x}^{(1)}}^T \cdot \nabla^2 F(\mathbf{x}) \cdot \mathbf{x}^{(2)} + \nabla F(\mathbf{x}) \cdot \mathbf{x}^{(1,2)} \end{pmatrix}.$$



$$\mathbf{x}^{(1)^T} \cdot \nabla^2 F(\mathbf{x}) \cdot \mathbf{x}^{(2)}$$

... accumulation of the whole Hessian element-wise by *seeding* input directions $\mathbf{x}^{(1)} \in \mathbb{R}^n$ $\mathbf{x}^{(2)} \in \mathbb{R}^n$ independently with the Cartesian basis vectors in $\mathbb{R}^n$ for $\mathbf{x}^{(1,2)} = 0$; harvesting from $y^{(1,2)}$.

Note: Approximate Tangents of Tangents

A second derivative code

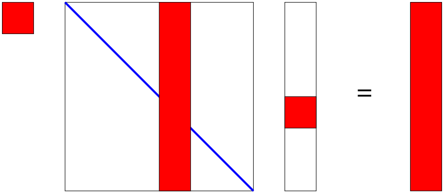
$$F^{(2)}_{(1)} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n,$$

generated in Tangent-over-Adjoint (ToA) mode computes

$$\begin{pmatrix} y \\ y^{(2)} \\ \mathbf{x}_{(1)} \\ \mathbf{x}^{(2)}_{(1)} \end{pmatrix} = F^{(2)}_{(1)}(\mathbf{x}, \mathbf{x}^{(2)}, \mathbf{x}_{(1)}, \mathbf{x}^{(2)}_{(1)}, y_{(1)}, y^{(2)}_{(1)}),$$

as follows:

$$\begin{pmatrix} y \\ y^{(2)} \\ \mathbf{x}_{(1)} \\ \mathbf{x}^{(2)}_{(1)} \end{pmatrix} := \begin{pmatrix} F(\mathbf{x}) \\ \nabla F(\mathbf{x}) \cdot \mathbf{x}^{(2)} \\ \mathbf{x}_{(1)} + \nabla F(\mathbf{x})^T \cdot y_{(1)} \\ \mathbf{x}^{(2)}_{(1)} + y_{(1)} \cdot \nabla^2 F(\mathbf{x}) \cdot \mathbf{x}^{(2)} + \nabla F(\mathbf{x})^T \cdot y^{(2)}_{(1)} \end{pmatrix}$$



$$y_{(1)} \cdot \nabla^2 F(\mathbf{x}) \cdot \mathbf{x}^{(2)}$$

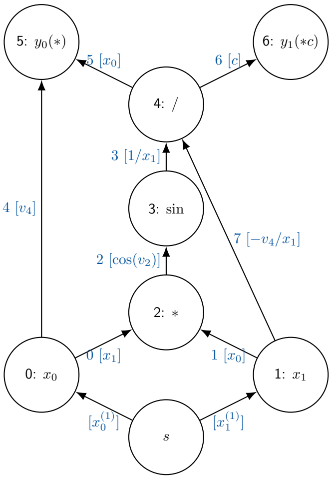
... accumulation of the whole Hessian column-wise by seeding input directions $\mathbf{x}^{(2)} \in \mathbb{R}^n$ with the Cartesian basis vectors in $\mathbb{R}^n$ for $\mathbf{x}_{(1)}^{(2)} = 0$, $y_{(1)} = 1$ and $y_{(1)}^{(2)} = 0$; harvesting from $\mathbf{x}_{(1)}^{(2)}$.

Note: Approximate Tangents of Adjoints

Progress

dco/c++ features

- tangents and adjoints of arbitrary order through recursive template instantiation for numerical simulation code implemented in C++
- front-ends for Fortran, C#, Matlab, Python (3x alpha)
- optimized assignment-level gradient code through expression templates
- cache-optimized internal representation in various incarnations
- vector modes / detection and exploitation of sparsity
- external adjoint / Jacobian interfaces
- user-defined intrinsics
- intrinsic NAG Library functions (e.g. Linear Algebra, Interpolation, Root Finding, Nearest Correlation Matrix)
- support for parallelism: thread-safe data structures, adjoint MPI library, GPU coupling, meta adjoint programming (map)



Tangent IDAG

We consider

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 * \sin(x_0 * x_1)/x_1 \\ \sin(x_0 * x_1)/x_1 * c \end{pmatrix}$$

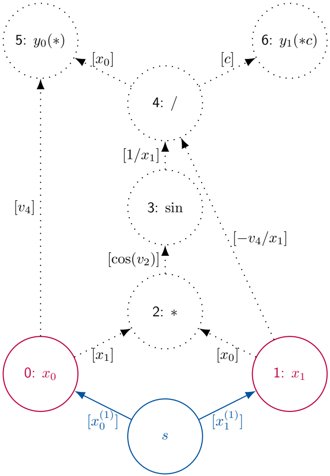
implemented as

$$t := \sin(x_0 * x_1)/x_1$$
$$y_0 := x_0 * t;\; y_1 := t * c$$

yielding SAC

$$v_2 := x_0 * x_1$$
$$v_3 := \sin(v_2)$$
$$v_4 := v_3/x_1$$
$$y_0 := x_0 * v_4;\; y_1 := v_4 * c$$

for some *passive* value c, i.e, no derivatives of or with respect to required; $\mathbf{x}, \mathbf{y}$, and t are *active*.

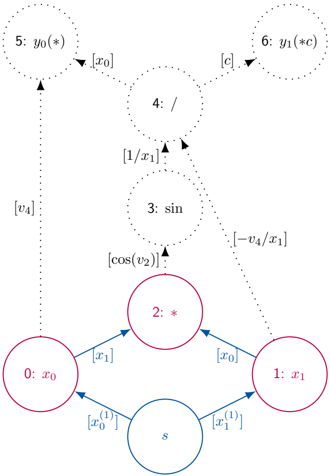


$$x_0 := ?$$
$$x_1 := ?$$

$$x_0^{(1)} := ?$$
$$x_1^{(1)} := ?$$

Tangents by Overloading

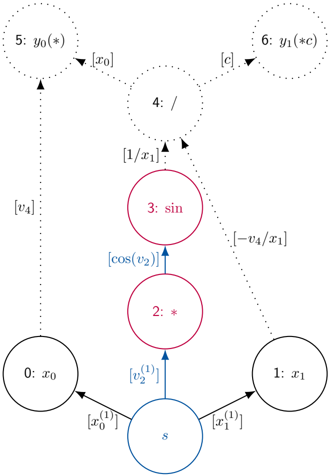
Propagate (Local Directional Derivatives)



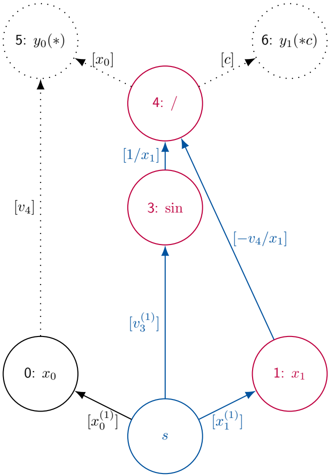
$$v_2 := x_0 * x_1$$
$$v_2^{(1)} := x_1 * x_0^{(1)} + x_0 * x_1^{(1)}$$

Tangents by Overloading

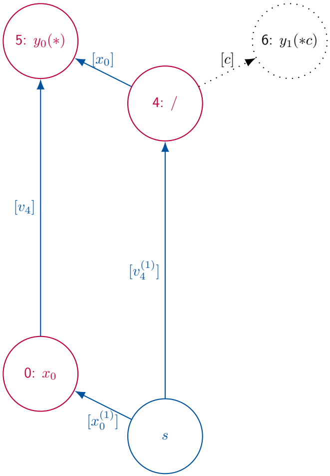
Propagate



$$v_2 := x_0 * x_1$$
$$v_2^{(1)} := x_1 * x_0^{(1)} + x_0 * x_1^{(1)}$$
$$v_3 := \sin(v_2)$$
$$v_3^{(1)} := \cos(v_2) * v_2^{(1)}$$



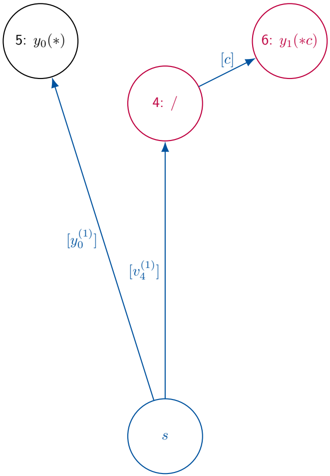
$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_2^{(1)} &:= x_1 * x_0^{(1)} + x_0 * x_1^{(1)} \\
v_3 &:= \sin(v_2) \\
v_3^{(1)} &:= \cos(v_2) * v_2^{(1)} \\
v_4 &:= v_3 / x_1 \\
v_4^{(1)} &:= (v_3^{(1)} - v_4 * x_1^{(1)}) / x_1
\end{aligned}
$$



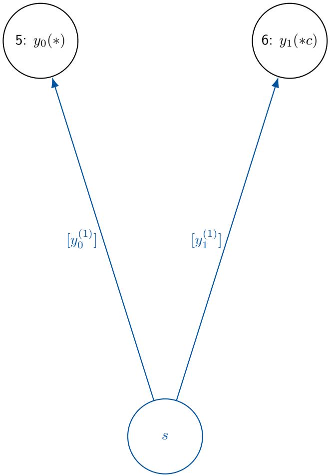
$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_2^{(1)} &:= x_1 * x_0^{(1)} + x_0 * x_1^{(1)} \\
v_3 &:= \sin(v_2) \\
v_3^{(1)} &:= \cos(v_2) * v_2^{(1)} \\
v_4 &:= v_3 / x_1 \\
v_4^{(1)} &:= (v_3^{(1)} - v_4 * x_1^{(1)}) / x_1 \\
y_0 &:= x_0 * v_4 \\
y_0^{(1)} &:= v_4 * x_0^{(1)} + x_0 * v_4^{(1)}
\end{aligned}
$$

Tangents by Overloading

Propagate



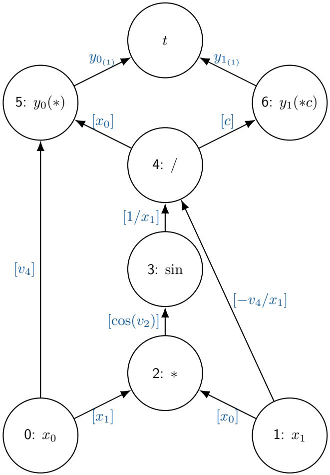
$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_2^{(1)} &:= x_1 * x_0^{(1)} + x_0 * x_1^{(1)} \\
v_3 &:= \sin(v_2) \\
v_3^{(1)} &:= \cos(v_2) * v_2^{(1)} \\
v_4 &:= v_3 / x_1 \\
v_4^{(1)} &:= (v_3^{(1)} - v_4 * x_1^{(1)}) / x_1 \\
y_0 &:= x_0 * v_4 \\
y_0^{(1)} &:= v_4 * x_0^{(1)} + x_0 * v_4^{(1)} \\
y_1 &:= v_4 * c \\
y_1^{(1)} &:= c * v_4^{(1)}
\end{aligned}
$$



$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_2^{(1)} &:= x_1 * x_0^{(1)} + x_0 * x_1^{(1)} \\
v_3 &:= \sin(v_2) \\
v_3^{(1)} &:= \cos(v_2) * v_2^{(1)} \\
v_4 &:= v_3 / x_1 \\
v_4^{(1)} &:= (v_3^{(1)} - v_4 * x_1^{(1)}) / x_1 \\
y_0 &:= x_0 * v_4 \\
y_0^{(1)} &:= v_4 * x_0^{(1)} + x_0 * v_4^{(1)} \\
y_1 &:= v_4 * c \\
y_1^{(1)} &:= c * v_4^{(1)}
\end{aligned}
$$

User Guide: $\mathbf{y}^{(1)} := \nabla F(\mathbf{x}) \cdot \mathbf{x}^{(1)}$

```
void driver(const vector<double>& xv, double &yv, vector<double> &g) {
  typedef dco::gt1s<double>::type DCO_T;
  size_t n=xv.size();
  DCO_T y=0;
  for (size_t i=0;i<n;i++) {
    vector<DCO_T> x(n,0);
    for (size_t j=0;j<n;j++) x[j]=xv[j];
    dco::derivative(x[i])=1; // seed directions
    f(x,y); // overloaded primal
    g[i]=dco::derivative(y); // harvest directional derivatives
  }
  yv=dco::value(y); // extract function value
}
```



Adjoint IDAG

We consider

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 * \sin(x_0 * x_1)/x_1 \\ \sin(x_0 * x_1)/x_1 * c \end{pmatrix}$$

implemented as

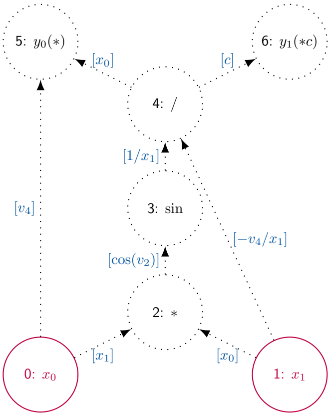
$$t := \sin(x_0 * x_1)/x_1$$
$$y_0 := x_0 * t$$
$$y_1 := t * c$$

yielding SAC

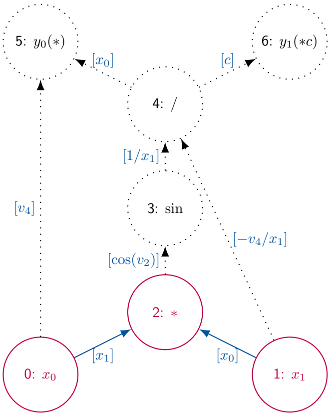
$$v_2 := x_0 * x_1$$
$$v_3 := \sin(v_2)$$
$$v_4 := v_3/x_1$$
$$y_0 := x_0 * v_4$$
$$y_1 := v_4 * c$$

for some passive value c.

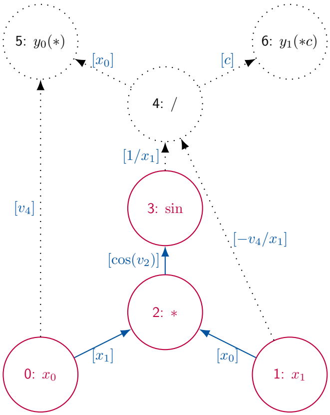
Register (Independent Inputs with Tape)



$$x_0 := ?$$
$$x_1 := ?$$



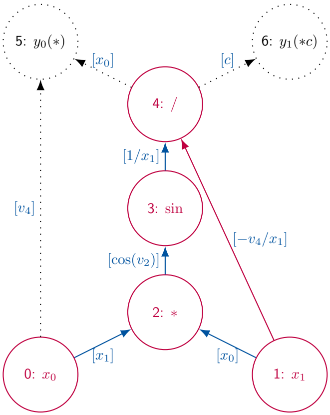
$$v_2 := x_0 * x_1$$



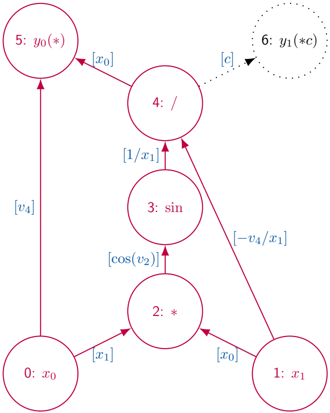
$$v_2 := x_0 * x_1$$
$$v_3 := \sin(v_2)$$

Adjoints by Overloading

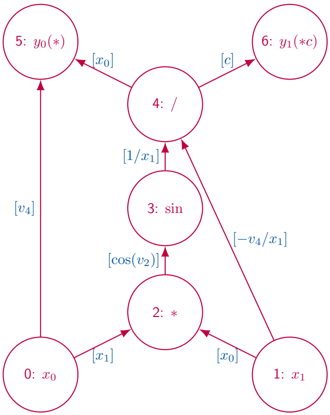
Record (Tape)



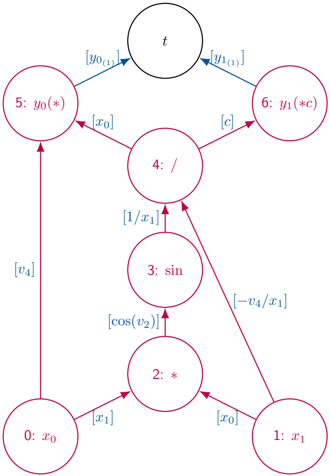
$$v_2 := x_0 * x_1$$
$$v_3 := \sin(v_2)$$
$$v_4 := v_3/x_1$$



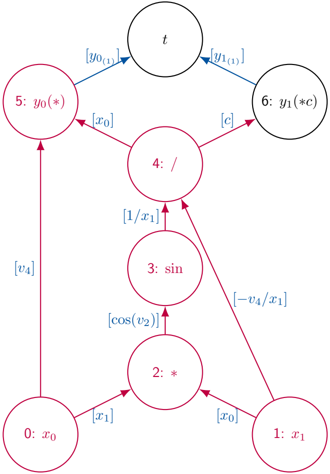
$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_3 &:= \sin(v_2) \\
v_4 &:= v_3 / x_1 \\
y_0 &:= x_0 * v_4
\end{aligned}
$$



$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_3 &:= \sin(v_2) \\
v_4 &:= v_3 / x_1 \\
y_0 &:= x_0 * v_4 \\
y_1 &:= v_4 * c
\end{aligned}
$$



$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_3 &:= \sin(v_2) \\
v_4 &:= v_3 / x_1 \\
y_0 &:= x_0 * v_4 \\
y_1 &:= v_4 * c \\
y_{0_{(1)}} &:= ? \\
y_{1_{(1)}} &:= ? \\
x_{0_{(1)}} &:= ? \\
x_{1_{(1)}} &:= ? \\
v_{2_{(1)}} &:= 0 \\
v_{3_{(1)}} &:= 0 \\
v_{4_{(1)}} &:= 0
\end{aligned}
$$



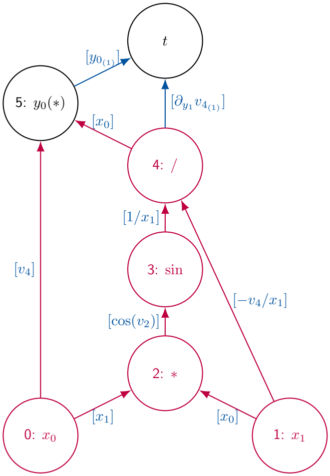
$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_3 &:= \sin(v_2) \\
v_4 &:= v_3 / x_1 \\
y_0 &:= x_0 * v_4 \\
y_1 &:= v_4 * c \\
v_{4_{(1)}} &+= c * y_{1_{(1)}}
\end{aligned}
$$

Note C++ Syntax:

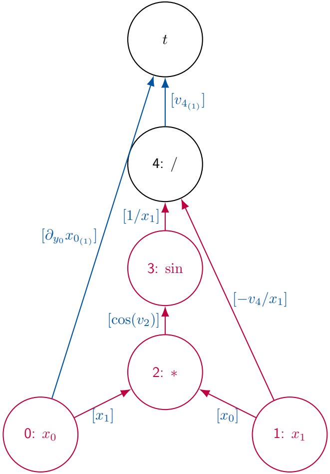
$$v_{4_{(1)}} += c * y_{1_{(1)}}$$

$$\Leftrightarrow$$

$$v_{4_{(1)}} := v_{4_{(1)}} + c * y_{1_{(1)}} .$$

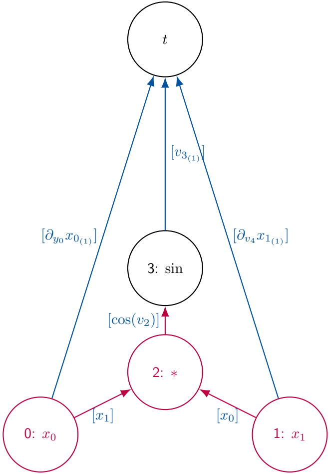


$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_3 &:= \sin(v_2) \\
v_4 &:= v_3 / x_1 \\
y_0 &:= x_0 * v_4 \\
y_1 &:= v_4 * c \\
v_{4_{(1)}} &+= c * y_{1_{(1)}} \\
v_{4_{(1)}} &+= x_0 * y_{0_{(1)}} \\
x_{0_{(1)}} &+= v_4 * y_{0_{(1)}}
\end{aligned}
$$



$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_3 &:= \sin(v_2) \\
v_4 &:= v_3 / x_1 \\
y_0 &:= x_0 * v_4 \\
y_1 &:= v_4 * c \\
v_{4_{(1)}} &+= c * y_{1_{(1)}} \\
v_{4_{(1)}} &+= x_0 * y_{0_{(1)}} \\
x_{0_{(1)}} &+= v_4 * y_{0_{(1)}} \\
u &:= 1/x_1 \\
v_{3_{(1)}} &+= u * v_{4_{(1)}} \\
x_{1_{(1)}} &-= v_4 * u * v_{4_{(1)}}
\end{aligned}
$$

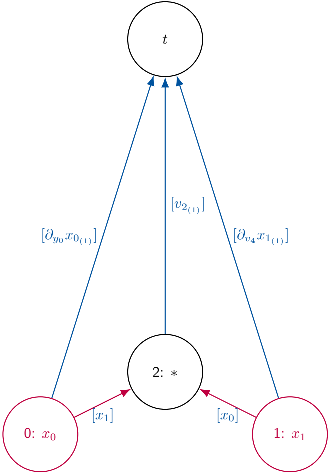
Interpret (Tape)



$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_3 &:= \sin(v_2) \\
v_4 &:= v_3 / x_1 \\
y_0 &:= x_0 * v_4 \\
y_1 &:= v_4 * c \\
v_{4_{(1)}} &+= c * y_{1_{(1)}} \\
v_{4_{(1)}} &+= x_0 * y_{0_{(1)}} \\
x_{0_{(1)}} &+= v_4 * y_{0_{(1)}} \\
u &:= 1/x_1 \\
v_{3_{(1)}} &+= u * v_{4_{(1)}} \\
x_{1_{(1)}} &-= v_4 * u * v_{4_{(1)}} \\
v_{2_{(1)}} &+= \cos(x_2) * v_{3_{(1)}}
\end{aligned}
$$

Adjoints by Overloading

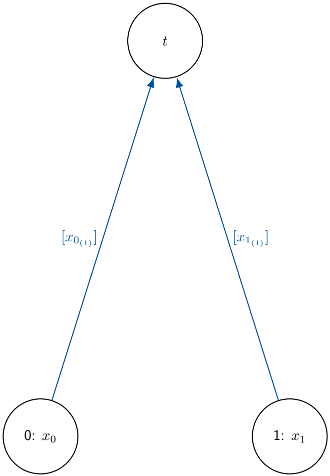
Interpret (Tape)



$$v_2 := x_0 * x_1$$
$$v_3 := \sin(v_2)$$
$$v_4 := v_3 / x_1$$
$$y_0 := x_0 * v_4$$
$$y_1 := v_4 * c$$
$$v_{4_{(1)}} += c * y_{1_{(1)}}$$
$$v_{4_{(1)}} += x_0 * y_{0_{(1)}}$$
$$x_{0_{(1)}} += v_4 * y_{0_{(1)}}$$
$$u := 1/x_1$$
$$v_{3_{(1)}} += u * v_{4_{(1)}}$$
$$x_{1_{(1)}} -= v_4 * u * v_{4_{(1)}}$$
$$v_{2_{(1)}} += \cos(x_2) * v_{3_{(1)}}$$
$$x_{0_{(1)}} += x_1 * v_{2_{(1)}}$$
$$x_{1_{(1)}} += x_0 * v_{2_{(1)}}$$

Adjoints by Overloading

Harvest



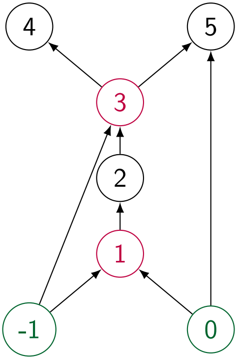
$$
\begin{aligned}
v_2 &:= x_0 * x_1 \\
v_3 &:= \sin(v_2) \\
v_4 &:= v_3 / x_1 \\
y_0 &:= x_0 * v_4 \\
y_1 &:= v_4 * c \\
v_{4_{(1)}} &+= c * y_{1_{(1)}} \\
v_{4_{(1)}} &+= x_0 * y_{0_{(1)}} \\
x_{0_{(1)}} &+= v_4 * y_{0_{(1)}} \\
u &:= 1/x_1 \\
v_{3_{(1)}} &+= u * v_{4_{(1)}} \\
x_{1_{(1)}} &-= v_4 * u * v_{4_{(1)}} \\
v_{2_{(1)}} &+= \cos(x_2) * v_{3_{(1)}} \\
x_{0_{(1)}} &+= x_1 * v_{2_{(1)}} \\
x_{1_{(1)}} &+= x_0 * v_{2_{(1)}}
\end{aligned}
$$

Adjoints with dco/c++

Driver

User Guide: $\mathbf{x}_{(1)} := \nabla F(\mathbf{x})^T \cdot \mathbf{y}_{(1)}$

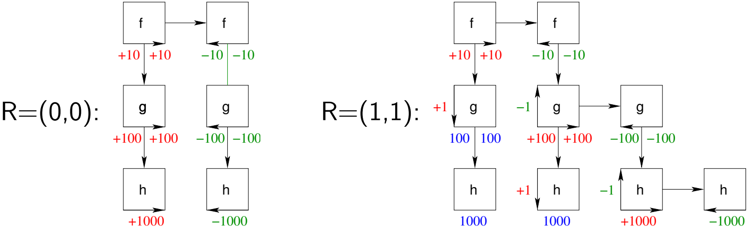
```
void driver(const vector<double> &xv, double &yv, vector<double> &g) {
  typedef dco::ga1s<double> DCO_M; // dco mode
  typedef DCO_M::type DCO_T; // dco type
  typedef DCO_M::tape_t DCO_TAPE_T; /dco tape type
  size_t n=xv.size();
  vector<DCO_T> x(n); DCO_T y;
  DCO_M::global_tape=DCO_TAPE_T::create(); // tape creation
  for (size_t i=0;i<n;i++) { // independent tape entries
    x[i]=xv[i]; DCO_M::global_tape->register_variable(x[i]);
  }
  f(x,y); // overloaded primal
  DCO_M::global_tape->register_output_variable(y); // dependent tape entry
  yv=dco::value(y); dco::derivative(y)=1; // seed
  DCO_M::global_tape->interpret_adjoint(); // tape interpretation
  for (size_t i=0;i<n;i++) { g[i]=dco::derivative(x[i]); } // harvest
  DCO_TAPE_T::remove(DCO_M::global_tape); // release tape
}
```



$v_5, v_4, \ldots, v_{-1}$

- U.N.: DAG REVERSAL is NP-Complete, J. Disc. Alg. 7(4), 402-410 (2009).
- U.N.: CALL TREE REVERSAL is NP-Complete, LNCSE 64, 13-22 (2008).
- J. Lotz et al.: Mixed Integer Programming for Call Tree Reversal, SIAM CSC (2016).

Example: Let $\overline{\text{MEM}} = 1110$ ...



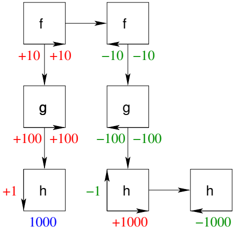
MEM=1220, OPS=0

MEM=1110, OPS=2200

Example: Let $\overline{\text{MEM}} = 1110$ ... Greedy Heuristics

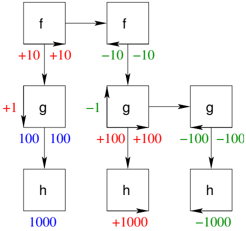
Smallest Memory Increase starts from R $= \mathbf{1}$ and yields ...
Largest Memory Decrease (LMD) starts from R $= \mathbf{0}$ and yields ...

R=(0,1):



MEM=1110, OPS=1000

R=(1,0):



MEM=1120, OPS=1200

Largest Memory Increase (LMI) remains at R $= \mathbf{1}$ as R $= (1, 0)$ infeasible

Algorithmic Differentiation (AD) is based on Symbolic Differentiation (SD).
AD approaches vary in terms of choice of SD level.

- U.N. et al.: Algorithmic differentiation of numerical methods: Tangent and adjoint solvers for parameterized systems of linear equations, RWTH Technical Report AIB-2012-10 (2012).
- U.N. et al.: Algorithmic differentiation of numerical methods: Tangent and adjoint solvers for parameterized systems of nonlinear equations, ACM TOMS 41 (4), 26 (2015).
- N. Safiran et al.: Algorithmic Differentiation of Numerical Methods: Second-Order Adjoint Solvers for Parameterized Systems of Nonlinear Equations, Procedia Computer Science 80, 2231-2235 (2016).
- J. Lotz et al.: A Case Study in Adjoint Sensitivity Analysis of Parameter Calibration, Procedia Computer Science 80, 201-211 (2016).

(Embedded) Symbolic Adjoints

Case Study: Root Finding

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
x^2 - p = 0 \arrow[r, "x_{(1)} \cdot \frac{d}{dp}", "\text{differentiate}"'] \arrow[d, "(x^*, G) = \overrightarrow{S}(x^0, p)"', "\text{record}"] \arrow[dr, "\text{perturb}"', "\dot{x}^* = S(x^0, p+h)"] & x_{(1)} \cdot \left(2 \cdot x \cdot \frac{dx}{dp} - 1\right) = 2 \cdot x \cdot \underbrace{\left(x_{(1)} \cdot \frac{dx}{dp}\right)}_{=p_{(1)}} - x_{(1)} = 0 \arrow[d, "\tilde{p}^* = \tilde{S}(\tilde{p}^0, x^*, x_{(1)})"] \\
x^* \approx \sqrt{p} \arrow[r, "\text{interpret}", "p^*_{(1)} = \overleftarrow{S}(G, x_{(1)})"'] & p_{(1)} = \frac{x_{(1)}}{2 \cdot \sqrt{p}} = \frac{x_{(1)}}{2 \cdot x} \approx p^*_{(1)} \approx \tilde{p}^* \approx x_{(1)} \cdot \frac{\dot{x}^* - x^*}{h}
\end{tikzcd}
$$

$$\underset{x(p)}{\operatorname{argmin}} \ \frac{1}{3}\cdot x^3 - p\cdot x + q$$

differentiate first-order optimality condition

$x_{(1)} \cdot \frac{d}{dp}$

$$x_{(1)} \cdot \left(2\cdot x \cdot \frac{dx}{dp} - 1\right) = 2\cdot x \cdot \underbrace{\left(x_{(1)}\cdot\frac{dx}{dp}\right)}_{=p_{(1)}} - x_{(1)} = 0$$

$(x^*, G) = \overrightarrow{S}(x^0, p)$ record

perturb $\dot{x}^* = S(x^0, p+h)$

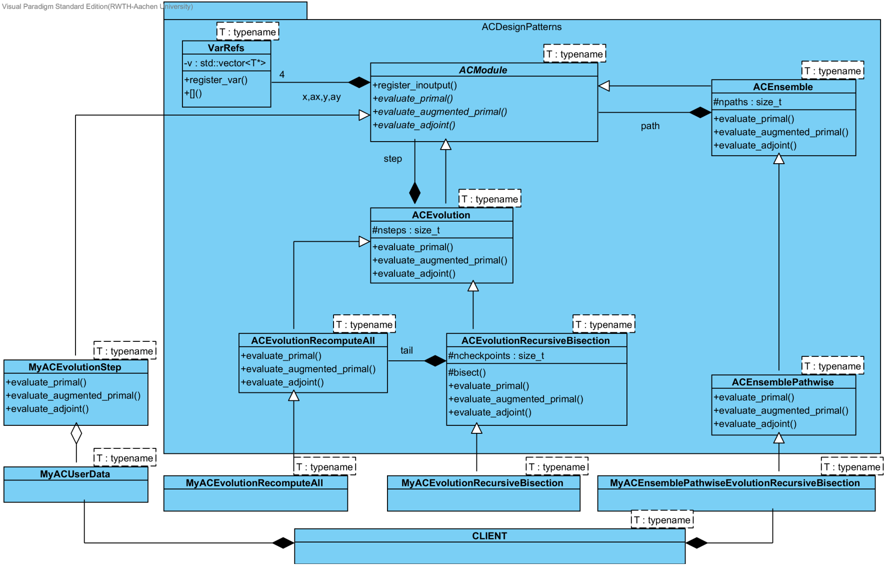
$\tilde{p}^* = \tilde{S}(\tilde{p}^0, x^*, x_{(1)})$

$$x^* \approx \sqrt{p}$$

interpret

$p^*_{(1)} = \overleftarrow{S}(G, x_{(1)})$

$$p_{(1)} = \frac{x_{(1)}}{2\cdot\sqrt{p}} = \frac{x_{(1)}}{2\cdot x} \approx p^*_{(1)} \approx \tilde{p}^* \approx x_{(1)} \cdot \frac{\dot{x}^* - x^*}{h}$$



Visual Paradigm Standard Edition(RWTH-Aachen University)
ACDesignPatterns
T : typename
VarRefs
-v : std::vector<T*>
+register_var()
+[]()
4
x,ax,y,ay
T : typename
ACModule
+register_inoutput()
+evaluate_primal()
+evaluate_augmented_primal()
+evaluate_adjoint()
T : typename
ACEnsemble
#npaths : size_t
+evaluate_primal()
+evaluate_augmented_primal()
+evaluate_adjoint()
path
step
T : typename
ACEvolution
#nsteps : size_t
+evaluate_primal()
+evaluate_augmented_primal()
+evaluate_adjoint()
T : typename
ACEvolutionRecomputeAll
+evaluate_primal()
+evaluate_augmented_primal()
+evaluate_adjoint()
tail
T : typename
ACEvolutionRecursiveBisection
#ncheckpoints : size_t
#bisect()
+evaluate_primal()
+evaluate_augmented_primal()
+evaluate_adjoint()
T : typename
ACEnsemblePathwise
+evaluate_primal()
+evaluate_augmented_primal()
+evaluate_adjoint()
T : typename
MyACEvolutionStep
+evaluate_primal()
+evaluate_augmented_primal()
+evaluate_adjoint()
T : typename
MyACUserData
T : typename
MyACEvolutionRecomputeAll
T : typename
MyACEvolutionRecursiveBisection
T : typename
MyACEnsemblePathwiseEvolutionRecursiveBisection
T : typename
CLIENT