Martingale Measures & Change of Measure Explained

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Abstract

An option is a financial instrument that allows the holder to buy or sell an underlying security in the future at an agreed strike or price set today. European options are often priced under the assumption of constant interest rates as seen in the Black-Scholes (1973) model.

In interest rate markets however the underlying security is an interest rate, which cannot be assumed constant. Likewise bond markets have a similar requirement. To relax such an assumption option payoffs and prices can be evaluated as the expectation of a stochastic martingale process.

In this paper we illustrate how to use the change of measure technique to evaluate the dynamics of a stochastic process. Firstly we discuss the preliminaries, namely Martingale measures and numeraires. Secondly we model interest rates as a Vasicek short rate process. Finally we outline how to apply a change of measure technique, where it can be seen that a change of measure to the terminal-forward measure allows us to evaluate model dynamics and simplify the calculation.

1 Martingale Measures & Numeraires

Consider any option with a generic payoff denoted $\mathcal{H}_T$. The martingale representation theorem provides us with a framework to evaluate the price of an option using the below formula, whereby the price V_t at time t of such an option is evaluated with respect to a numeraire N and corresponding probability measure $\mathbb{Q}_N$.

$$\frac{V_t}{N_t} = \mathbb{E}^{\mathbb{Q}_N}\left[\frac{\mathcal{H}_T}{N_T}|\mathcal{F}_t\right] \tag{1}$$

which can also be written as

$$V_t = N_t.\mathbb{E}^{\mathbb{Q}_N}\left[\frac{\mathcal{H}_T}{N_T}|\mathcal{F}_t\right] \tag{2}$$

where
V_t is the option price evaluated at time t
N_t is the numeraire evaluated at time t
$\mathbb{E}^{\mathbb{Q}_N}[.]$ is an expectation with respect to the measure of numeraire N (discussed below)
$\mathcal{H}_T$ is a generic option payoff evaluated at time T

Under the 'Martingale Representation' approach an arbitrage free portfolio is formed to replicate the option using both the underlying and numeraire, which is a tradable asset. Together the underlying and numeraire form a perfect hedge. If the numeraire is to be part of a hedge portfolio it must be a positive tradable asset, which pays no dividends. The later condition ensures that we have a smooth continuous price process without jumps.

Each numeraire can be represented as a stochastic process[1] and therefore has a probability measure assigned to it. The probability measure corresponds to the probability density function governing the likelihood of price changes of the numeraire.

The numeraire also determines the denomination of the option price V_t or pricing units. If for example the option price is 100, the numeraire determines the units, e.g. pounds, euros, dollars. The numeraire is typically a cash instrument or bond, however it may be a completely different instrument such as, for example, a commodity or stock, provided it pays no dividends.

The most popular choice of numeraire would be a $savings\ account$, sometimes referred to as a cash bond, the associated equivalent probability measure is called the $risk$-$neutral$ measure and denoted $\mathbb{Q}$. A dollar savings account numeraire would denominate option prices V_t in dollars and would imply that the option replicating portfolio would comprise of the underlying and a dollar cash bond.

1.1 Savings Account Numeraire

Under the martingale representation theorem an option price is unique regardless of the choice of numeraire. Therefore option prices can be evaluated using any numeraire, subject to the conditions above, namely that the numeraire is a positive tradable non-dividend paying asset. However in many cases it is not convenient and in some cases not possible to evaluate the expectation term in equations (1) and (2).

A savings account[2] is formed by holding cash in a risk-free account that accrues continuously compounded interest. Consider such a savings account process B_T[3] with dynamics

$$B_T = e^{\int_t^T r_u\ du} \tag{3}$$

[1] One could also consider a numeraire as a random variable with a corresponding probability density function.

[2] A savings account can also be thought of as investing in a cash bond or depositing funds in a savings account, which is assumed risk-free and accumulating continuously compounded interest.

[3] Note that time t represents the filtration time or pricing date, T the maturity of the cash bond and $(T - t)$ the amount of time of the savings account funds are held on deposit.

or in difference form

$$dB_t = r_t B_t dt \tag{4}$$

Applying the martingale representation formula using the savings account numeraire gives

$$\frac{V_t}{B_t} = \mathbb{E}^Q \left[\frac{\mathcal{H}_T}{B_T} | \mathcal{F}_t \right] \tag{5}$$

Rerranging this gives

$$\begin{aligned} V_t &= B_t \mathbb{E}^Q \left[\frac{\mathcal{H}_T}{B_T} | \mathcal{F}_t \right] \\ V_t &= \mathbb{E}^Q \left[\frac{B_t}{B_T} \mathcal{H}_T | \mathcal{F}_t \right] \\ V_t &= \mathbb{E}^Q \left[e^{-\int_t^T r_u \, du} \mathcal{H}_T | \mathcal{F}_t \right] \end{aligned} \tag{6}$$

The rates process r_u in equation (6) under the savings account numeraire is stochastic and not trivial to evaluate, so at this point consider a change of numeraire.

The price of a derivative is invariant regardless of the choice of numeraire and therefore a numeraire can be chosen to simplify the calculation of the Expectation term within equation (5). Other considerations relating to the choice of measure include:

1. **Analytical Tractability**
   Can a closed form solution be reached?

2. **Implementation**
   Is the solution compatible with a recombining tree[4] and monte carlo pricing methods?

3. **Behaviour**
   Does the solution exhibit mean reversion[5]?

4. **Dynamics**
   Do the solution dynamics imply positive interest rates at all times?
   Can rates go negative[6]?

1.2 Choice of Measure

Firstly recall from section (1) that our replication portfolio consists of an underlying and a numeraire, and note that the underlying and numeraire could both be similiar instruments, both

[4]Trees allow us allow us to price American options.

[5]Mean Reversion is an empirical market observation whereby over time certain instruments, such as interest rates, revert back to a mean average level.

[6]Interest rates can certainly be negative, but not as frequent or for such prolonged periods as suggested by a normal distribution say.

bond instruments perhaps. Secondly remembering that a measure or numeraire must be a tradable instrument, consider the tradable instruments available for selection.

In the interest rate market rates are derived from fixed income bonds. Typically these form the basis for our choice of numeraire. There are several bond instruments to choose from and the most popular choice is the savings account or cash bond as outlined above in section (1.1). Other choices of numeraire and measure are outlined below.

1.2.1 Risk-Neutral Measure, $\mathbb{Q}$

The tradable numeraire is a riskless cash bond or rolling savings account. The associated measure is called the risk-neutral measure $\mathbb{Q}$. This measure plays a key role in the Black-Scholes (1993) model.

1.2.2 Terminal-Forward Measure, $\mathbb{Q}_T$

The tradable numeraire is a zero coupon bond of maturity T, which is chosen to match the maturity of the underlying instrument to be priced. Hence the associated measure is called the terminal-forward measure $\mathbb{Q}_T$. This numeraire is used to price bonds, forwards and the like.

1.2.3 Forward Measure, $\mathbb{Q}_F$

The tradable numeraire is a zero coupon bond of maturity S, where $S > T$. That is to say the numeraire maturity S is greater than the maturity of the underlying instrument T. The associated measure is called the forward measure $\mathbb{Q}_F$.

1.2.4 Annuity Measure, $\mathbb{Q}_A$

The tradable numeraire is an annuity. The associated measure is called the annuity measure $\mathbb{Q}_A$. This numeraire is used to price swaptions.

Finally for completeness and reference purposes we should mention the real-world measure $\mathbb{P}$, which is not typically used for derivatives pricing.

1.2.5 Real-World Measure, $\mathbb{P}$

The real-world measure $\mathbb{P}$ gives the real-world probability of an event occurring. If an experiment were to be repeated many times this probability measure would be helpful in determining the long term average result.

Example: Real-World Measure
For example if we wanted to know the probability of rolling a fair die and landing on the number

six. The real-world probability gives $P(die = 6) = \frac{1}{6}$, but this result would only be of use if we were to repeat the experiment rolling the die many times.

That is to say the number 6 should appear on average one time out of six. For a small number of die throws this often not the case. However as we increase the number of die throws we converge to the result and more so as the number of throws increases to infinity.

As far as derivatives pricing is concerned the real-world measure $\mathbb{P}$ provides an indication of the long term average price of a derivative but would not give an arbitrage free price.

1.3 Change of Measure

To change between measures the Radon-Nikodym Derivative is used, which is often encountered when changing from the real-world measure to the risk-neutral measure and denoted[7] $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

Consider two numeraires N and M with associated equivalent martingale measures $\mathbb{Q}_N$ and $\mathbb{Q}_M$. Under the $\mathbb{Q}_N$ measure we have

$$V_t = N_t \mathbb{E}^{\mathbb{Q}_N}\left[\frac{\mathcal{H}_T}{N_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_N}\left[\frac{N_t}{N_T}\mathcal{H}_T|\mathcal{F}_t\right] \tag{7}$$

and under the $\mathbb{Q}_M$ measure we have

$$V_t = M_t \mathbb{E}^{\mathbb{Q}_M}\left[\frac{\mathcal{H}_T}{M_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\frac{M_t}{M_T}\mathcal{H}_T|\mathcal{F}_t\right] \tag{8}$$

equating equations (7) and (8) gives

$$\mathbb{E}^{\mathbb{Q}_N}\left[\mathcal{H}_T\frac{N_t}{N_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\mathcal{H}_T\frac{M_t}{M_T}|\mathcal{F}_t\right] \tag{9}$$

Stochastic Terms
The N_T and M_T terms in equations (7) to (9) above are stochastic and must remain within the expectation operator, however N_t and M_t are known values at the filtration time t and could be treated as constants.

The Radon-Nikodym derivative is a ratio of probability measures $\left(\frac{d\mathbb{Q}_{New}}{d\mathbb{Q}_{Old}}\right)$ such that we divide (and eliminate) the old measure and multiply (and introduce) the new measure.

We define the Radon-Nikodym derivative of $d\mathbb{Q}_M$ with respect to $d\mathbb{Q}_N$ as below

$$\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N} = \frac{\left(\frac{M_t}{M_T}\right)}{\left(\frac{N_t}{N_T}\right)} = \left(\frac{M_t}{M_T}\frac{N_T}{N_t}\right) \tag{10}$$

[7] where $d\mathbb{Q}$ represents the risk-neutral measure we are changing to and $d\mathbb{P}$ is the real-world measure we are changing from.

multiplying the left-hand side LHS of equation (9) by the Radon-Nikodym derivative changes the LHS measure from $\mathbb{Q}_{\mathbb{N}}$ to $\mathbb{Q}_{\mathbb{M}}$ as demonstrated below

$$\mathbb{E}^{\mathbb{Q}_M}\left[\frac{N_t}{N_T}\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N}\mathcal{H}_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\mathcal{H}_T\frac{N_t}{N_T}\left(\frac{M_t}{M_T}\frac{N_T}{N_t}\right)|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\mathcal{H}_T\frac{M_t}{M_T}|\mathcal{F}_t\right] \tag{11}$$

which leads to and implies

$$\mathbb{E}^{\mathbb{Q}_N}\left[\frac{N_t}{N_T}\mathcal{H}_T|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_M}\left[\frac{N_t}{N_T}\frac{d\mathbb{Q}_M}{d\mathbb{Q}_N}\mathcal{H}_T|\mathcal{F}_t\right] \tag{12}$$

Equation (12) demonstrates how to move from one measure $\mathbb{Q}_N$ to another $\mathbb{Q}_M$, namely from equation (7) to (8).

2 Summary of the Vasicek Short Rate Model

In our previous paper [4] 'An Overview of the Vasicek Short Rate Model' the Vasicek model was outlined and reviewed, below we summarize the the main points.

2.1 Short Rate Process

The Vasicek short rate model has an SDE with the following functional form

$$dr_t = (\theta - ar_t)dt + \sigma dB_t \tag{13}$$

which can also be represented as

$$dr_t = a(b - r_t)dt + \sigma dB_t \tag{14}$$

where
a = Speed of Mean Reversion, $0 \leq a \leq 1$
b = Mean Reversion Level
θ = ab
r_t = Short Rate at time, t
σ = Short Rate Volatility
B_t = Brownian Motion Process at time, t

2.2 Short Rate Solution

The solution to the Vasicek SDE in equation (13) follows, whose derivation can be found in [4]. It is important to note that this solution is under the savings account numeraire with the corresponding risk-neutral measure $\mathbb{Q}$.

$$r_t = e^{-a(t-s)}r_s + \frac{\theta}{a}\left(1 - e^{-a(t-s)}\right) + \sigma\int_{u=s}^{t} e^{-a(t-u)}\, dB_u \tag{15}$$

2.3 Dynamics

The distribution of the short rate solution in equation (15) is primarily determined by the Brownian process, which is Gaussian having dynamics $B_t \sim \mathcal{N}(0, t)$.

$$r_t \sim N\left(\frac{\theta}{a,}, \frac{\sigma^2}{2a}\right) \tag{16}$$

2.4 Why Change to the Terminal Forward Measure?

The price of an option at time t using the numeraire N with a corresponding risk-neutral measure $\mathbb{Q}_N$ is defined in equations (1) and (2) as

$$V_t = N_t \mathbb{E}^{\mathbb{Q}_N}\left[\frac{\mathcal{H}_T}{N_T}|\mathcal{F}_t\right] \tag{17}$$

Recalling that the option price is measure invariant[8], careful attention is paid to the $\left(\frac{\mathcal{H}_T}{N_T}\right)$ term within the expectation of (17) above and a measure is chosen to simplify the expectation as much as possible.

Using the cash account measure $\mathbb{Q}$ with a cash bond numeraire B_t as defined in equations (3) and (4) the option price is determined as

$$V_t = \underbrace{B_t}_{=1} \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathcal{H}_T}{B_T}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathcal{H}_T}{e^{\int_t^T r(u)\ du}}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\ du}\mathcal{H}_T|\mathcal{F}_t\right] \tag{18}$$

In this scenario, when using the cash account measure, the bond term B_t outside the expectation in (18) simplifies to unity i.e. $B_t = 1$. However the bond term B_T inside the expectation remains and is stochastic. This stochastic term cannot be factored outside of the expectation operator. The cash measure does not simplify the calculation.

Under the terminal forward measure $\mathbb{Q}_T$ all bonds $P(t, T)$ both pure discount and coupon bearing mature at par[9]. Therefore at maturity $P(T, T) = 1$ and changing to this measure would conveniently lead to a simplified expression for the option price as follows.

$$V_t = P(t, T)\mathbb{E}^{\mathbb{Q}_T}\left[\frac{\mathcal{H}_T}{\underbrace{P(T, T)}_{=1}}|\mathcal{F}_t\right] = P(t, T)\mathbb{E}^{\mathbb{Q}_T}\left[\mathcal{H}_T|\mathcal{F}_t\right] \tag{19}$$

Specifically for a European style option the payoff $\mathcal{H}_T$ would be specified as

$$\mathcal{H}_T = max\left(\phi\left(P(t, T) - K\right), 0\right) \tag{20}$$

[8] That is the price is constant regardless of the choice of numeraire. Furthermore since the choice of measure is discretionary, one typically selects a measure to simplify the calculation.

[9] This means that at maturity we receive back 100% of the bond's notional or face value.

or equivalently

$$\mathcal{H}_T = \phi\left(P(t,T) - K\right)^+ \tag{21}$$

where

$$\phi = \begin{cases} +1 & \text{for a call option} \\ -1 & \text{for a put option} \end{cases} \tag{22}$$

Using the terminal forward measure $\mathbb{Q}_T$ it follows that the European option price V_t at time t on an underlying bond $P(t,T)$ having maturity T with $t < T$ is

$$V_t = P(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\phi\left(P(t,T) - K\right)^+ |\mathcal{F}_t\right] \tag{23}$$

This expression is easier to evaluate than that under the cash measure.

2.5 How to Change to the Terminal Forward Measure

Having established that a switch of measure to the terminal forward measure is desirable, attention is drawn to the fact that at this point the solution to the Vasicek SDE has been determined under the cash account measure. Via the change of measure process we proceed to demonstrate how to change to the terminal measure and follow-up by determining the Vasicek solution under this new measure.

The risk-neutral measure $\mathbb{Q}$ is associated with the risk free cash account numeraire B_t. Consider another general numeraire N_t with the associated equivalent martingale measure $\mathbb{Q}_N$ and the below dynamics.

Notation

To avoid confusion, between the cash bond and Brownian motion, W has been used in this section to denote the Brownian / Wiener process instead of the usual B.

$$dB_t = r_t B_t dt \tag{24}$$

$$dN_t = r_t N_t dt + \sigma_t^N N_t dW_t^{\mathbb{Q}} \tag{25}$$

Using Itô's Lemma to evaluate the dynamics for the ratio of B to N gives

$$d\left(\frac{B_t}{N_t}\right) = B_t d\left(\frac{1}{N_t}\right) + \left(\frac{1}{N_t}\right) dB_t \tag{26}$$

Let $X_t = d\left(\frac{1}{N_t}\right)$ and evaluate using Itô's Lemma

$$\begin{aligned} dX_t &= \frac{dX_t}{dN_t}dN_t + \frac{1}{2}\frac{d^2X_t}{dN_t^2}dN_t^2 \\ &= -\frac{1}{N_t^2}\left(r_t N_t dt + \sigma_t^N N_t dW_t^{\mathbb{Q}}\right) + \frac{1}{2}\left(\frac{2}{N_t^3}\right)\left(\sigma_t^N\right)^2 N_t^2 dt \\ &= -\frac{r_t}{N_t}dr - \frac{\sigma_t^N}{N_t}dW_t^{\mathbb{Q}} + \frac{\left(\sigma_t^N\right)^2}{N_t}dt \\ &= \left(\frac{\left(\sigma_t^N\right)^2 - r_t}{N_t}\right)dt - \left(\frac{\sigma_t^N}{N_t}\right)dW_t^{\mathbb{Q}} \end{aligned} \tag{27}$$

substituting (27) into (26)

$$\begin{aligned} d\left(\frac{B_t}{N_t}\right) &= B_t dX_t + \left(\frac{r_t B_t}{N_t}\right) dt \\ &= B_t\left[\left(\frac{\left(\sigma_t^N\right)^2 - r_t}{N_t}\right) dt - \left(\frac{\sigma_t^N}{N_t}\right) dW_t^{\mathbb{Q}}\right] + \left(\frac{r_t B_t}{N_t}\right) dt \\ &= \frac{\left(\sigma_t^N\right)^2 B_t}{N_t} dt - \frac{r_t B_t}{N_t} dt - \frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}} + \frac{r_t B_t}{N_t} dt \\ &= \frac{\left(\sigma_t^N\right)^2 B_t}{N_t} dt - \frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}} \end{aligned} \tag{28}$$

we know that $d\left(\frac{B_t}{N_t}\right)$ in equation (28) is a martingale under $\mathbb{Q}_N$ therefore

$$d\left(\frac{B_t}{N_t}\right) = -\frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}_N} \tag{29}$$

comparing (28) and (29) leads to

$$\frac{\left(\sigma_t^N\right)^2 B_t}{N_t} dt - \frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}} = -\frac{\sigma_t^N B_t}{N_t} dW_t^{\mathbb{Q}_N} \tag{30}$$

simple factorization and rearrangement gives

$$dW_t^{\mathbb{Q}_N} = dW_t^{\mathbb{Q}} - \sigma_t^N dt \tag{31}$$

We should recognize (31) as the Girsanov result with σ_t^N as the market price of risk λ.

Cameron-Martin-Girsanov Theorem

The Cameron-Martin-Girsanov theorem states that if we have an existing $\mathbb{Q}$-Brownian motion and a new equivalent $\mathbb{Q}_N$-Brownian motion then there exists a previsible $\mathcal{F}_t$ measurable process λ that provides a mechanism to change from the existing measure to the new one, such that

$$dW_t^{\mathbb{Q}_N} = dW_t^{\mathbb{Q}} + \lambda dt \tag{32}$$

and the corresponding λ[10] *to change from the old measure to the new one is quoted below*

$$\begin{aligned} \lambda &= \left(\frac{\mu_{\text{Old}} - \mu_{\text{New}}}{\sigma}\right) \\ &= \left(\frac{\mu^{\mathbb{Q}} - \mu^{\mathbb{Q}_N}}{\sigma}\right) \end{aligned} \tag{33}$$

noting from (28) and (29) that $\mu^{\mathbb{Q}} = \left(\frac{\left(\sigma_t^N\right)^2 B_t}{N_t}\right)$, $\mu^{\mathbb{Q}_N} = 0$ and that $\sigma = -\left(\frac{\sigma_t^N B_t}{N_t}\right)$ gives

$$\lambda = \left(\frac{\left(\frac{\left(\sigma_t^N\right)^2 B_t}{N_t}\right) - 0}{-\left(\frac{\sigma_t^N B_t}{N_t}\right)}\right) = -\sigma_t^N \tag{34}$$

[10]It is important to note that we must use $-\lambda$ or equivalently $-\sigma$ when the original Brownian process has a negative diffusion term, as outlined below.

substituting (34) into (32) leads to (31) confirming that which was stated above.

$$dW_t^{\mathbb{Q}_N} = dW_t^{\mathbb{Q}} - \sigma_t^N dt \tag{35}$$

Negative Diffusion Terms

It is important to note that when the diffusion term $dW^{\mathbb{Q}}$ in the original stochastic process is negative we must use $-\lambda$ or equivalently $-\sigma$ in order to successfully change measure. Alternatively we could make a positive variable substitution for the negative drift or even absorb the negative sign into the symmetric Brownian process, which all amount to the same course of action. Rearranging and substituting $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{Q}_N} + \sigma_t^N dt$ from equation (35) into the original stochastic process (28) confirms this to be the correct course of action to change the drift from that under the original measure to that of the new measure whilst at the same time making no change to the diffusion term.

When the numeraire N_t is pure discount bond $Z(t,T)$ with associated equivalent terminal-forward martingale measure $\mathbb{Q}_T$ then (35) becomes

$$dW^{\mathbb{Q}_T} = dW^{\mathbb{Q}} - \sigma_t^Z dt \tag{36}$$

Recalling the definition of the pure discount bond from [4]

$$Z(t,T) = A(t,T)e^{-r_t B(t,T)} \tag{37}$$

where

$$A(t,T) = e^{\left((B(t,T)-(T-t))\left(\frac{\theta}{a}-\frac{\sigma^2}{2a^2}\right)-\left(\frac{\sigma^2 B(t,T)^2}{4a}\right)\right)}$$

and

$$B(t,T) = \left(\frac{1-e^{-a(T-t)}}{a}\right)$$

Since r_t is normally distributed as shown in (16) this implies that $Z(t,T)$ is lognormally distributed[11]. Consequently $\ln(Z(t,T))$ has normal dynamics giving

$$\begin{aligned}
Var(\ln(Z(t,T))) &= Var(ln(A(t,T)e^{-B(t,T)r_t})) \\
&= Var(\ln(A(t,T)) - B(t,T)r_t) \\
&= Var(\ln(A(t,T)) + Var(-B(t,T)r_t) \\
&= B(t,T)^2 Var(r_t) \\
&= B(t,T)^2\sigma^2
\end{aligned} \tag{38}$$

Defining $\sigma_Z^2 = Var(\ln(Z(t,T)))$ and taking the square root leads to an expression for the volatility

$$\sigma_Z = \pm\sqrt{B(t,T)^2\sigma^2} \tag{39}$$

from which we take the negative root[12] to get

$$\sigma_Z = -B(t,T)\sigma \tag{40}$$

[11] This comes from the fact that r_t is stochastic and normally distributed. The exponential of any normal process is lognormal. The A(t,T) and B(t,T) terms are deterministic.

[12] Since the diffusion term containing $dW^{\mathbb{Q}}$ in (28) is negative.

it follows by substituting (40) into (36) that

$$dW^{\mathbb{Q}_T} = dW^{\mathbb{Q}} + B(t,T)\sigma dt \tag{41}$$

Notation
Attention is drawn to the fact that $B(t,T)$ above should not be confused with the cash account numeraire B_t.

Furthermore as discussed above can confirm the correctness of this measure change by substituting (41) into (28) to nullify the drift and make the process a martingale under the terminal forward measure. This will also confirm our intuition surrounding the use of the negative volatility parameter in (40).

2.6 Short Rate Solution using the Terminal Forward Measure

Substituting the change of measure kernel for the Brownian process, namely equation (36) from section (2.4) the Vasicek short rate SDE in equation (13) becomes

$$\begin{aligned} dr_t &= (\theta - ar_t)\,dt + \sigma dB_t^{\mathbb{Q}} \\ &= (\theta - ar_t)\,dt + \sigma\left(dB_t^{\mathbb{Q}_T} - \sigma B(t,T)dt\right) \\ &= \left(\theta - ar_t - \sigma^2 B(t,T)\right)dt + \sigma dB_t^{\mathbb{Q}_T} \end{aligned} \tag{42}$$

This new SDE under the terminal measure can also be solved using the integrating factor shorthand from [4]. Rearranging (42) and multiplying by the Integrating Factor, $I_t = e^{at}$ gives

$$\begin{aligned} dr_t + ar_t dt &= \left(\theta - \sigma^2 B(t,T)\right)dt + \sigma dB_t^{\mathbb{Q}_T} \\ \underbrace{Idr_t + Iar_t dt}_{=d(Ir_t)} &= I\left(\theta - \sigma^2 B(t,T)\right)dt + \sigma I dB_t^{\mathbb{Q}_T} \\ d(Ir_t) &= I\left(\theta - \sigma^2 B(t,T)\right)dt + \sigma I dB_t^{\mathbb{Q}_T} \\ d(e^{at}r_t) &= e^{at}\left(\theta - \sigma^2 B(t,T)\right)dt + \sigma e^{at} dB_t^{\mathbb{Q}_T} \end{aligned} \tag{43}$$

substituting for $B(t,T)$ from equation (37) gives

$$\begin{aligned} d(e^{at}r_t) &= e^{at}\left(\theta - \sigma^2\left(\frac{1 - e^{-a(T-t)}}{a}\right)\right)dt + \sigma e^{at} dB_t^{\mathbb{Q}_T} \\ &= \theta e^{at}dt - \sigma^2\left(\frac{e^{at} - e^{-a(T-2t)}}{a}\right)dt + \sigma e^{at} dB_t^{\mathbb{Q}_T} \\ &= \theta e^{at}dt - \left(\frac{\sigma^2}{a}\right)e^{at}dt + \left(\frac{\sigma^2}{a}\right)e^{-a(T-2t)}dt + \sigma e^{at} dB_t^{\mathbb{Q}_T} \end{aligned} \tag{44}$$

integrating over (s, t), where $0 < s < t$

$$\begin{aligned} e^{at}r_t - e^{as}r_s = &\frac{\theta}{a}\left(e^{at} - e^{as}\right) - \frac{\sigma^2}{a^2}\left(e^{at} - e^{as}\right) \\ &+ \frac{\sigma^2}{2a^2}\left(e^{-a(T-2t)} - e^{-a(T-2s)}\right) + \sigma\int_s^t e^{au} dB_u^{\mathbb{Q}_T} \end{aligned} \tag{45}$$

rearranging terms

$$e^{at}r_t = e^{as}r_s + \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right)\left(e^{at} - e^{as}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-2t)} - e^{-a(T-2s)}\right) + \sigma\int_s^t e^{au}dB_u^{\mathbb{Q}_T} \quad (46)$$

leading to

$$r_t = e^{-a(t-s)}r_s + \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right)\left(1 - e^{-a(t-s)}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right) + \sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T} \quad (47)$$

for convenience we can factorize and express this as

$$r_t = e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t) + \sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T} \quad (48)$$

where

$$F^{\mathbb{Q}_T}(s,t) = \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right)\left(1 - e^{-a(t-s)}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)} - e^{-a(T+t-2s)}\right) \quad (49)$$

This solution is identical to that under the cash measure, except that it contains an additional factor $F^{\mathbb{Q}_T}$.

2.7 Short Rate Dynamics under the Terminal Measure

The dynamics of the Vasicek solution under the terminal forward measure outlined in section (2.6) and equations (48) and (49) in particular are derived as follows.

Firstly observe that the short rate solution under the terminal measure is Gaussian, since the Brownian term is normally distributed by definition, whereby $B_t \sim \mathcal{N}(0, t)$.

The distribution mean μ can be found by taking the expectation of equation (48) and noting that the drift terms are deterministic and that both the expected value of the diffusion term and the stochastic integral are zero.

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) &= \mathbb{E}^{\mathbb{Q}_T}\left[e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t)|\mathcal{F}_s\right] + \underbrace{\mathbb{E}^{\mathbb{Q}_T}\left[\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_\mathbb{T}}|\mathcal{F}_s\right]}_{=0} \\ &= \mathbb{E}^{\mathbb{Q}_T}\left[e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t)|\mathcal{F}_s\right] \\ &= e^{-a(t-s)}r_s + F^{\mathbb{Q}_\mathbb{T}}(s,t) \end{aligned} \quad (50)$$

Similarly the variance σ^2 is

$$\begin{aligned} Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) &= \underbrace{Var^{\mathbb{Q}_T}\left(e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t)|\mathcal{F}_s\right)}_{=0} + Var^{\mathbb{Q}_T}\left(\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_\mathbb{T}}|\mathcal{F}_s\right) \\ &= Var^{\mathbb{Q}_T}\left(\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T}|\mathcal{F}_s\right) \\ &= \underbrace{\mathbb{E}^{\mathbb{Q}_T}\left[\left(\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T}\right)^2|\mathcal{F}_s\right]}_{\text{Apply Itô Isometry}} - \underbrace{\mathbb{E}^{\mathbb{Q}_T}\left[\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T}|\mathcal{F}_s\right]^2}_{=0} \end{aligned} \quad (51)$$

The variance of the drift term in (51) was zero and likewise the term expectation of the stochastic integral is zero. What remains is to apply expand and solve the squared stochastic term using Itô's isometry rule, which eliminates the randomness, since dB_t^2 becomes dt.

$$\begin{aligned} Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) &= \mathbb{E}^{\mathbb{Q}_T}\left[\left(\sigma\int_s^t e^{-a(t-u)}dB_u^{\mathbb{Q}_T}\right)^2|\mathcal{F}_s\right] \\ &= \mathbb{E}^{\mathbb{Q}_T}\left[\sigma^2\int_s^t e^{-2a(t-u)}du|\mathcal{F}_s\right] \\ &= \sigma^2\int_s^t e^{-2a(t-u)}du \\ &= \frac{\sigma^2\left(1-e^{-2a(t-s)}\right)}{2a} \end{aligned} \tag{52}$$

Hence under the terminal forward measure the Vasicek short rate has dynamics

$$r_t^{\mathbb{Q}_T} \sim \mathcal{N}\left(e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t), \frac{\sigma^2\left(1-e^{-2a(t-s)}\right)}{2a}\right) \tag{53}$$

where

$$F^{\mathbb{Q}_T}(s,t) = \left(\frac{\theta}{a}-\frac{\sigma^2}{a^2}\right)\left(1-e^{-a(t-s)}\right) + \frac{\sigma^2}{2a^2}\left(e^{-a(T-t)}-e^{-a(T+t-2s)}\right) \tag{54}$$

Finally the dynamics for the limiting case for t and T are considered. We examine the dynamics when the filtration time[13] $t \to \infty$ and bond maturity $T \to \infty$.

Limits under the Terminal Forward Measure

When using the savings account numeraire and the corrsponding cash measure $\mathbb{Q}$ the filtration time t had no upperbound and therefore we considered the limiting case to be $t \to \infty$. However when using a bond numeraire and the terminal forward measure $\mathbb{Q}_T$ the filtration time t cannot exceed the maturity of the underlying bond i.e. $t < T$. As a result under the terminal forward measure the limiting case for t becomes $t \to T$.

Knowing from (53) and (54) that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_T}[r_t|\mathcal{F}_s] &= e^{-a(t-s)}r_s + F^{\mathbb{Q}_T}(s,t) \\ &= e^{-a(t-s)}r_s + \left(\frac{\theta}{a}-\frac{\sigma^2}{a^2}\right)\left(1-e^{-a(t-s)}\right) + \left(\frac{\sigma^2}{2a^2}\right)\left(e^{-a(T-t)}-e^{-a(T+t-2s)}\right) \end{aligned} \tag{55}$$

the limit as $t \to \infty$ is given by

$$\begin{aligned} \lim_{t\to T}\left(\ \mathbb{E}^{\mathbb{Q}_T}[r_t|\mathcal{F}_s]\ \right) &= e^{-a(T-s)}r_s + \left(\frac{\theta}{a}-\frac{\sigma^2}{a^2}\right)\left(1-e^{-a(T-s)}\right) + \left(\frac{\sigma^2}{2a^2}\right)\left(\underbrace{e^{-a(T-T)}}_{=1} - e^{-a(T+T-2s)}\right) \\ &= e^{-a(T-s)}r_s + \left(\frac{\theta}{a}-\frac{\sigma^2}{a^2}\right)\left(1-e^{-a(T-s)}\right) + \left(\frac{\sigma^2}{2a^2}\right)\left(1-e^{-2a(T-s)}\right) \end{aligned} \tag{56}$$

[13] or pricing date

by further considering the limiting case where the bond maturity $T \to \infty$ we observe that the exponential terms in (56) tend to zero giving

$$\lim_{\substack{t\to T\\ T\to\infty}} \left(\ \mathbb{E}^{\mathbb{Q}_T}[r_t|\mathcal{F}_s] \ \right) = \left(\frac{\theta}{a} - \frac{\sigma^2}{a^2}\right) + \left(\frac{\sigma^2}{2a^2}\right) \\ = \left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right) \tag{57}$$

and similarly the variance is given by (53)

$$Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) = \frac{\sigma^2\left(1 - e^{-2a(t-s)}\right)}{2a} \tag{58}$$

letting $t \to T$ produces the below the limiting case

$$\lim_{t\to T} Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) = \frac{\sigma^2\left(1 - e^{-2a(T-s)}\right)}{2a} \tag{59}$$

and by further taking the limit $T \to \infty$ this becomes

$$\lim_{\substack{t\to T\\ T\to\infty}} Var^{\mathbb{Q}_T}(r_t|\mathcal{F}_s) = \frac{\sigma^2}{2a} \tag{60}$$

Therefore in the limiting case where $t \to T$ and $T \to \infty$ under the terminal forward measure has the following dynamics

$$r_t^{\mathbb{Q}_T} \sim N\left(\left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right), \frac{\sigma^2}{2a}\right) \tag{61}$$

or equivalently

$$r_t^{\mathbb{Q}_T} \sim N\left(\left(b - \frac{\sigma^2}{2a^2}\right), \frac{\sigma^2}{2a}\right) \tag{62}$$

Model Dynamics under Different Measures
When changing measures only the mean of the model distribution is transformed and the variance term being measure invariant remains unchanged.

Comparing the dynamics under the cash and terminal measures, as expected, only the distribution mean is different and the variance is unchanged and measure invariant. It can be seen that the expected value tends to $\left(b - \frac{\sigma^2}{2a^2}\right)$. Hence r_t in the limit tends to the reversion level less some factor $\frac{\sigma^2}{2a^2}$. The variance tends to $\left(\frac{\sigma^2}{2a}\right)$, which is the model variance scaled by the speed of mean reversion.

3 Option Pricing

To evaluate the price of a European option we can discount the expected value of the payoff using (19)

$$V_t = P(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\underbrace{\frac{\mathcal{H}_T}{P(T,T)}}_{=1}|\mathcal{F}_t\right] = P(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[\mathcal{H}_T|\mathcal{F}_t\right] \tag{63}$$

In the case when our underlying process is log-normal we can evaluate the price of a European option as the expected value of a log-normal process using the equation below and substituting the mean and variance from the model dynamics derived above. We provide a detailed example of how to do this in the case of bond options in our paper [6].

3.1 Expected Value of a Log-Normal Process

If X is a random variable that is lognormally distributed then let us define $Y := ln(X)$ with mean μ and variance σ^2 with $Y \sim N(\mu, \sigma^2)$. Knowing that the expectation of a random variable, X is defined as $\mathbb{E}(X) = \int_{-\infty}^{+\infty} X f(x)\, dx$, where f(x) denotes the probability density function of X we deduce that

$$\mathbb{E}^{\mathbb{Q}_T}\left[\phi\,(X-K)^+\right] = \int_{-\infty}^{\infty}\left[\phi\,(X-K)\right]^+ \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2}\,d\log x \tag{64}$$

Since $Y := ln(X)$ this can be written in terms of Y as

$$\mathbb{E}^{\mathbb{Q}_T}\left[\phi\,(X-K)^+\right] = \int_{-\infty}^{\infty}\left[\phi\,(e^y-K)\right]^+ \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}\,dy \tag{65}$$

4 Conclusion

In conclusion we have discussed numeraires, measures and how and why sometimes we apply a change of measure. We reviewed athe Vasicek short rate model for interest rates and evaluated the dynamics of the process, whereby changing from a risk neutral measure to a terminal-forward measure made the calculations easier. Finally we outlined how to proceed to value an option using the model dynamics derived, citing a detailed example.

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